

January 21, 2000

termination of the simplex algorithm

Might the process continue forever, going from dictionary to dictionary without ever arriving at a solution?

No. Because:

There are only a finite number of basic feasible solutions and therefore only finitely many vertices.

Once you leave a vertex (and thereby increase the value of the objective function), there is no way of ever returning because the value of the objective function cannot decrease.

So, the only way to get into trouble is if you stay at the same vertex forever. This CAN happen, and is known as cycling. But there are easy ways to avoid it by judicious choice of entering and leaving variables.

how bad can the simplex algorithm be?

Klee-Minty example

pivoting

example: reduce  $\begin{pmatrix} 3 & -1 & 1 & 5 \\ -2 & 5 & 8 & 14 \\ 1 & 1 & 3 & 7 \end{pmatrix}$  to  $\begin{pmatrix} 1 & 0 & 1 & 3 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ .

tableau format

Let's return to a problem we looked at earlier ("Example showing all basic solutions"):

$$\begin{aligned} &\text{min or max} \\ &x_1 + 2x_2 + 3x_3 \\ &\text{subject to} \\ &2 \leq x_1 + x_2 \leq 3 \\ &4 \leq x_1 + x_3 \leq 5 \\ &x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0 \end{aligned}$$

Instead of starting with a feasible basic solution in dictionary format, we start with one in tableau format. It is really the same thing, except that we move all the variables, basic and nonbasic, to the left. So, whereas we started in dictionary format with

$$\begin{aligned} x_1 &= 2 - x_2 + x_5 \\ x_3 &= 2 + x_2 - x_5 + x_7 \\ x_4 &= 1 - x_5 \\ x_6 &= 1 - x_7 \\ z &= 8 + 4x_2 - 2x_5 + 3x_7 \end{aligned}$$

we now start instead with the tableau

|   |    |   |   |    |   |    |  |    |       |
|---|----|---|---|----|---|----|--|----|-------|
| 1 | 1  | 0 | 0 | -1 | 0 | 0  |  | 2  | BASIS |
| 0 | -1 | 1 | 0 | 1  | 0 | -1 |  | 2  | 1     |
| 0 | 0  | 0 | 1 | 1  | 0 | 0  |  | 1  | 3     |
| 0 | 0  | 0 | 0 | 0  | 1 | 1  |  | 1  | 4     |
| 0 | 4  | 0 | 0 | -2 | 0 | 3  |  | -8 | 6     |

At each iteration, we examine the bottom row (which represents the objective function) looking for a positive entry. If there is none, we

have arrived at the maximizer. If there is one, we pick it as the entering variable. We look at the positive entries, if any, above that positive entry. If there is none, the problem is unbounded. If there is one or more, then for each one, we compute the ratio with the right hand side and pick one for which the ratio is smallest. Pivot on that entry. The pivot operation is identical to that performed in the Gauss-Jordan method for solving systems of linear equations. This produces the next tableau.

$$\begin{array}{ccccccc|c}
 1 & 1 & 0 & 0 & -1 & 0 & 0 & 2 & 2 \\
 1 & 0 & 1 & 0 & 0 & 0 & -1 & 4 & 3 \\
 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 4 \\
 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 6 \\
 -4 & 0 & 0 & 0 & 2 & 0 & 3 & -16 & 
 \end{array}$$

$$\begin{array}{ccccccc|c}
 1 & 1 & 0 & 0 & -1 & 0 & 0 & 2 & 2 \\
 1 & 0 & 1 & 0 & 0 & 1 & 0 & 5 & 3 \\
 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 4 \\
 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 7 \\
 -4 & 0 & 0 & 0 & 2 & -3 & 0 & -19 & 
 \end{array}$$

$$\begin{array}{ccccccc|c}
 1 & 1 & 0 & 1 & 0 & 0 & 0 & 3 & 2 \\
 1 & 0 & 1 & 0 & 0 & 1 & 0 & 5 & 3 \\
 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 5 \\
 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 7 \\
 -4 & 0 & 0 & -2 & 0 & -3 & 0 & -21 & 
 \end{array}$$

HW#12 Execute phase I of the simplex algorithm for the following problem (it is the same problem as exercise 11) using tableau format.

$$\begin{array}{l}
 \max x_2 \text{ subject to} \\
 x_1 + x_2 \geq 1 \\
 x_1 \leq 2 \\
 x_2 \leq 2 + 0.5 x_1 \\
 x_1, x_2 \geq 0
 \end{array}$$

Use your result to write down the initial tableau for phase II.

A look at the Fundamental Theorem (p. 42)

Every LP problem in standard form has the following three properties:

- i. if it has no optimal solution then it is infeasible or unbounded
- ii. if it has a feasible solution then it has a basic feasible solution
- iii. if it has an optimal solution then it has a basic optimal solution

Dual linear programming problems

see p. 56: correction! I had the inequalities wrong way during lecture

PRIMAL

$$\max c_1x_1 + \dots + c_nx_n$$

subject to

$$a_{11}x_1 + \dots + a_{1n}x_n \leq b_1$$

...

$$a_{m1}x_1 + \dots + a_{mn}x_n \leq b_m$$

$$\text{all } x_j \geq 0$$

DUAL

$$\min b_1y_1 + \dots + b_my_m$$

subject to

$$a_{11}y_1 + \dots + a_{m1}y_m \geq c_1$$

...

$$a_{1n}y_1 + \dots + a_{mn}y_m \geq c_n$$

$$\text{all } y_i \geq 0$$