

min path problem

source: R.T. Rockafellar, [Network Flows and Monotropic Optimization](#), John Wiley & Sons 1984

some terminology

Let v be an arc function (there is a number $v(j)$ assigned to every arc).

We will refer to $v(j)$ as the **tension** in arc j . If P is a path from node

i to node i' , the **spread of v relative to P** is $\sum_{j \in P^+} v_j - \sum_{j \in P^-} v_j$

In the case where v is the differential of a potential, $v = \Delta u$, we have the following observations:

1. spread of v relative to $P = u(i') - u(i)$ ("integration rule")
2. the spread relative to P depends only on the endpoints of P .
3. the spread relative to a circuit is zero.
4. u can be reconstructed from v (up to a constant on each component)

Associate to each arc a **span interval** $[d^-(j), d^+(j)]$ (**upper span** and **lower span**).

A tension v is **feasible with respect to spans** if $v(j) \in [d^-(j), d^+(j)]$ for every j .

The **upper span** and the **lower span** of a path P are defined as

$$d^+(P) = \sum_{j \in P^+} d^+(j) - \sum_{j \in P^-} d^-(j)$$

$$d^-(P) = \sum_{j \in P^+} d^-(j) - \sum_{j \in P^-} d^+(j)$$

If v is feasible with respect to spans then

$$d^-(P) \leq \text{spread of } v \text{ relative to } P \leq d^+(P).$$

The min path problem

is to minimize $d^+(P)$ over all paths $P : N^+ \rightarrow N^-$.

Example 1: shortest paths

$$D(j) = [-l(j), +l(j)]$$

Example 2: longest positive paths

$$D(j) = [-\infty, -l(j)]$$

Example 3: knapsack problem

Example 5: PERT

The max tension problem

is to maximize the spread of u from N^+ to N^- over all potentials u that are constant on N^+ , constant on N^- , and such that $v = \Delta u$ is feasible with respect to spans.

in this context, the "spread of u " just means the value of $u(i') - u(i)$ for any $i' \in N^-$ and $i \in N^+$. It is also equal to the spread of Δu with respect to any path from N^+ to N^- .

A **cut of unlimited span** is a cut Q separating N^+ from N^- such that $d^+(j) = +\infty$ for all $j \in Q^+$ and $d^-(j) = -\infty$ for all $j \in Q^-$.

If a cut $Q = [S, N \setminus S]$ of unlimited span exists, and if a potential u exists that is feasible with respect to spans, then the sup in the max tension problem is $+\infty$. To see this, just note that redefining u on all nodes not in S by adding an arbitrary positive constant c

$$u(i) := u(i) + c \text{ for all } i \notin S$$

$$u(i) \text{ unchanged for all } i \in S$$

one obtains a potential that is still feasible with respect to spans, but for which the spread of u from N^+ to N^- has been increased by c . Since c can be chosen arbitrarily large, this shows that the sup in the max tension problem is $+\infty$.

Max tension min path theorem

THEOREM: Suppose there is at least one potential satisfying the constraints of the max tension problem. Then

$$\text{sup in max tension problem} = \text{min in min path problem}$$

This common value is $+\infty$ if there is a cut Q separating N^+ from N^- of unlimited span, whereas if there is no such cut, it is finite, and the max tension problem has a solution.

A path from N^+ to N^- that are feasible for the min path problem are those whose upper span is finite (that is, $d^+(P) < \infty$). Those paths are precisely the solutions to a certain painted path problem. If we color the arcs of the network

green if the upper and lower span are both finite

red if the lower span is $-\infty$ and the upper span is $+\infty$

white if the lower span is $-\infty$ and the upper span is finite

black if the lower span is finite and the upper span is ∞

then the solutions P to the painted path problem are those whose upper spans $d^+(P)$ are finite.

If there are no such paths (if the painted path problem has no solution), then all paths from N^+ to N^- have upper span $=\infty$. When this happens, we know that the painted cut problem has a solution. That is, there is a cut $Q = [S, N \setminus S]$ separating N^+ and N^- such that every arc in Q^+ is black or red and every arc in Q^- is white or red. Notice that such a cut is a cut of unlimited span.