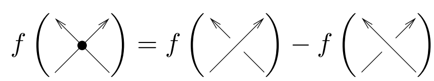
BEADS: FROM LIE ALGEBRAS TO LIE GROUPS

Abstract. The Kontsevich integral of a knot is a powerful invariant which takes values in an algebra of trivalent graphs with legs. Given a Lie algebra, the Kontsevich integral determines an invariant of knots (the so-called colored Jones function) with values in the symmetric algebra of the Lie algebra. Recently A. Kricker and the author constructed a rational form of the Kontsevich integral which takes values in an algebra of trivalent graphs with beads. After replacing beads by an exponential legs, this rational form recovers the Kontsevich integral. The goal of the paper is to explain the relation between beads and functions defined on a Lie group. As an application, we provide a rational form for the colored Jones function of a knot, conjectured by Rozansky.

1. Introduction

1.1. **Finite type invariants of knots, Feynmann diagrams and Lie algebras.** A *knot* is a (smooth) imbedding of a circle S^1 in 3-space. An *immersed knot* is a (smooth) immersion of S^1 in 3-space with transverse double points. Any numerical invariant f of knots which takes values in an abelian group A can be extended to an invariant of immersed knots via the rule:



In the early nineties, Vassiliev (and independently, Goussarov) introduced the notion of a *finite type invariant* of knots, that is of a function f : Knots $\longrightarrow A$, which, when extended to a function on the set of immersed knots, vanishes on immersed knots with sufficiently many double points. Shortly afterwards, Kontsevich constructed an invariant

$$Z : \text{Knots} \longrightarrow \mathcal{A}(\star)$$

(the so-called *Kontsevich integral*) where $\mathcal{A}(\star)$ is the vector space over the rational numbers $\mathbb{Q}$ generated by unitrivalent graphs (with vertex orientations), modulo the well-known antisymmetry AS and IHX relations. The Kontsevich integral Z has two key properties:

- Z is a *universal* finite type invariant. That is, it determines every $\mathbb{Q}$-valued finite type invariant of knots.
- Z determines the *Jones polynomial* of a knot, and more generally the invariants of knot that are defined using *quantum groups* and their representations.

Let us briefly recall the latter property. Given a Lie algebra $\mathfrak{g}$ with an invariant inner product, there is a map (often called a *weight system*)

$$W^h_{\mathfrak{g}} : \mathcal{A}(\star) \longrightarrow S(\mathfrak{g})^{\mathfrak{g}}[[h]], \tag{1}$$

where $S(\mathfrak{g})$ is the *symmetric* algebra of the vector space $\mathfrak{g}$.

Definition 1.1. We will call the map

$$W^h_{\mathfrak{g}} \circ Z : \text{Knots} \longrightarrow S(\mathfrak{g})^{\mathfrak{g}}[[h]]$$

the $\mathfrak{g}$-*colored Jones function* of a knot, and we will denote it by $J_{\mathfrak{g}}$.

In a sense, the colored Jones function of a knot is a generating function of the quantum group invariants of a knot. More precisely, given an irreducible representation V of $\mathfrak{g}$, its evaluation at its dominant weight gives rise to a linear map

$$S(\mathfrak{g})^{\mathfrak{g}}[[h]] \longrightarrow \mathbb{Q}[[h]]$$

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The image of the Kontsevich integral under the composition $W_{\mathfrak{g},V}$ of the above two maps is an element of the ring $\mathbb{Z}[q^{\pm 1}]$ (where $q = e^h$) and coincides with the quantum group invariant of knots, using the $(\mathfrak{g}, V)$ data. If $\mathfrak{g} = \mathfrak{sl}_2$ and $V = \mathbb{C}^2$ is the defining representation of $\mathfrak{sl}_2$, then $W_{\mathfrak{sl}_2,\mathbb{C}^2} \circ Z$ coincides with the Jones polynomial.

1.2. **A rational form of the Kontsevich integral, Feynmann diagrams and beads.** Recently, A. Kricker and the author [GK] constructed a *rational form* Z^{rat} of the Kontsevich integral of a knot

$$Z^{\mathrm{rat}} : \mathrm{Knots} \longrightarrow \mathcal{A}^0(\Lambda_{\mathrm{loc}}) := \mathcal{B}(\Lambda \to \mathbb{Z}) \times \mathcal{A}(\Lambda_{\mathrm{loc}})$$

which consists of a 'matrix part' and a 'graph-part', where

- $\Lambda = \mathbb{Z}[t, t^{-1}]$, Λ_{loc} is the localization of Λ with respect to elements $f \in \Lambda$ such that $f(1) = 1$.
- $\mathcal{B}(\Lambda \to \mathbb{Z})$ is a quotient of the set of Hermitian matrices over $\Lambda = \mathbb{Z}[t, t^{-1}]$ which are invertible over $\mathbb{Z}$, modulo the equivalence $A \sim B$ iff $A \oplus D = P(B \oplus E)P^\star$ for diagonal matrices D, E with monomials in t on the diagonal and for P invertible over Λ.
- $\mathcal{A}(\Lambda_{\mathrm{loc}})$ is the completed vector space of trivalent graphs with oriented vertices and edges, and with elements (often called *beads*) of Λ_{loc} associated to each edge, modulo the AS, IHX, Orientation Reversal, Linearity and Holonomy relations shown in Figure 1. $\mathcal{A}(\Lambda_{\mathrm{loc}})$ is a graded vector space, where the degree of a diagram is half the number of trivalent vertices, and the completion refers to the above grading.

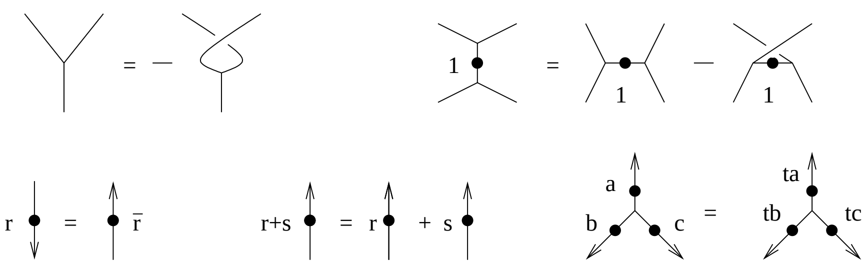


Figure 1. The AS, IHX (for arbitrary orientations of the edges), Orientation Reversal, Linearity and Holonomy Relations.

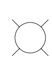
The rational form Z^{rat} has two key properties:

- It is a *universal finite type invariant of knots*, where the analogue of crossing change (in the definition of finite type invariants) is replaced by the *null move* described in terms of surgery on claspers with nullhomologous leaves in a knot complement. For details, see [GR] and [GK].
- Z^{rat} determines the Kontsevich integral of a knot, after we *replace beads by hair*.

Let us explain the last phrase. Given a trivalent graph s with beads in Λ, let $\mathrm{Hair}(s) \in \mathcal{A}(\star)$ denote the result of replacing each bead t by an exponential of hair (i.e., legs):

$$t \to \sum_{n=0}^{\infty} \frac{1}{n!} \quad n \text{ legs}$$

This map can be defined even if the beads are elements of $\hat{\Lambda} = \mathbb{Q}[[t-1]]$, and in particular if the beads are elements of $\Lambda_{\mathrm{loc}} \subseteq \hat{\Lambda}$. Notice that $\mathrm{Hair}(s)$ is a series of unitrivalent graphs that contain no wheels, where, for example, a *wheel* with 4 hair is:



Next, we add wheels from the matrix part of $\mathcal{A}^0(\Lambda_{\mathrm{loc}})$ as follows. We define

$$\mathrm{Hair}^\nu : \mathcal{A}^0(\Lambda_{\mathrm{loc}}) \longrightarrow \mathcal{A}(\star)$$

by

$$(A, s) \in \mathcal{B}(\Lambda \to \mathbb{Z}) \times \mathcal{A}(\Lambda_{\mathrm{loc}}) \mapsto \nu \exp\left(-\frac{1}{2} \operatorname{tr} \log(A)(e^h)|_{h^n \to w_n}\right) \mathrm{Hair}(s)$$

where $\nu = Z(\text{unknot})$, w_n is the wheel with n legs and we think of $\mathcal{A}(\star)$ as an algebra with the disjoint union multiplication of graphs.

[GK, Theorem 1.3] states that

$$Z = \text{Hair}^\nu \circ Z^{\text{rat}}. \tag{2}$$

Let us end this introduction to the Z^{rat} invariant with three comments. For knots with a fixed Alexander polynomial Δ, the graph-part of the Z^{rat} invariant takes values in graphs with beads in the abelian subgroup

$$\Lambda_\Delta = \frac{1}{\Delta}\Lambda$$

of Λ. Furthermore, if we restrict Z^{rat} to knots with trivial Alexander polynomial, then the matrix part of Z^{rat} is trivial and there is no need to consider graphs with beads in Λ_{loc}. Namely, we have:

$$Z^{\text{rat}} : \text{Knots with trivial Alexander polynomial} \longrightarrow \mathcal{A}(\Lambda).$$

As a first approximation to understanding of the rational form Z^{rat}, the reader may restrict their attention to knots with trivial Alexander polynomial.

A last comment: the Kontsevich integral Z and its rational form Z^{rat} can be defined for knots K in integral homology 3-spheres M, i.e., manifolds M such that $H_\star(M,\mathbb{Z}) = H_\star(S^3,\mathbb{Z})$.

1.3. **Weight systems for diagrams with beads.** The paper is concerned with defining a concept of a weight system for diagrams with beads. As an application, we will deduce a rational form for the colored Jones function of a knot. As we will see, the analogue of the weight system map (1) uses the Lie group, rather than its Lie algebra. This explains the title of the paper.

For a reference on Lie groups and their representations, see [CSM]. Consider a compact connected (not necessarily simply connected) Lie group G with semisimple Lie algebra $\mathfrak{g}_\mathbb{R}$, complexification $\mathfrak{g} = \mathfrak{g}_\mathbb{R} \otimes \mathbb{C}$, and let $C(G)$ the algebra of continuous complex-valued functions on G. G acts on $C(G)$ by conjugation. Consider the subspace $C_{\text{alg}}(G)$ of *almost invariant functions*, that is functions $f \in C(G)$ whose image under G lies in a finite dimensional subspace of $C(G)$. It turns out that $C_{\text{alg}}(G)$ is a subalgebra of $C(G)$. Since $C_{\text{alg}}(G)$ is perhaps not a familiar enough object to a topologist, we discuss several alternative views of it in the following:

Lemma 1.2. *There are canonical isomorphisms between $C_{alg}(G)$ and*

(a) $\oplus_{E\in\hat{G}} E' \otimes E$*, where $\hat{G}$ is the set of unitary irreducible representations of G.*
(b) *The ring of* representative functions *on G, where a representative function on G is a function $G \to \mathbb{C}$ of the form*

$$f_{M,\xi,\eta}(g) = \langle \eta, g\xi \rangle$$

where M is a finite dimensional unitary representation of G and $\xi, \eta \in M$. In other words, $f_{M,\xi,\eta}(g)$ is a matrix element of the action of g on M with respect to a suitable basis.
(c) *The coordinate ring of the linear algebraic group $G_\mathbb{C}$.*

Proof. All these are restatements of the *Peter-Weyl* theorem. For an excellent discussion of these ideas, we refer the reader to [CSM]. For (a) see [CSM, pp.91-100]. For (b) and (c), see [CSM, p.92]. □

Let $R(G)$ denote the algebra of *characters* of unitary representations of G, a subalgebra of the *class functions* $C(G)^G$, and let $R(T)^W$ denote the algebra of *Weyl-invariant* characters of a *maximal torus* T of G.

Corollary 1.3. *We have isomorphisms of algebras:*

$$C_{\text{alg}}(G)^G \cong R(G) \cong R(T)^W.$$

Proof. Characters are invariant functions on G, thus there is an inclusion $R(G) \to C_{\text{alg}}(G)^G$. The first description of $C_{\text{alg}}(G)$ above and Shur's lemma (in the form $(E'\otimes E)^G \cong \mathbb{C}$) implies that $C_{\text{alg}}(G)^G \cong \oplus_{E\in\hat{G}}\mathbb{C}$, from which follows that the inclusion $R(G) \to C_{\text{alg}}(G)^G$ is an isomorphism.

The restriction of characters from G to T induces an isomorphism $R(G) \cong R(T)^W$. This is the content of Chevalley's theorem [Hu, Section 23.1] and also [BGV, Theorem 7.28], and is a consequence of the fact that given $g \in G$, there exist $h \in G$ and $t \in T$ unique up to the action of $W = N_G(T)/T$ (where $N_G(T)$ is the normalizer of T in G) such that $g = h^{-1}th$. □

We will need a subspace of the algebra $C_{\mathrm{alg}}(G)$ of almost invariant functions. Given a finite set S of conjugacy classes in G, let $R_S(G)$ denote the class functions on $G - S$, and $R_R(T)^W$ denote the W-invariant characters on $T - S$. We define $C_{\mathrm{alg},S}(G)^G = R_S(G)$.

The above corollary implies that

Corollary 1.4. *For every S as above, we have vector space isomorphisms:*

$$C_{\mathrm{alg},S}(G)^G \cong R_S(G) \cong R_S(T)^W.$$

Given an element $f \in \Lambda$, consider the subset S of elements in G such that $\det(f(\mathrm{Ad}(g))) = 0$. It is easy to see that S is a finite union of conjugacy classes, and that S is empty if f has *no roots* on the unit circle S^1. We will denote the corresponding vector spaces by $C_{\mathrm{alg},f}(G)^G$, $R_f(G)$ and $R_f(T)^W$ respectively.

Remark 1.5. For computational reasons, it is useful to know the structure of the algebra $R(T)^W$. $R(T)$ can be identified with the group-ring $\mathbb{C}[\Lambda_w]$ of the *weight lattice* $\Lambda_w \subset \mathfrak{t}^\star$

$$R(T) \cong \mathbb{C}[\Lambda_w].$$

For $\lambda \in \Lambda_w$, we will denote by e_λ the associated element of $\mathbb{C}[\Lambda_w]$ and $R(T)$. Notice that $e_\lambda e_{\lambda'} = e_{\lambda+\lambda'}$. The corresponding function

$$e_\lambda : T \to \mathbb{C}$$

is given by $e_\lambda(e^x) = e^{\lambda(x)}$ for $x \in \mathfrak{t}$, the Lie algebra of T. Although we will not use, a theorem of Chevalley states that $R(T)^W$ is freely generated by polynomials in the variables e_λ. □

We now have the preliminaries to define a notion of weight system for diagrams with beads.

Definition 1.6. Given G as above, define a map

$$W_G^h : \mathcal{A}(\Lambda_\Delta) \longrightarrow C_{\mathrm{alg},\Delta}(G)^G[[h]] \tag{3}$$

as follows: fix an element $g \in G$, and a trivalent graph Γ with beads. Cut Γ into a union of Y graphs together with oriented edge segments which contain the beads. Color each of the Y graphs with elements of a basis e_a of the Lie algebra $\mathfrak{g}$ of G, and replace an oriented segment with a bead $f(t) \in 1/\Delta\Lambda$ with $f(\mathrm{Ad}(g)) \in \mathrm{Aut}(\mathfrak{g})$. This can be done as long as $\det(\Delta(\mathrm{Ad}(g))) \neq 0$, i.e., as long as $g \in G - \Delta$, using the above notation. Then, contract the indices, using structure constants on the Y graphs. Varying g defines an element $W_G(\Gamma) \in C(G)$ which is independent of the basis of $\mathfrak{g}$, and further lies in the subalgebra of $C(G)$ generated by the entries of matrices of the Adjoint representation of G. This implies that $W_G(\Gamma) \in C_{\mathrm{alg}}(G)$. It is easy to see that the AS, IHX and Holonomy relations of $\mathcal{A}(\Lambda)$ are satisfied and that $W_G(\Gamma)$ lies in the G-invariant part $C_{\mathrm{alg}}(G)^G$ of $C_{\mathrm{alg}}(G)$. Finally, let $W_G^h = W_G h^{\deg}$, where deg is half the number of trivalent vertices of a diagram. Next, we extend the above map, namely we define

$$W_G^h : \mathcal{A}^0(\Lambda_\Delta) \longrightarrow C_{\mathrm{alg},\Delta}(G)^G[[h]]$$

by

$$(A, s) \in \mathcal{B}(\Lambda \to \mathbb{Z}) \times \mathcal{A}(\Lambda_\Delta) \mapsto \frac{1}{\det(\det(A)|_{t\to Ad(g)})^{1/2}} W_G^h(s)$$

Next we discuss how to compare the weight systems $W_{\mathfrak{g}}^h$ and W_G^h of Equations (1) and (3). To achieve this, we need to discuss the map Hair on the level of Lie algebras. Let us denote by

$$\exp^\star : C_{\mathrm{alg}}(G)^G[[h]] \longrightarrow S(\mathfrak{g})^{\mathfrak{g}}[[h]]$$

the map defined by

$$\exp^\star(f)(\lambda) = f(e^{h\lambda}) \in P(\mathfrak{g})[[h]] \cong P(\mathfrak{g}^\star)[[h]] \cong S(\mathfrak{g})[[h]]$$

for $f \in C_{\mathrm{alg}}(G)^G$ and $\lambda \in \mathfrak{g}$, where $P(V)$ denotes the polynomial functions on a vector space V and where $e : \mathfrak{g} \to G$ is the exponential function and where $\mathfrak{g} \cong \mathfrak{g}^\star$ via an invariant inner product. Notice the h in the $e^{h\lambda}$ term, which can be interpreted by considering the 1-parameter family $\mathfrak{g}_h$ of Lie algebras on the vector space $\mathfrak{g}$ with Lie bracket $[a, b]_h = [a, b]$, where $[\cdot, \cdot]$ is the Lie bracket on $\mathfrak{g}$.

Since $\exp^\star(f) \in S(\mathfrak{g})^{\mathfrak{g}}[[h]]$ is invertible when $f(1) = 1$, it follows that $\exp^\star$ descends to a map

$$\exp^\star : C_{\mathrm{alg},\Delta}(G)^G[[h]] \longrightarrow S(\mathfrak{g})^{\mathfrak{g}}[[h]]$$

Definition 1.7. A slightly normalized version of $\exp^\star$ is the map

$$\Phi : C_{\mathrm{alg},\Delta}(G)^G[[h]] \longrightarrow S(\mathfrak{g})^{\mathfrak{g}}[[h]]$$

given by

$$\Phi(f)(\lambda) = \frac{j^{1/2}(h\lambda)}{j^{1/2}(h\rho)} \exp^\star(f)(\lambda).$$

where $j^{1/2} : \mathfrak{g} \longrightarrow \mathbb{C}$ is defined by

$$j^{1/2}(x) = \det\left(\frac{\sinh \mathrm{ad}_{x/2}}{\mathrm{ad}_{x/2}}\right)^{1/2}.$$

(The square root is well-defined, see [Du]).

Observe that our map Φ differs from the map Φ of Kashiwara-Verge [KV] by the factor $j^{1/2}(h\rho)$, which is independent of a knot. Φ turns out to coincide with the map Hair^ν on the level of Lie algebras, as is revealed by the next theorem.

Definition 1.8. We will call the map

$$W^h_G \circ Z^{\mathrm{rat}} \longrightarrow C_{\mathrm{alg},\Delta}(G)^G[[h]]$$

the *G-colored Jones function* of a knot, and we will denote it by J_G.

The next theorem, which compares the G and $\mathfrak{g}$-colored Jones function of a knot, is often called the *Rationality Conjecture* for the $\mathfrak{g}$-colored Jones function. The conjecture was formulated by Rozansky in [R2] and proven by entirely different means for $\mathfrak{sl}_2$ in [R1].

We are now ready to state the main result of the paper.

Theorem 1. (a) The weight systems $W^h_{\mathfrak{g}}$ and W^h_G fit in a commutative diagram:

$$\begin{tikzcd} \mathcal{A}^0(\Lambda_{\mathrm{loc}}) \arrow[r, "W^h_G"] \arrow[d, "\mathrm{Hair}^\nu"] & C_{\mathrm{alg},\mathrm{loc}}(G)^G[[h]] \arrow[d, "\Phi"] \\ \mathcal{A}(\star) \arrow[r, "W^h_{\mathfrak{g}}"] & S(\mathfrak{g})^{\mathfrak{g}}[[h]] \end{tikzcd}$$

In other words, the image of the map Hair^ν coincides with the Kashiwara-Vergne map Φ on the level of Lie algebras.

(b) We have:

$$J_{\mathfrak{g}} = \Phi \circ J_G.$$

In other words, if we write

$$J_G = \sum_{n=0}^{\infty} Q_{G,n} h^n$$

for $Q_{G,n} \in C_{\mathrm{alg},\Delta}(G)^G[[h]]$, then we have

$$J_{\mathfrak{g}}(\lambda) = [\lambda - \rho] \sum_{n=0}^{\infty} Q_{G,n}|_{e_\mu \to e^{h(\lambda,\mu)}} h^n \in S(\mathfrak{g})^{\mathfrak{g}}[[h]]$$

where $[\lambda - \rho]$ is the *quantum dimension* of the representation of $\mathfrak{g}$ of highest weight $\lambda - \rho$ given by

$$[\lambda - \rho] = \prod_{\alpha \succ 0} \frac{\sinh h(\alpha, \lambda)}{\sinh h(\alpha, \rho)}.$$

In other words, J_G is equivalent to a sequence $\{Q_{G,n}\}_n$ of (partially defined) G-invariant functions on G associated to a knot. If a knot has trivial Alexander polynomial, then it follows from Remark 1.5 that J_G is equivalent to a sequence of polynomials, i.e., elements of $\mathbb{C}[\Lambda_w]$. It is well known that these polynomials actually lie in $\mathbb{Q}[\Lambda_w]$, see also [A].

At any rate, any evaluation of these sequences of function gives a knot invariant. There are two natural evaluations to consider. Evaluation at 1 and average over G. Let us end with a definition.

Definition 1.9. Given a Lie group G as above, let us define two invariants

$$\kappa^h_G : \text{Knots} \longrightarrow \mathbb{Q}[[h]]$$
$$\tau^h_G : \text{Knots with trivial Alexander polynomial} \longrightarrow \mathbb{C}[[h]]$$

by

$$\kappa^h_G = J_G(1) \qquad \text{and} \qquad \tau^h_G = \int_G J_G d\mu$$

where μ is the *Haar measure* on G. We will call τ^h_G the *Kashaev invariant* of the knot.

Unpublished work of Thang Le, [Le] combined with the results of this paper shows that when $G = SU(2)$, the Kasahev invariant is related to the usual Kashaev invariant of the knot. More precisely, Le constructs a function

$$\kappa : \text{Knots} \longrightarrow \widehat{\mathbb{Z}[q]}$$

where $\widehat{\mathbb{Z}[q]} = \lim_{\leftarrow n} \mathbb{Z}[q]/((1-q)(1-q^2)\dots(1-q^n))$ is the *cyclotomic completion* of the ring $\mathbb{Z}[q]$ such that

- κ evaluates to the Kashaev invariants of the knot, i.e., for all n we have
$$\kappa(e^{2\pi i/n}) = \bar{J}_{\mathfrak{sl}_2,\mathbb{C}^n}(e^{2\pi i/n})$$
where $\bar{J}_{\mathfrak{g}} = J_{\mathfrak{g}}/J_{\mathfrak{g}}(\text{unknot})$.
- When composed with the map $\widehat{\mathbb{Z}[q]} \longrightarrow \mathbb{Q}[[h]]$, κ coincides with our invariant $\kappa^h_{SU(2)}$.

1.4. **Acknowledgement.** The author wishes to thank A. Kricker for numerous conversations and TTQ. Le for pointing an error in an ealier version of the paper.

2. Proof of Theorem 1

(a) It suffices to show that the following diagrams commute:

$$\begin{tikzcd} \mathcal{A}(\Lambda_\Delta) \arrow[r, "W^h_G"] \arrow[d, "\mathrm{Hair}^\nu"] & C_{\mathrm{alg},\Delta}(G)^G[[h]] \arrow[d, "\Phi"] \\ \mathcal{A}(\star) \arrow[r, "W^h_{\mathfrak{g}}"] & S(\mathfrak{g})^{\mathfrak{g}}[[h]] \end{tikzcd} \qquad \begin{tikzcd} \mathcal{A}(\Lambda_\Delta) \arrow[r, "W^h_G"] \arrow[d, "\mathrm{Hair}^\nu"] & C_{\mathrm{alg},\Delta}(G)^G[[h]] \arrow[d, "\Phi"] \\ \mathcal{A}(\star) \arrow[r, "W^h_{\mathfrak{g}}"] & S(\mathfrak{g})^{\mathfrak{g}}[[h]] \end{tikzcd}$$

Both weight system maps W_G and $W_{\mathfrak{g}}$ are defined in terms of contractions of indices of tensors. The commutativity of the left diagram follows from the following relation between the Ad and ad

$$\mathrm{Ad}(\exp(x)) = \exp(\mathrm{ad}_x)$$

valid for $x \in \mathfrak{g}$. It remains to deal with the 'wheels factor', that is with the second diagram. This is dealt by Lemma 2.1 below. (b) follows immediately from (a) using the Definitions of $J_{\mathfrak{g}}$ and J_G. □

Lemma 2.1. *The right diagram above commutes.*

Proof. We will identify $C_{\mathrm{alg},\Delta}(G)^G[[h]]$ with $R_\Delta(T)^W[[h]]$ and $S(\mathfrak{g})^{\mathfrak{g}}[[h]]$ with $S(\mathfrak{t})^W[[h]]$. Then, we need to show that the following diagram

$$\begin{tikzcd} \mathcal{A}(\Lambda_\Delta) \arrow[r, "W^h_G"] \arrow[d, "\mathrm{Hair}^\nu"] & R_\Delta(T)^W[[h]] \arrow[d, "\Phi"] \\ \mathcal{A}(\star) \arrow[r, "W^h_{\mathfrak{g}}"] & S(\mathfrak{t})^W[[h]] \end{tikzcd}$$

commutes. We will think of $S(\mathfrak{t})$ as the algebra of polynomials on $\mathfrak{t}^\star$. Fix a point $\lambda \in \mathfrak{t}^\star$ and a matrix $A \in B(\Lambda \to \mathbb{Z})$. We need to show that for every $\lambda \in \mathfrak{g}$ we have:

$$W^h_{\mathfrak{g}} \circ \mathrm{Hair}^\nu(A)(\lambda) = \Phi \circ W^h_G(A)(\lambda).$$

It follows by definition that

$$W^h_{\mathfrak{g}} \circ \mathrm{Hair}^\nu(A)(\lambda) = W^h_{\mathfrak{g}} \circ Z(\text{unknot})(\lambda) \cdot W^h_{\mathfrak{g}} \circ \mathrm{Hair}(A)(\lambda).$$

It is well known (see for example [A, Lemma 2.4]) that

$$W^h_{\mathfrak{g}} \circ Z(\text{unknot})(\lambda) = [\lambda - \rho] = \frac{j^{1/2}(h\lambda)}{j^{1/2}(h\rho)}.$$

Furthermore, we have

$$\text{Hair}(A) = \exp\left(-\frac{1}{2}\sum_{n=1}^{\infty} a_{2n}\omega_{2n}\right)$$

where ω_n denotes the wheel with n legs and

$$\log(\det(A)(e^h)) = \sum_{n=1}^{\infty} a_{2n}h^{2n} \in \mathbb{Q}[[h]].$$

Observe that $W^h_{\mathfrak{g}} : \mathcal{A}(\star) \to S(\mathfrak{g})^{\mathfrak{g}}[[h]]$ is multiplicative and the value of $W^h_{\mathfrak{g}}$ on wheels is given by:

$$W^h_{\mathfrak{g}}(w_{2n})(x) = \operatorname{tr}\operatorname{ad}_x^{2n} = 2\sum_{\alpha \succ 0} \alpha(x)^{2n}$$

for $x \in \mathfrak{t}$. Combining this, it follows that

$$\begin{aligned}
(W^h_{\mathfrak{g}} \circ \text{Hair})(A)(e_\lambda) &= W^h_{\mathfrak{g}} \circ \exp\left(-\frac{1}{2}\sum_{n=0}^{\infty} a_{2n}\omega_{2n}\right) \\
&= \exp\left(-\frac{1}{2}\sum_{n=0}^{\infty} a_{2n}W^h_{\mathfrak{g}}(\omega_{2n})\right) \\
&= \exp\left(-\sum_{n=0}^{\infty} a_{2n}\operatorname{tr}\operatorname{ad}_x^{2n}h^{2n}\right) \\
&= \prod_{\alpha \succ 0}\exp\left(-\sum_{n=0}^{\infty} a_{2n}(\alpha,\lambda)^{2n}h^{2n}\right) \\
&= \prod_{\alpha \succ 0}\exp\Big(-\log(\det(A)(e^{h(\alpha,\lambda)}))\Big) \\
&= \prod_{\alpha \succ 0}\frac{1}{\det(A)(e^{h(\alpha,\lambda)})}.
\end{aligned}$$

Thus,

$$W^h_{\mathfrak{g}} \circ \text{Hair}^\nu(A)(\lambda) = [\lambda - \rho]\prod_{\alpha \succ 0}\frac{1}{\det(A)(e^{h(\alpha,\lambda)})}.$$

On the other hand, consider the element $e^{h\lambda} \in T$ (where we identify $\mathfrak{t}$ with $\mathfrak{t}^\star$ using the W-invariant inner product) and its Adjoint action $\text{Ad}(e^{h\lambda}) \in \text{Aut}(\mathfrak{g})$. It follows that the matrix of $\text{Ad}(e^{h\lambda})$ is diagonal with respect to the *weight decomposition* of $\mathfrak{g}$:

$$\mathfrak{g} = \oplus_{\alpha \succ 0}\mathfrak{g}_{-\alpha} \oplus \mathfrak{t} \oplus \oplus_{\alpha \succ 0}\mathfrak{g}_\alpha$$

with eigenvalue $e^{h(\alpha,\lambda)}$ on each root space $\mathfrak{g}_\alpha$ and eigenvalue 1 on the Cartan algebra $\mathfrak{t}$. Thus, for every Laurrent polynomial $f \in \Lambda$, we have that $f(\text{Ad}(e^{h\lambda}))$ is a diagonal matrix with eigenvalue $f(e^{h(\alpha,\lambda)})$ on $\mathfrak{g}_\alpha$ and $f(1)$ on $\mathfrak{t}$. Since $A(1) = 1$ and $A(t) = A(t^{-1})$, it follows that

$$\begin{aligned}
\Phi \circ W^h_G(A) &= \Phi\left(\frac{1}{\det(\det(A)|_{t\to\text{Ad}(g)}})^{1/2}\right)(\lambda) \\
&= [\lambda - \rho]\frac{1}{\det(\det(A)|_{t\to e^{h(\alpha,\lambda)}})^{1/2}} \\
&= [\lambda - \rho]\prod_{\alpha \succ 0}\frac{1}{\det(A)(e^{h(\alpha,\lambda)})}.
\end{aligned}$$

The result follows. □

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