max flow problem

additional source: R.T. Rockafellar, Network Flows and Monotropic Optimization, John Wiley & Sons 1984

some terminology

If x is a flow in a network with node-arc incidence matrix A, then div x = -Ax is called the divergence of x. It is a vector whose components specify the quantities leaving each node. If $S \subset N$ is a collection of nodes, the total quantity leaving S is div(S)= $\Sigma_{i \in S}\, y_i$, where y = div x. This is also called the divergence of x from S.

This terminology is somewhat confusing -- div x is a vector, but div(S) is a number. Note also that it would probably have made sense to write div(x,S).

The divergence of x from N is always zero.

proof: div(N)= $\Sigma_{i \in N}\, y_i = (1 \ldots 1)y = -(1 \ldots 1)Ax = 0$, since every column of A sums to zero. Recall that the ij-entry of A is -1 if arc j originates at node i, +1 if arc j terminates at node i, and 0 otherwise. *QED*

Proposition A: For any flow x, for any set $S \subset N$, div S = – div N\S.

Proof: We have

$$\text{div } S + \text{div } N\backslash S = \Sigma_{i \in S}\, y_i + \Sigma_{i \in N\backslash S}\, y_i = \Sigma_{i \in S}\, y_i = 0. \quad QED$$

cuts

Let $S \subset N$. The collections Q^+ of all arcs from S to N\S and Q^- of all arcs from N\S to S will be referred to as the cut Q=[S,N\S].

For the example in network A, let S={#3}. Then the cut Q contains three arcs: Q^+ contains arcs #3 and #5, and Q^- contains arc #2.

Again, for the example in network A, suppose S={#1,#4} (so that S contains nodes #1 and #4). Then Q contains four arcs: Q^+ contains arcs #1 and #2 and Q^- contains arcs #4 and #5.

Again, for the example in network A, suppose S={#1,#3,#4}. Then Q contains three arcs: Q^+ contains arcs #1 and #3 and Q^- contains arc #4.

If x is a flow and Q=[S,N\S] is a cut, the flux of x across Q is the quantity $\Sigma_{j \in Q^+}\, x(j) - \Sigma_{j \in Q^-}\, x(j)$.

The divergence principle: For any flow x and any cut Q=[S,N\S],

div(S) = flux of x across Q

Proof of the divergence principle: Letting a_i denote row i of A,

$$\text{div}(S) = \Sigma_{i \in S}\, (\text{div}\, x)_i = -\Sigma_{i \in S}\, a_i' x = -\Sigma_{i \in S}\, \Sigma_{j \in A}\, a_{ij} x(j)$$

$$= -\Sigma_{j \in A}\, \Sigma_{i \in S}\, a_{ij} x(j) = -\Sigma_{j \in A}\, x(j) \left(\Sigma_{i \in S}\, a_{ij} \right).$$

Observe:

If arc j goes from a node of S to a node of S or from a node of N\S to a node of N\S, then $\sum_{i\in S} a_{ij} = 0$.

If arc j goes from a node of S to a node of N\S (that is, $j\in Q^+$), then $\sum_{i\in S} a_{ij} = -1$.

If arc j goes from a node of N\S to a node of S (that is, $j\in Q^-$), then $\sum_{i\in S} a_{ij} = +1$.

Hence $\mathrm{div}(S) = \sum_{j\in Q^+} x(j) - \sum_{j\in Q^-} x(j)$. *QED*

max-flow problem

To each arc j in a network is assigned a capacity interval $C(j)= [c^-(j),c^+(j)]$ (Capacities can be infinite.)

A flow x is feasible with respect to these capacities if $x(j)\in C(j)$ for each j.

The upper capacity $c^+(Q)$ and lower capacity $c^-(Q)$ of a cut $Q=[S,N\setminus S]$ are defined to be

$$c^+(Q) = \sum_{j\in Q^+} c^+(j) - \sum_{j\in Q^-} c^-(j)$$

and

$$c^-(Q) = \sum_{j\in Q^+} c^-(j) - \sum_{j\in Q^-} c^+(j).$$

Proposition B: For any flow x feasible with respect to arc capacities and any cut $Q=[S,N\setminus S]$,

$$c^-(Q) \le \text{flux of } x \text{ across } Q \le c^+(Q).$$

Proof: The flux of x across Q is

$$\sum_{j\in Q^+} x(j) - \sum_{j\in Q^-} x(j) \le \sum_{j\in Q^+} c^+(j) - \sum_{j\in Q^-} c^-(j).$$

Likewise,

$$\sum_{j\in Q^+} x(j) - \sum_{j\in Q^-} x(j) \ge \sum_{j\in Q^+} c^-(j) - \sum_{j\in Q^-} c^+(j).$$ *QED*

To state the max flow problem, we need a network with arc capacities assigned, a set N^+ of source nodes, and a set N^- of destination nodes, with $N^+\cap N^-=\varnothing$.

Max flow problem (MF): Among all flows x that are conservative on $N\setminus(N^+\cup N^-)$ and feasible with respect to arc capacities, find one that maximizes $\mathrm{div}(N^+) = -\mathrm{div}(N^-)$.

How MF differs from the transportation problem

This differs from the Transporation Problem in several ways:

In the TP, exact supplies are specified at each node. In MF, the supply is specified to be zero at all nodes in $N\setminus(N^+\cup N^-)$ and is unspecified at the nodes in $N^+\cup N^-$.

In the TP, an arbitrary linear function of x is maximized. In MF, one specific linear function is minimized. The fact that one problem is minimization and the other is maximization is not significant since negating the objective function always turns a max problem into a min problem, or vice versa. The significant difference is the fact that the linear function is arbitrary in TP, whereas in MF it is always the same function, namely $\mathrm{div}(N^+)$.

To see that $\mathrm{div}(N^+)$ really is a linear function of x, note that

$$\mathrm{div}(N^+) = (\text{the flux of } x \text{ across } Q=[N^+, N\setminus N^+])$$
$$= \textstyle\sum_{j\in Q^+} x(j) - \sum_{j\in Q^-} x(j).$$

min-cut problem

Let $N^+\cap N^-=\varnothing$. We say that the cut $Q=[S,N\setminus S]$ separates N^+ from N^- provided $S\supset N^+$ and $S\cap N^-=\varnothing$.

min-cut problem (MC): Minimize $c^+(Q)$ over all cuts Q separating N^+ from N^-.

The min in MC must always be achieved, because there are only finitely many cuts.

The min can equal $+\infty$.

This happens if there is a path of unlimited capacity from N^+ to N^-, one whose forward arcs have upper capacity $+\infty$ and whose backward arcs have lower capacity $-\infty$. If such a path exists, then every cut separating N^+ from N^- coontains such an arc and therefore has upper capacity $+\infty$.

Observe, in this case, that $\sup(MF)=\min(MC)=+\infty$.

A solution to MC can be found by enumerating all possible cuts.

But, realistically speaking, this is not a practical thing to do.

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theorem of Ford and Fulkerson (max-flow min-cut)

Theorem: If the MF problem is feasible then sup(MF)=min(MC). Furthermore, this value is $+\infty$ if and only if there exists a path of unlimited capacity path of unlimited capacity from N^+ to N^-.

For now, we will just prove the easy direction, which is that sup(MF) $\leq$ min(MC).

Proposition C: If x is a flow that is conservative on $N\backslash(N^+ \cup N^-)$, $N^+ \cap N^- = \varnothing$, and $Q=[S,N\backslash S]$ is a cut separating N^+ from N^- then the flux of x across Q is equal to div(N^+).

Proof:

$$\text{The flux of } x \text{ across } Q = \text{div}(S) \qquad \text{(divergence principle)}$$
$$= \sum_{i\in S} (\text{div}\, x)_i = \sum_{i\in N^+} (\text{div}\, x)_i$$
$$= \text{div}(N^+),$$

where we have used the fact that $(\text{div } x)_i=0$ for all nodes in $S\backslash N^+$. *QED*

Proposition D: If x is a flow that is conservative on $N\backslash(N^+ \cup N^-)$, $N^+ \cap N^- = \varnothing$, and $Q=[S,N\backslash S]$ is a cut separating N^+ from N^- then

$$\text{div}(N^+) \leq c^+(Q).$$

Proof:

$$c^+(Q) \geq \text{flux of } x \text{ across } Q \qquad \text{(proposition B)}$$
$$= \text{div}(N^+) \qquad \text{(proposition C)}$$

Corollary E: sup(MF) $\leq$ min(MC).

Corollary F: If x is a feasible flow for the MF problem and Q is a cut feasible for the MC problem and such that $\text{div}(N^+)=c^+(Q)$ then x solves the max flow problem and Q solves the min cut problem.

max flow algorithm (simplified version)

step 0: Start with a flow that is feasible with respect to arc capacities, and is conservative on $N\backslash(N^+ \cup N^-)$.

step 1: Does an augmenting path from N^+ to N^- exist?

step 2: If YES, then

i. if path has unlimited capacity, STOP; sup(MF)=$+\infty$.

ii. otherwise, send additional flow from N^+ to N^- through augmenting path (as much as possible, consistent with arc capacities). Repeat step 1.

If NO, STOP; the flow is optimal.

Remarks:

More needs to be said about how an augmenting path is to be found in step one.

When the search for an augmenting path fails, we would like to get something more than just that information -- we would like to also get a solution to the min-cut problem.

finding an augmenting path

At the moment when we are about to look for an augmenting path, we have a feasible flow for the MF problem. In the augmenting path, an arc j can be used ...

(white arc)	in the forward direction only if $c^-(j) = x(j) < c^+(j)$
(black arc)	in the reverse direction only if $c^-(j) < x(j) = c^+(j)$
(green arc)	in either direction if $c^-(j) < x(j) < c^+(j)$
(red arc)	in neither direction if $c^-(j) = x(j) = c^+(j)$

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painted paths and painted cuts

Given a network with two nonempty disjoint node sets N^+ and N^-, and a painting of the arcs by the colors green, white, black and red, the painted network algorithm solves *either* the painted path problem *or* the painted cut problem

a solution to the painted path problem is a path from N^+ to N^- such that every forward arc is green or white and every reverse arc is green or black.

a solution to the painted cut problem is a cut $Q=[S,N\backslash S]$, $S\supset N^+$, $S\cap N^-=\emptyset$ (a cut separating N^+ and N^-) such that every arc in Q^+ is red or black and every arc in Q^- is red or white.

The painted network algorithm: At each step in this algorithm, we have a set $S\supset N^+$ and a function $\rho:S\backslash N^+\to S$.

step 0: Start with $S=N^+$. The domain of ρ ($=S\backslash N^+$) starts out empty.

step 1: Check each arc joining a node $j_1\in S$ with a node $j_2\in N\backslash S$ until one of the following is found:

a green or white arc j_1~j_2 from $j_1\in S$ to $j_2\in N\backslash S$

or

a green or black arc j_2~j_1 from $j_2\in N\backslash S$ to $j_1\in S$.

step 2: If no such arc is found, STOP; $Q=[S,N\backslash S]$ solves the painted cut problem.

step 3: If such an arc is found, let $S = S\cup\{j_2\}$ and define $\rho(j_2)=j_1$.

If $j_2\in N^-$, STOP; the path j_2, $\rho(j_2)$, $\rho(\rho(j_2))$, $\rho(\rho(\rho(j_2)))$,… (in the opposite order) solves the painted path problem.

If $j_2\notin N^-$, go to step 1.

max flow algorithm with painted network subroutine

What happens when a solution to the painted cut problem is found?

Every arc in Q^+ is red or black and every arc in Q^- is red or white.

This means that the upper capacity of the cut is

$$c^+(Q) = \sum_{j \in Q^+} c^+(j) - \sum_{j \in Q^-} c^-(j)$$

$$= \sum_{j \in Q^+} x(j) - \sum_{j \in Q^-} x(j)$$

= flux of x across Q

By corollary F, x solves the max flow problem and Q solves the min cut problem.

termination with commensurability assumption

Assume there is some cut of finite upper capacity. Then there is an upper bound on how large div(N^+) can become. Suppose all the problem data is integral (arc capacities and initial feasible flow). Then improvement in flux is ≥ 1 at each iteration, so termination must occur after finitely many iterations.

further refinement is possible

as nodes are added, you can keep track of value of additional flow

possible variations in method for choosing next node to add to S during painted network subroutine

breadth-first vs. depth-first