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Basic solutions for LPs with upper and lower bounds

Let us consider this **EXAMPLE A**

$$\text{maximize } -x_1+2x_2+6x_3$$
$$\text{subject to}$$
$$x_1+x_2+x_3=1.5$$
$$0\le x_1\le 1,\ 0\le x_2\le 1,\ 0\le x_3\le 1$$

In tableau form, we represent this problem as before:

$$\begin{array}{ccc|c} 1 & 2 & 3 & \\ 1 & 1 & 1 & 3/2 \\ -1 & 2 & 6 & 0 \end{array}$$

In this situation, a basis consists of a choice of just one column. For example,

$$\begin{array}{ccc|cc} 1 & 2 & 3 & & \\ 1 & 1 & 1 & 3/2 & \mathbf{2} \\ -3 & 0 & 4 & -3 & \end{array}$$

is the tableau corresponding the the basis {2}. As before, setting the nonbasic variables equal to zero gives us a basic solution (perhaps feasible, perhaps not). Since

$$x_2 = 3/2-x_1-x_3$$

setting $x_1=x_3=0$ gives us the basic solution (0,3/2,0), which is not feasible. However, in problems with upper bounds, we also allow the nonbasic variables to be equal to the upper bounds in a basic solution. Thus there are four basic solutions corresponding to this tableau:

$x_1=x_3=0$ gives (0,3/2,0) (infeasible)
$x_1=0$, $x_3=1$ gives (0,1/2,1) (feasible, optimal)
$x_1=1$, $x_3=0$ gives (1,1/2,0) (feasible)
$x_1=x_3=1$ gives (1,-1/2,1) (infeasible)

To specify a particular basic solution, it is necessary to specify values (0 or 1 in this case) for each nonbasic variable. Thus the optimal tableau would be expressed like this:

OPTIMAL TABLEAU

$$\begin{array}{ccc|cc} 1 & 2 & 3 & & \\ 1 & 1 & 1 & 3/2 & \mathbf{2} \\ -3 & 0 & 4 & -3 & \\ \mathbf{=0} & & \mathbf{=1} & \text{obj}=3-3x_1+4x_3=3-3(0)+4(1)=7 & \end{array}$$

How do we know this is optimal? Answer: because the bottom row contains only negative numbers where nonbasic variable is zero and positive numbers where nonbasic variable is one.

Simplex method for LPs with upper and lower bounds

Execution of simplex algorithm, starting with basic feasible solution (1,1/2,0):

$$\begin{array}{ccc|cc} 1 & 2 & 3 & & \\ \mathbf{1} & 1 & 1 & 3/2 & \mathbf{2} \\ -3 & 0 & 4 & -3 & \end{array}$$

$$\begin{array}{ccc} \mathbf{=1} & & \mathbf{=0} \end{array} \qquad obj=3-3x_1+4x_3=3-3(1)+4(0)=0$$

Note that, although the tableau is the same, we are actually starting at the vertex at the opposite side of the hexagon, so three pivot steps should be expected.

Either x_1 or x_3 is a candidate to change in value. The +4 in column 3 tells us that the objective function will increase if we are able to increase x_3. The -3 in column 1 tells us that the objective function will increase if we are able to *decrease* x_1. Let's choose x_1. The question to ask is this: how much can x_1 decrease (from its current value of 1) while maintaining feasibility? The equation $x_2 = 3/2-x_1-x_3=3/2-x_1$ gives us the answer. x_1 can decrease until $x_1=1/2$. It cannot decrease more than that because x_2 cannot be allowed to be greater than 1. So as x_1 enters the basis, x_2 will leave with the value 1. Pivoting, we get the new tableau that corresponds to the vertex (1/2,1,0):

$$\begin{array}{ccc|ccc} 1 & 2 & 3 & & & \\ 1 & 1 & \mathbf{1} & 3/2 & & \mathbf{1} \\ 0 & 3 & 7 & 3/2 & & \\ & \mathbf{=1} & \mathbf{=0} & & & \end{array} \qquad obj=-3/2+3x_2+7x_3=-3/2+3(1)+7(0)=3/2$$

Notice that the value of the objective function has increased. Notice that we now have only one choice which nonbasic variable can change in value: x_3. To see how much x_3 can be increased (from its current value of 0), look at the equation $x_1 = 3/2-x_2-x_3 = 3/2-1-x_3 = 1/2-x_3$. It can increase only to 1/2 when x_1 becomes zero. So x_1 leaves the basis with the value 0 and x_3 enters with the value 1/2. Pivoting, we get the new tableau that corresponds to the vertex (0,1,1/2):

$$\begin{array}{ccc|ccc} 1 & 2 & 3 & & & \\ 1 & \mathbf{1} & 1 & 3/2 & & \mathbf{3} \\ -7 & -4 & 0 & -9 & & \\ \mathbf{=0} & \mathbf{=1} & & & & \end{array} \qquad obj=9-7x_1-4x_2=9-7(0)-4(1)=5$$

and the objective function has increased again.

At this point, x_2 is the only nonbasic variable that is a candidate to change in value. To see how much x_2 can be decreased (from its current value of 1), look at the equation $x_3=3/2-x_1-x_2=3/2-0-x_2=3/2-x_2$, which tells us that we can decrease x_2 until it equals 1/2, at which point x_3 becomes zero. So x_2 enters the basis and x_3 becomes nonbasic with value zero. After pivoting, we arrive at the optimal tableau.

Oh yes, one more thing!

There is an important consideration that is not illustrated by the above example. At each iteration, we decide how much to increase or decrease the chosen nonbasic variable, the answer is provided by the tighter of two restrictions:

i. The nonbasic variable must remain in [0,1]

ii. The basic variable must remain in [0,1]

In the example, the second restriction proved to be the tighter one. When that happens, the basic variable becomes nonbasic and the nonbasic becomes basic. However, if the first restriction had turned out to be the tighter one, then things would have been handled quite differently -- the basic variable would remain basic, and the nonbasic variable would remain nonbasic, but with the other value -- it would flip-flop from 0 to 1 or from 1 to 0. The tableau would otherwise be unchanged. Although this possibility did not occur in the example, it does occur in the homework problem below, so watch out for it!

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EXAMPLE B

Consider this problem: Maximize x_3 subject to

$$\begin{aligned} 13x_1 - 4x_2 + 16x_3 \quad\quad &= 16\\ -13x_1 - 2x_2 \quad\quad + 6x_4 &= -6\\ \text{all } x_i \in [0,1] \end{aligned}$$

Starting with

$$\begin{array}{cccc|c} 1 & 2 & 3 & 4 & \\ 13 & -4 & 16 & 0 & 16\\ -13 & 2 & 0 & 6 & -6\\ 0 & 0 & 1 & 0 & 0 \end{array}$$

we can pivot twice to start with the basic feasible solution (12/13,0,1/4,1) obj=1/4

$$\begin{array}{cccc|cc} 1 & -2/13 & 0 & -6/13 & 6/13 & 1\\ 0 & -1/8 & 1 & 3/8 & 5/8 & 3\\ 0 & 1/8 & 0 & -3/8 & -5/8 & \\ & =0 & & =1 & & \end{array}$$

There are two candidates for change: x_2 and x_4. We will try to change (decrease) x_4. The dictionary is:

$x_1 = 6/13 + 2/13\ x_2 + 6/13\ x_4 = 6/13 + 6/13\ x_4$

$x_3 = 5/8 + 1/8\ x_2 - 3/8\ x_4 = 5/8 - 3/8\ x_4$

x_4 can be decreased until it is 0, x_4 remains nonbasic

Next tableau is same for feasible solution (6/13,0,5/8,0) obj=5/8

$$\begin{array}{cccc|cc} 1 & -2/13 & 0 & -6/13 & 6/13 & 1\\ 0 & -1/8 & 1 & 3/8 & 5/8 & 3\\ 0 & 1/8 & 0 & -3/8 & -5/8 & \\ & =0 & & =0 & & \end{array}$$

There is one candidate for change (increase): x_2. Dictionary:

$x_1 = 6/13 + 2/13\ x_2 + 6/13\ x_4 = 6/13 + 2/13\ x_2$

$x_3 = 5/8 + 1/8\ x_2 - 3/8\ x_4 = 5/8 + 1/8\ x_2$

x_2 can be increased until it is one, remains nonbasic

Next tableau is same for optimal solution (8/13,1,3/4,0) obj=3/4

$$\begin{array}{cccc|cc} 1 & -2/13 & 0 & -6/13 & 6/13 & 1\\ 0 & -1/8 & 1 & 3/8 & 5/8 & 3\\ 0 & 1/8 & 0 & -3/8 & -5/8 & \\ & =1 & & =0 & & \end{array}$$

Also read example, pp. 119-123 in Chvátal

Optimality conditions

Theorem : For a linear programming problem of the form $\max\ \mathbf{c}'\mathbf{x}$ subject to $\mathbf{a}_1\cdot\mathbf{x} = b_1, \dots, \mathbf{a}_m\cdot\mathbf{x} = b_m, \mathbf{a}_{m+1}\cdot\mathbf{x}\le b_{m+1}, \dots, \mathbf{a}_r\cdot\mathbf{x}\le b_r$ and fixed constraints of the form $\mathbf{x}\in C$, a necessary and sufficient condition for optimality of $\mathbf{x}^*$ is that:

There exist numbers $y_1,\dots,y_r$ such that $y_{m+1},\dots,y_r\ge 0$,

$$\mathbf{c} - y_1\mathbf{a}_1 - \dots - y_r\mathbf{a}_r \in N_C(x^*)$$

$$\text{for } i=m+1,\dots,r,\ y_i\ge 0 \text{ and } y_i=0 \text{ if } \mathbf{a}_i\cdot\mathbf{x}=b_i$$

Application of optimality conditions to example

For the above **EXAMPLE A**, we note that

$$\begin{pmatrix}-1\\2\\6\end{pmatrix} - 2\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}-3\\0\\4\end{pmatrix} \in N_C(0,1/2,1)$$

where C is the cube. This proves that (0,1/2,1) is the optimal solution.

For the above **EXAMPLE B**, we note that

$$\begin{pmatrix}0\\0\\1\\0\end{pmatrix} - 1/16\begin{pmatrix}13\\-4\\16\\0\end{pmatrix} - 1/16\begin{pmatrix}-13\\2\\0\\6\end{pmatrix} = \begin{pmatrix}0\\1/8\\0\\-3/8\end{pmatrix} \in N_C(8/13,1,3/4,0)$$

where C is the cube. This proves that (8/13,1,3/4,0) is the optimal solution.

Homework 24: Consider the LP: maximize $-12x_1-5x_2+x_3$ subject to $6x_1+3x_2+2x_3=7$, $0\le x_i\le 1$ $(i=1,2,3)$.

a) Draw a picture showing the polygon on the cube.

b) Write down the tableau corresponding to each vertex of the polygon.

c) Identify the optimal tableau

d) Solve via the simplex algorithm, starting with the tableau that corresponds to the minimizer and taking the longer path around to the optimal solution.

e) Verify the optimality of the optimal solution by showing that the optimality conditions hold there for some appropriate y.

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normal cone examples

The normal cone to $\{x\in R^3 : x\ge 0\}$

at (2,0,3) consists of all vectors of the form (0,-,0)
at (0,0,0) consists of all vectors of the form (-,-,-)
at (5,1,3) consists of all vectors of the form (0,0,0)

The normal cone to $\{x\in R^3 : 0\le x_i\le 1,\ i=1,2,3\}$ (unit cube)

at (1,1,1) consists of all vector sof the form (+,+,+)
at (1,1/2,1) consists of all vector sof the form (+,0,+)
at (0,1/2,1) consists of all vector sof the form (-,0,+)
at (1/2,1/3,3/7) consists of all vector sof the form (0,0,0)

The normal cone to $\{x\in R^4 : x_3\ge 0, x_4\ge 0\}$

at (-5,17,3,4) consists of all vectors of the form (0,0,0,0)
at (-5,17,0 8) consists of all vectors of the form (0,0,-,0)

at (-5,17,0,0) consists of all vectors of the form (0,0,-,-)

The normal cone to $\{x\in R^4 : x_2\le 5,\ -1\le x_3\le 3, 2\le x_4\}$

at (-5,5,-1,2) consists of all vectors of the form (0,+,-,-)
at (-5,3,3,3) consists of all vectors of the form (0,0,+,0)

p. 147, #9.5:

Determine whether x*=(3,-1,0,2) is optimal for the problem:

$$\text{maximize } 6x_1 + x_2 - x_3 - x_4$$

subject to

$$\begin{aligned} x_1 + 2x_2 + x_3 + x_4 &\le 5 \\ 3x_1 + x_2 - x_3 &\le 8 \\ x_2 + x_3 + x_4 &= 1 \\ x_3 \ge 0,\ x_4 &\ge 0 \end{aligned}$$

First check that x* satisfies all the constraints. It does. Note that the first inequality is not active at x* but the second is. x* is optimal if and only if numbers $\lambda\ge 0$ and $\mu\in R$ exist such that

$$\begin{pmatrix} 6 \\ 1 \\ -1 \\ -1 \end{pmatrix} - \lambda \begin{pmatrix} 3 \\ 1 \\ -1 \\ 0 \end{pmatrix} - \mu \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \le 0 \\ 0 \end{pmatrix}$$

The values λ=2 and μ=-1 work for all but the third component. Since this system has no solution, x* is not optimal.

homework 25: Determine whether or not x*=(2,1,5,3,0) is optimal for the following LP:

maximize $3x_1 + 8x_2 + 4x_3 - x_4$ subject to the constraints

$$\begin{aligned} 2x_1 + 2x_2 + x_3 + x_4 - x_5 &\le 14 \\ x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 &\le 1000 \\ 3x_1 - 3x_2 + 2x_3 + 4x_4 &= 25 \\ 1\le x_2\le 3,\quad 2\le x_3\le 5,\quad 0\le x_4,&\quad x_5\le 0 \end{aligned}$$