ON FINITE TYPE 3-MANIFOLD INVARIANTS I

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ABSTRACT. Recently Ohtsuki [Oh2], motivated by the notion of finite type knot invariants, introduced the notion of finite type invariants for oriented, integral homology 3-spheres ($\mathbb{Z}HS$ for short). In the present paper we propose another definition of finite type invariants of $\mathbb{Z}HS$ and give equivalent reformulations of our notion. We show that our invariants form a filtered commutative algebra and are of finite type in in the sense of Ohtsuki and thus conclude that the associated graded algebra is a priori finite dimensional in each degree. We discover a new set of restrictions that Ohtsuki's invariants satisfy and give a set of axioms that characterize the Casson invariant. Finally, we pose a set of questions relating the finite type 3-manifold invariants with the (Vassiliev) knot invariants.

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1. Introduction

1.1. History. In recent years there has been a lot of progress defining (geometrically and combinatorially) knot and 3-manifold invariants. A unifying approach to these invariants is the concept of a topological quantum field theory ($TQFT$ for short) in $2+1$ dimensions, [At]. Witten [Wi2], using path integrals with a Chern-Simons action (a not yet defined infinite dimensional integration) gave examples of such theories depending on a semisimple compact Lie group and an integer.

Shortly afterwards, Reshetikhin-Turaev [RT1], [RT2], (and simultaneously many other authors [Koh], [Ku], [KR], [Po], [TW]), used equivalent initial data (namely semisimple Lie algebra and a primitive complex root of unity) as in Witten's Chern-Simons theory and combinatorially defined $TQFT$ in $2+1$ dimensions. $TQFTs$ in $2+1$ dimensions give rise to (complex valued) invariants of oriented, closed 3-manifolds, and invariants of framed colored links in 3-manifolds.

The path integral approach to topological quantum field theories suggests the existence of nonperturbative and perturbative knot and 3-manifold invariants. Examples of nonperturbative knot invariants are the values at roots of unity of colored Jones polynomials of knots, [RT1]. Examples of nonperturbative 3-manifold invariants are the Reshetikhin-Turaev invariants [RT2]. Examples of perturbative (or finite type) knot invariants are the Vassiliev invariants, [B-N1], [BL], [Va]. For the Vassiliev invariants of knots in S^3 one has:

- an axiomatic definition,
- a general existence theorem [B-N1], [Ko1], [LM],
- a comparison theorem to the above mentioned nonperturbative kont invariants [B-N1], [Dr], [Ka], and to the Chern-Simons theory perturbative knot invariants [BT], and finally
- ways of calculating them, from combinatorics of chord diagrams [B-N2].

The situation with perturbative (or finite type) 3-manifold invariants is puzzling. On the one hand perturbative Chern-Simons theory predicts the existence of invariants of a pair (M, ρ) where M is a rational homology 3-sphere and $\rho \in Hom(\pi_1(M), G)$

(G is a fixed compact semisimple Lie group here). In cases of acyclic ρ one has such invariants [AxS1], [AxS2], [Ko2]. However, these invariants do not solve any of the above mentioned problems,essentially due to the absence of surgery formulas.

1.2. A review of Ohtsuki's definition. On the other hand, Ohtsuki [Oh1] recently defined finite type invariants for integral homology 3-spheres ($\mathbb{Z}HS$ for short). His definition was inspired by the notion of finite type knot invariants. Let us review his definition and introduce some notation. Let $\mathcal{M}$ denote the vector space (over $\mathbb{Q}$) on the set of oriented integral homology 3-spheres ($\mathbb{Z}HS$ for short). A link $L \subseteq M$ in a $\mathbb{Z}HS$ sphere is called algebraically split if the linking numbers between any two components vanish. A framing $f = (f_1, \dots, f_n)$ for an n component link is a sequence of integers associated to each component. [1]

A framed link (L, f) in a $\mathbb{Z}HS$ M is called unimodular if $f_i = \pm 1$ for all i. A link $L \subseteq M$ is called boundary if each component bounds a Seifert surface, and the Seifert surfaces are disjoint from each other. A framed link (L, f) is called AS (respectively B) admissible if it is algebraically split (respectively, boundary) and unimodular. For every framed link (L, f) in M we denote by $\chi(L, f; M)$ the result of doing Dehn surgery on L in M, [Ro]. For a framed link (L, f) in M we denote

$$(L, f; M) := \sum_{L' \subseteq L} (-1)^{|L'|} \chi(L', f(L'); M) \in \mathcal{M} \tag{1}$$

Let us define two decreasing filtrations $\mathcal{F}^{Oh}_{\star}$ (respectively, $\mathcal{F}_{\star}$) on the vector space $\mathcal{M}$ as follows: $\mathcal{F}^{Oh}_{m}\mathcal{M}$ (respectively, $\mathcal{F}_m\mathcal{M}$) is the subspace spanned by (L, f, M) for all B (respectively, AS) admissible unimodular links of $m+1$ components. We can now state the following definition, due to Ohtsuki [Oh1]:

Definition 1.1. [Oh1] $\lambda \in \mathcal{F}^{Oh}_{m}\mathcal{L}$ is an invariant of $\mathbb{Z}HS$ of type m if $\lambda(\mathcal{F}^{Oh}_{m+1}\mathcal{L}) = 0$, i.e. if for every admissible AS link L of $m+1$ components in a $\mathbb{Z}HS$ M, we have that

$$\sum_{L' \subseteq L} (-1)^{|L'|} \chi(L', f(L'); M) = 0 \tag{2}$$

Let $\mathcal{F}^{Oh}\mathcal{L} := \cup_{m \geq 0} \mathcal{F}^{Oh}_{m}\mathcal{L}$. It is easy to show that $\mathcal{F}^{Oh}\mathcal{L}$ is a filtered commutative algebra (with pointwise multiplication). Let $\mathcal{G}^{Oh}_{\star}\mathcal{L}$ (and more generally $\mathcal{G}_{\star}\mathcal{O}$) denote the associated graded algebra of $\mathcal{F}^{Oh}\mathcal{L}$ (or more generally, of a filtered object $\mathcal{F}_{\star}\mathcal{O}$).

[1] Usually a framing for a link is a choice of a simply closed curve γ_i on the tubular neighborhood of each component L_i such that the linking number between γ_i and a meridian of L_i is 1. Any two framings of a single component differ by an integer number. Since the 3-manifolds that we consider are oriented integral homology spheres, canonical (otherwise called zero) framings exist, hence the identification of the possible framings with the integer numbers.

1.3. Variations for finite type 3-manifold invariants. In the present paper we introduce another notion of finite type invariants of $\mathbb{Z}HS$. We compare our definition with Ohtsuki's, (theorem 2) and with the finite type knot invariants (corollary 1.3) and show that the associated graded vector space is a priori finite dimensional in each degree. We discover a new set of restrictions that Ohtsuki's invariants satisfy (theorem 4). As an application, we deduce a nonexistence theorem for 3-manifold invariants (proposition 1.4) and a characterization for the Casson invariant (theorem 5).

We begin with the following definition.

Definition 1.2. Let $\mathcal{F}_m\mathcal{L}$ denote the set of all invariants λ (with values in a field, assumed to be $\mathbb{Q}$) of $\mathbb{Z}HS$ satisfying the following property (with the notation as in section 1.2):

- For every oriented embedded surface $\Sigma \hookrightarrow M$ in a $\mathbb{Z}HS$ M and every choice $\gamma_1, \dots, \gamma_{m+1}$ of oriented, separating, nonintersecting simple closed curves on Σ we have

$$\sum_{\epsilon_i \in \{0,1\}} \prod_i (-1)^{\epsilon_i} \lambda(\chi(\gamma_1^{\epsilon_1}, \dots, \gamma_{m+1}^{\epsilon_{m+1}}; M)) = 0 \tag{3}$$

We call such λ type m invariants of $\mathbb{Z}HS$.

Let $\mathcal{L} = \cup_m \mathcal{F}_m\mathcal{L}$ denote the space of finite type invariants of $\mathbb{Z}HS$. $\mathcal{L}$ has an increasing filtration $\mathcal{F}_.$ and a pointwise multiplication that respects the increasing filtration and gives $\mathcal{L}$ the structure of a filtered commutative algebra.

1.4. Statement of the results. We begin by giving an equivalent definition of type m $\mathbb{Z}HS$ invariants.

Theorem 1. *The following are equivalent:*

1. $\lambda \in \mathcal{F}_m\mathcal{L}$ *(i.e. it satisfies the property of definition* 1.2*)*
2. λ *satisfies the property of definition* 1.2 *with the extra assumption that* $\Sigma \hookrightarrow M$ *is a surface of a Heegaard splitting*
3. *with the notation of section* 1.2*, for every* b *admissible link* $L \subseteq M$ *of* $m+1$ *components we have*

$$\sum_{L' \subseteq L} (-1)^{|L'|} \chi(L', f(L'); M) = 0 \tag{4}$$

i.e. $\lambda \in \mathcal{F}_m\mathcal{L}$.

As far as comparing the two notions of finite type invariants of $\mathbb{Z}HS$, we have the following theorem:

Theorem 2. *There is a one-to-one map*

$$\mathcal{F}_m\mathcal{L} \hookrightarrow \mathcal{F}^{Oh}_{3m+3}\mathcal{L} \tag{5}$$

I.e., with the notation of 1.2 *we have that* $\mathcal{F}_m\mathcal{M} \supseteq \mathcal{F}^{Oh}_{3m+3}\mathcal{M}$. *Therefore, since Ohtsuki* [Oh2] *proved that* $\mathcal{G}^{Oh}_m\mathcal{L}$ *is a finite dimensional vector space, it follows that so is* $\mathcal{G}_m\mathcal{L}$.

We now introduce a map from knots to (linear combinations of) 3-manifolds defined by $K \to (K, S^3) := \chi(K, +1; S^3) - (S^3)$. Dually this map induces a map from 3-manifold invariants to knot invariants. This map allows us to compare finite type 3-manifold invariants and knot invariants as follows:

Proposition 1.3. *The above mentioned map descends to a map* $\psi : \mathcal{F}^{Oh}_m\mathcal{L} \to \mathcal{F}_{m-1}\mathcal{V}$. *where* $\mathcal{F}_m\mathcal{V}$ *is the space of type* m *knot invariants (see section* 2*)*

In particular, for $m = 1, 2$ we can compare finite type $\mathbb{Z}HS$ invariants and knot invariants as follows:

Theorem 3. *For* $m = 1, 2$

(1) *the map* $\mathcal{F}_m\mathcal{L} \to \mathcal{F}^{Oh}_{3m+3}\mathcal{L}$ *of theorem* 2 *factors through a map*

$$\mathcal{F}_m\mathcal{L} \hookrightarrow \mathcal{F}_{3m}\mathcal{L}^{Oh} \tag{6}$$

(2) *Furthermore, using proposition* 1.3 *the associated composite map* $\mathcal{F}_m\mathcal{L} \hookrightarrow \mathcal{F}^{Oh}_{3m}\mathcal{L} \to \mathcal{F}_{3m-1}\mathcal{V}$ *factors through a map* $\mathcal{F}_m\mathcal{L} \to \mathcal{F}^{Special}_{2m}\mathcal{V}$ *where* $\mathcal{F}^{Special}_{2m}\mathcal{V}$ *is the space of special Vassiliev invariants.*

Ohtsuki [Oh1] gave the following dimensions for the graded vector spaces $\mathcal{F}^{Oh}_m\mathcal{L}$:

m	0	1	2	3
$dim\mathcal{G}_m\mathcal{L}^{Oh}$	1	0	0	1

We give a new set of restrictions that the type m invariants of Ohtsuki satisfy thus deducing the following

Theorem 4. *Every type* 4 *Ohtsuki's invariant is of type* 3, *i.e.* $\mathcal{G}^{Oh}_4\mathcal{L} = 0$.

Proposition 1.4. *If* V *is a type* 3 *knot invariant which can be extended to a* $\mathbb{Z}HS$ *invariant so that it satisfies the following property:*

$$V(K) = V(\chi(K^{+1}; S^3)) \tag{7}$$

(for all knots K *in* S^3*), then* V *is a type* 2 *knot invariant.*

This in turn proves the following characterization of the Casson invariant:

Theorem 5. *For an invariant* λ *of* $\mathbb{Z}HS$ *the following are equivalent:*

(1) $\lambda \in \mathcal{F}^{Oh}_2\mathcal{L}$ *of definition* 1.2
(2) *with the notation of section* 1.2, $\lambda(\mathcal{F}_2\mathcal{M}) = 0$

(3) $\lambda \in \mathcal{F}_3^{Oh}\mathcal{L}$
(4) $\lambda \in \mathcal{F}_4^{Oh}\mathcal{L}$
(5) *There are constants a, b such that $\lambda = a + b\lambda_{Casson}$ where λ_{Casson} is the Casson invariant* [AM].

Remark 1.5. Ohtsuki [Oh2] had previously proved that (3) is equivalent to (5).

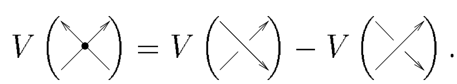
1.5. Plan of the proof. In section 2 we review the definition and a few properties of finite type knot invariants, otherwise known as Vassiliev invariants. In section 3.1 we review some terminology and notation from Dehn surgery of links. In section 3.2 we prove proposition 1.3, thus giving a map from finite type invariants of $\mathbb{Z}HS$ to (finite type) invariants of knots in S^3.In section 3.3 we prove theorem 2. In section 4.1 we show surgery properties that our and Ohtsuki's finite type 3-manifold invariants satisfy and in section 4.2 we prove theorem 1 that restates our definition 1.2. In section 5 we pose a set of questions relating the finite type knot and 3-manifold invariants. In section 6.1 we partially answer our questions and prove theorem 3 and in section 6.2 we give a new set of restrictions that Ohtsuki's invariants satisfy, thus showing theorem 4 and proposition 1.4. Finally, in section 7 we collect an equivalent set of properties that characterize the Casson invariant.

1.6. Acknowledgment. We wish to thank D. Auckly, L. Kauffman, E. Lerman, J. Levine, K. Millett and J. Roberts for many useful comments. Especially we wish to thank D. Bar-Natan and R. Kirby for many enlightening conversations.

2. Finite type knot invariants

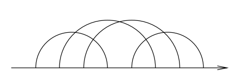
In this section we review finite type knot invariants, otherwise known as Vassiliev invariants [B-N1], [BL], [Va]. A standard reference for the next definitions and notation is [B-N1].

A *Vassiliev invariant* of type m is a knot invariant V which vanishes whenever it is evaluated on a knot with more than m double points, where the definition of V is extended to knots with double points via the formula



The algebra $\mathcal{V}$ of all Vassiliev invariants (with values in $\mathbb{Q}$) is filtered, with the type m subspace $\mathcal{F}_m\mathcal{V}$ containing all type m Vassiliev invariants. The associated graded space of $\mathcal{V}$ is isomorphic to the space $\mathcal{W}$ of all *weight systems*. A degree m weight system is a homogeneous linear functional of degree m on the graded vector space $\mathcal{A}^r$ of *chord diagrams* like in figure 1 divided by the $4T$ and framing independence relations explained in figures 2 and 3.

Figure 1. A chord diagram:



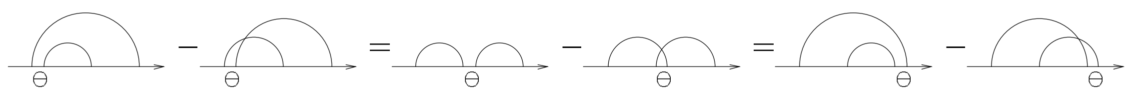
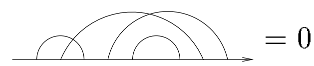


Figure 2. To get the $4T$ relations, add an arbitrary number of chords in arbitrary positions (only avoiding the short intervals marked by a 'no-entry' sign $\ominus$) to all six diagrams in exactly the same way.

Figure 3. The framing independence relation: any diagram containing a chord whose endpoints are not separated by the endpoints of other chords is equal to 0.



$\mathcal{A}^r$ is graded by the number of chords in a chord diagram. It is a commutative and co-commutative Hopf algebra with multiplication defined by juxtaposition, and with co-multiplication Δ defined as the sum of all possible ways of 'splitting' a diagram. The co-algebra structure of $\mathcal{A}^r$ defines an algebra structure on $\mathcal{W}$. The Hopf algebra $\mathcal{A}$ is defined in the same way as $\mathcal{A}^r$, only without imposing the framing independence relation.

There are natural maps $W_m : \mathcal{F}_m\mathcal{V} \to \mathcal{G}_m\mathcal{W} = \mathcal{G}_m\mathcal{A}^{r\star}$. For a type m Vassiliev invariant V it is natural to think of $W_m(V)$ as "the m'th derivative of V". W_m is not isomorphism, however its kernel is $\mathcal{F}_{m-1}\mathcal{V}$. It therefore follows that $\mathcal{F}_m\mathcal{V}$ is a finite dimensional vector space for every m.

In the present paper we are primarily interested in type 5 knot invariants about which much more is known.

We can summarize the results in the following proposition [B-N1]:

Proposition 2.1. *The dimensions of the space of type m Vassiliev invariants of knots are given in the following table*

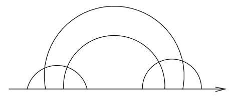
m	*0*	*1*	*2*	*3*	*4*	*5*
$dim\mathcal{G}_m\mathcal{W}$	*1*	*0*	*1*	*1*	*3*	*4*

Let us denote by $\Delta^{(m)}(K) := \frac{d^m}{dh_m}|_{h=0}\Delta(K)(e^h)$ the m^{th} derivative of the Alexander-Conway polynomial $\Delta(K)$ of a knot K [Ro] with the normalization as in [B-NG], example 2.8. It is clear that $\Delta^{(m)} \in \mathcal{F}_m\mathcal{V}$. Let us denote by $\mathcal{F}_m^{Special}\mathcal{V}$ the vector space of special Vassiliev invariants, i.e., these whose degree m part (i.e., whose image in $\mathcal{G}_m\mathcal{V}$) consists of products of $\Delta^{(m_i)}$ (for $\sum_i m_i = m$).

We also need the following lemma:

Lemma 2.2. *If a degree 4 weight system $W \in \mathcal{G}_4\mathcal{W}$ vanishes on the chord diagram $CD[4,1]$ of figure 4 then $W = aW(\Delta^{(4)}) + bW(\Delta^{(2)2})$ for some constants a, b.*

Proof. A little algebra shows that the intersection of the kernel of $W(\Delta^{(4)})$ and of $W(\Delta^{(2)2})$ is one dimensional spanned by the chord diagram $CD[4,1]$. The algebra can be performed by hand or using the program `NAT.m` of Bar-Natan [B-N2]. $\square$

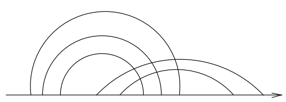


$CD[4,1]$

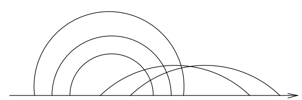
Figure 4. A chord diagram with 4 chords

Lemma 2.3. *If a degree 5 weight system $W \in \mathcal{W}$ vanishes on the chord diagrams of figure 5 then $W = 0$.*

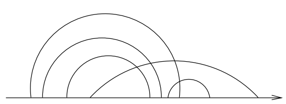
Proof. The claim follows from the fact that the chord diagrams in figure 5 form a basis for $\mathcal{G}_5\mathcal{A}^r$, a proof of which can be obtained using the program `NAT.m` of Bar-Natan [B-N2]. Notice that there are 36 chord diagrams of degree 5 but the quotient space $\mathcal{G}_5\mathcal{A}^r$ obtained modulo the $4T$ and framing independent relations is 4 dimensional. □



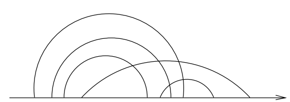
$CD[5,1]$



$CD[5,2]$



$CD[5,3]$



$CD[5,4]$

Figure 5. Some degree 5 chord diargams which form a basis for $\mathcal{G}_5\mathcal{A}^r$

3. Comparison with other approaches

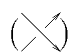
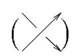
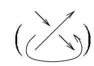
3.1. Preliminaries about links and 3-manifolds. In this section we review a few preliminaries about 3-manifolds. All 3-manifolds are oriented unless otherwise mentioned. In this paper we will restrict our attention to integral homology 3-spheres.

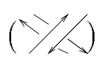
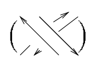
Let $K \subseteq M$ be a knot in a $\mathbb{Z}HS$ M. A framing f of K is a simply closed curve in the boundary of a tubular neighborhood of K in M that intersects a meridian m of K once positively. Recall that a canonical (otherwise called zero) framing f_0 of K in M exists: indeed since K is homologically trivial in M it bounds a Seifert surface, and let f_0 be a parallel of K in the surface. The result is independent of the surface chosen. For coprime integers p, q let $\chi(K^{p/q}; M)$ denote the (closed) 3-manifold obtained by doing p/q Dehn surgery on K i.e., the result of cutting M along the boundary of a tubular neighborhood of K and gluing back a standard solid torus $S^1 \times D^2$ in such a way that the standard meridian $a \times S^1$ gets identified with $pm + qf_0$. One obtains immediately that $H_1(\chi(K^{p/q}; M), \mathbb{Z}) \simeq \mathbb{Z}/|p|\mathbb{Z}$.

If $\Sigma \hookrightarrow M$ is an embedded surface and $\gamma \subseteq M$ is an oriented, simple closed curve, we denote by $\chi(\gamma^n; M)$ the 3-manifold obtained by cutting M across Σ, performing n Dehn twists along γ and gluing Σ back.

3.2. From knots to 3-manifolds. In this section we prove proposition 1.3. We begin by recalling the map from (framed) links to (linear combinations of) 3-manifolds of equation (1). We can compose this map with the one that sends algebraically split links L in S^3 to a pair (L, f) where f denotes the $+1$ framing on every component of L. We keep denoting the composed map from links in S^3 to $\mathcal{M}$ by $L \to (L)$. In the following lemma all links drown will be algebraically split (in S^3) and will have framing $+1$ on each component. With these notations following lemma

Lemma 3.1. *With the notation of section* 1.2 *the following identities hold in* $\mathcal{G}^{Oh}_{\star}\mathcal{M}$*:*

(8)  $-$  $=$  $\in \mathcal{G}^{Oh}_{m+1}\mathcal{M}$

(9)  $-$  $= 0 \in \mathcal{G}^{Oh}_{m}\mathcal{M}$

(10) $(K_0 \cup L) = 0 \in \mathcal{G}^{Oh}_{m}\mathcal{M}$

where $K_0 \cup L$ is the disjoint union of L with an unknot K_0. In the above equations, the left hand side represents links of m components. In the first equation, both strands belong to the same component, and in the second equation tqo stands of the same component go over/under two strands of another component.

Proof. A proof was first given by Ohtsuki in [Oh2]. It is a simple consequence of Kirby moves and the definition of the map (.). Note that we could also give a formula in $\mathcal{F}^{Oh}_{\star}\mathcal{M}$, rather than in the graded space $\mathcal{G}^{Oh}_{\star}\mathcal{M}$, however, the above form of the lemma suffices for our purposes. □

Proposition 1.3 now follows immediately by the first equation in (8). In the remaining part of the paper, it will be useful to describe the associated weight system of the the finite type knot invariant of proposition 1.3. This can be done as follows:

Remark 3.2. Let $\lambda \in \mathcal{F}^{Oh}_{m}\mathcal{L}$, $\psi_\lambda := \psi(\lambda) \in \mathcal{F}_{m-1}\mathcal{V}$ the associated knot invariant of proposition 1.3 and let $W_\lambda \in \mathcal{G}_{m-1}\mathcal{W}$ be the associated degree $m-1$ weight system. One way to calculate the value of W_λ on a chord diagram CD of $m-1$ chords is as follows: represent the chord diagram in a circle, resolve the crossing points between chords in any way and replace each chord with an unknot as in figure 6. This way we get an algebraically split m component link $L(CD)$ each component of which is an unknot. By definition and using lemma 3.1 we see that

$$W_\lambda(CD) = \psi_\lambda(L(CD)) \tag{11}$$

Note that even though $L(CD)$ depends on the way we choose to resolve the crossing points between the chords of the chord diagram, the value of ψ_λ is independent of that choice as follows by lemma 3.1.

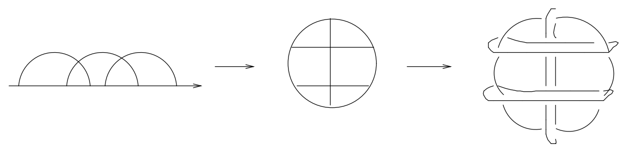


Figure 6. Reconstructing algebraically split m component links from linear chord diagrams of $m-1$ chords. Here we take $m=4$.

We believe that it is an interesting question (both for the sake of knot invariants, but also for the sake of 3-manifold invariants) to study the map of proposition 1.3.

3.3. A comparison theorem. In this section we prove theorem 2. With the notation of section 1.2 we will show that $\mathcal{F}_m\mathcal{M} \supseteq \mathcal{F}^{Oh}_{3m+3}\mathcal{M}$. In his fundamental paper [Oh2] Ohtsuki described a (finite) generating set of the vector space $\mathcal{G}^{Oh}_m\mathcal{M}$. Let us introduce some more notation before we recall his result. For $m \in \mathbb{N}$, let $G[m]$ denote the set of graphs with univalent and trivalent vertices with m edges.[2] We denote by $v_k(\Gamma)$ the number of k valent vertices, $e(\Gamma), v(\Gamma)$ the number of edges and vertices of such a graph Γ. Let $NI(\Gamma)$ denote the maximum number of nonintersecting edges. For such a graph $\Gamma \in G[m]$, Ohtsuki constructs an algebraically split link $L(\Gamma)$ in S^3 of m components (and framing $+1$ on each component) a follows: each edge is represented by an unknot, and each trivalent vertex is represented by a Borromean link). We can now state the following theorem of Ohtsuki:

Theorem 6. [Oh2] *$\mathcal{G}^{Oh}_m\mathcal{M}$ is generated (as a vector space) by the set $\{L(\Gamma)\}_{\Gamma\in G[m]}$.*

Theorem 2 now follows from theorem 6 and the following two lemmas:

Lemma 3.3. *One has the following lower bounds for $NI(\Gamma)$:*

- *If $\Gamma \in G[3m+1]$ then $NI(\Gamma) \geq m+1$.*
- *If $\Gamma \in G[3m+2]$ then $NI(\Gamma) \geq m$.*
- *If $\Gamma \in G[3m+3]$ then $NI(\Gamma) \geq m$.*

Lemma 3.4. *For every uni-trivalent graph Γ we have that*

$$(L(\Gamma)) \in \mathcal{F}^{Oh}_{NI(\Gamma)}\mathcal{M} \tag{12}$$

[2]and vertex orientations, as was communicated to the author by Ohtsuki

Proof. (of lemma 3.3) Let $\Gamma \in G[n]$, and let T be a spanning tree of it. Obviously we have that $e(T) = v(T) - 1 = v_1(\Gamma) + v_3(\Gamma) - 1$. Moreover, an easy induction shows that $NI(T) \geq \left[\frac{e(T)}{2}\right]$ (since T has at most trivalent vertices) and therefore that

$$NI(\Gamma) \geq NI(T) \geq \left[\frac{e(T)}{2}\right] = \left[\frac{v_1(\Gamma) + v_3(\Gamma) - 1}{2}\right] \tag{13}$$

We need to show that $\left[\frac{v_1(\Gamma)+v_3(\Gamma)-1}{2}\right] \geq \left[\frac{e(\Gamma)-1}{3}\right] + \epsilon(e(\Gamma))$ where $\epsilon(3k+1) = \epsilon(3k+2) = 0, \epsilon(3k) = 0$. Since $\Gamma \in G[n]$ we have that $2e(\Gamma) = v_1(\Gamma) + 3v_3(\Gamma)$ therefore it suffices to show that

$$\left[\frac{v_1(\Gamma) + v_3(\Gamma) - 1}{2}\right] \geq \left[\frac{(v_1(\Gamma) + 3v_3(\Gamma))/2 - 1}{3}\right] + \epsilon(e(\Gamma)) \tag{14}$$

Now a case by case argument for each class of n mod 3 shows the result. $\square$

Proof. (of lemma 3.4) We claim that the sublink L' of L that consists of a set of nonintersecting edges is a boundary sublink. In fact, we can attach discs with one handle to each unknot that corresponds to a set of nonintersecting edges, in such a way that each component of L' bounds a genus 1 surface and that every two surfaces are disjoint from each other. $\square$

The proof of theorem 2 is complete.

Remark 3.5. In fact, the above proof shows a bit more namely,

- if $\Gamma \in G[3m+2]$ satisfies $v_1(\Gamma) \neq 1$ then $NI(\Gamma) \geq m+1$, and
- if $\Gamma \in G[3m+3]$ satisfies $v_1(\Gamma) \neq 0$ then $NI(\Gamma) \geq m+1$

These lower bounds are sharp. For example the graph Γ of figure 7 shows that $\Gamma \in G[15]$ but $NI(\Gamma) = 4$ (and not 5).

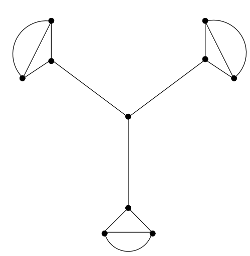


Figure 7. An annoying graph

4. Properties of finite type 3-manifold invariants

4.1. A surgery formula. In this section we prove a surgery formula for the invariants $\lambda \in \mathcal{F}_m\mathcal{L}$. Let K be a knot in a $\mathbb{Z}HS$ M, and $n \in \mathbb{N}$. Let $K_{((n))}$ denote the $(0, n)$ cable of K i.e., a link of n components parallel to K with linking numbers zero. We now have the following

Proposition 4.1. *If $\lambda \in \mathcal{F}_m\mathcal{L}$, $K \subseteq M$ as above and $n \in \mathbb{N}$, then with the notation of section 3.1 we have that*

$$\lambda(\chi(K^{1/n}; M)) = \sum_{j=0}^{m} \binom{n}{j} \psi_\lambda(K_{((j))}; M) \tag{15}$$

Proof. Figure 8 shows that $\chi(K^{1/n}; M)$ and $\chi(K_{((n))}^{1,1,\dots,1}; M)$ are diffeomorphic manifolds. Furthermore, for every $j \geq 0$ it follows by the definition of ψ_λ that

$$\psi_\lambda(K_{((j))}; M) = \sum_{k=0}^{j} (-1)^{j-k} \binom{j}{k} \lambda(K_{((k))}^{1,1,\dots,1}; M) \tag{16}$$

Furthermore $K_{((j))}$ is a boundary link of j components, therefore $\psi_\lambda(K_{((j))}; M) = 0$ for $j > m$. The result now follows by solving for λ from equation (16). $\square$

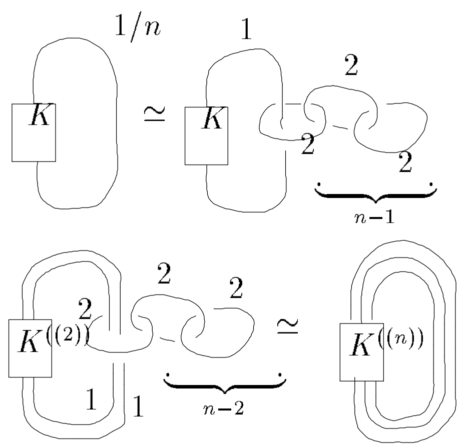


Figure 8. Some Kirby moves relating $1/n$ surgery on K

Exercise 4.2. Show that if $\lambda \in \mathcal{F}_m^{Oh}\mathcal{L}$ and $K \subseteq M$ a knot in a $\mathbb{Z}HS$ M, and $n \in \mathbb{N}$ then

$$\lambda(\chi(K^{1/n}; M)) = \sum_{j=0}^{3m} \binom{n}{j} \psi_\lambda(K_{((j))}; M) \tag{17}$$

4.2. A restatement of definition 1.2. In this section we prove theorem 1.

Proof. (of theorem 1) Obviously, (3) implies (1) which implies (2). We will show that (2) implies (3). Let $L \subseteq M$ be a boundary link of $m+1$ components in a $\mathbb{Z}HS$ M. Let E be an imbedded surface in M of $m+1$ components such that $\partial E = L$. Fix an identification of $E \times I$ with a bicollar of E in M. Let $\gamma := (\gamma_1, \dots, \gamma_{m+1})$ be a collar of ∂E in E. Then $E \times I$ is a (disconnected) handelbody, but $\overline{M \setminus E \times I}$ need not be. In any case, attach 1-handles on $E \times I$ away from $\gamma \times I$ to construct W such that $W, \overline{M \setminus W}$ are both (connected) handelbodies. Let $\Sigma \hookrightarrow M$ be the boundary of W. Note that $\gamma \times I$ is a disjoint union of separating annuli in Σ, and $1/1$ surgery on each component of L corresponds to cutting M along Σ, performing 1 *left*-handed Dehn twist along each component of $\gamma \times I$, and glueing back. Therefore (2) of theorem 1 implies (3). $\square$

5. Questions

5.1. A few questions. In this section we pose some questions relating our notion of finite type 3-manifold invariants with that of Ohtsuki (for 3-manifolds) and of Vassiliev (for knots).

Question 1. With the notation of section 1.2 is it true that $\mathcal{F}_m\mathcal{M} = \mathcal{F}^{Oh}_{3m}\mathcal{M}$?

Remark 5.1. Theorem 2 shows that $\mathcal{F}_m\mathcal{M} \supseteq \mathcal{F}^{Oh}_{3m+3}\mathcal{M}$. Note that if the above question had a positive answer, it would imply that $\mathcal{F}_m\mathcal{M} \supseteq \mathcal{F}^{Oh}\mathcal{M}$ and that $\mathcal{G}^{Oh}_m\mathcal{M} = 0$ for m not a multiple of 3.

Question 2. Does the map $\psi : \mathcal{F}^{Oh}_{3m}\mathcal{L} \to \mathcal{F}_{3m-1}\mathcal{V}$

- actually factor through a map

$$\mathcal{F}^{Oh}_{3m}\mathcal{L} \to \mathcal{F}_{2m}\mathcal{V} \tag{18}$$

  preserving the filtration?
- If so, is it true that the graded map

$$\mathcal{G}^{Oh}_{3m}\mathcal{L} \to \mathcal{G}_{2m}\mathcal{V} \tag{19}$$

  is one-to-one?
- Is it true that the image of (19) is the space $\mathcal{G}^{Special}_{2m}\mathcal{V}$ of special Vassiliev invariants, i.e., products of coefficients of the Conway polynomial?

Question 3. Is it true for the invariants λ_m defined in [Oh2] that $\lambda_m \in \mathcal{F}_m\mathcal{L}$? Also that $\lambda_m \in \mathcal{F}^{Oh}_{3m}\mathcal{L}$?

Question 4. Do (either of the two versions of) finite type invariants of $\mathbb{Z}HS$ separate them?

5.2. A general comment. We believe that the above mentioned questions will be helpful in understanding knot invariants as well as 3-manifold invariants. One feature of these questions is that they are (in principle) testable on a computer, which can decide about the fate of some of them. The experimental knowledge is small so far. Much remains to be done in analogy with the rather well developed theory of finite type knot invariants.

6. Calculations

6.1. Questions 1, 2 for $m = 1, 2$. In this section we partially answer questions 1, 2 in the case of $m = 1, 2$ and prove theorem 3.

Proof. (of part (1) of theorem 3) Following the notation and the proof of theorem 2 in section 3.3 and remark 3.5 we see that the result follows from the the following two claims (for $m = 1, 2$):

Claim 6.1. *For every $\Gamma \in G[3m+3]$ with $v_1(\Gamma) = 0$ we have $NI(\Gamma) \geq m+1$*

Claim 6.2. *For every $\Gamma \in G[3m+2]$ with $v_1(\Gamma) = 1$ we have $NI(\Gamma) \geq m+1$*

The proof of the above claims is given in the following lemmas 6.3 and 6.4. □

Lemma 6.3. *Claim* 6.1 *holds for $m = 1, 2$.*

Proof. For $m = 1$ it is easy to list all elements in $G[6]$ (see [Oh1]) and check it by hand. For $m = 2$ we could also list the relevant elements in $G[9]$ and check them by hand. Instead we prefer to give an alternative argument as follows: Let $\Gamma \in G[9]$ with $v_1(\Gamma) = 0$. Then $v_3(\Gamma) = 6$ and every spanning tree T of it has 5 edges. The possibilities for a spanning tree are shown in figure 9.

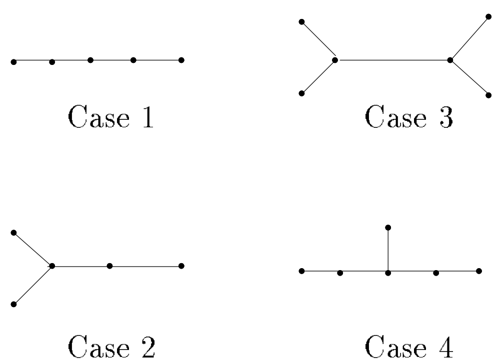


Figure 9. All possible trees with 5 edges

We now distinguish cases:
<u>Case 1</u> We are done since $NI(\Gamma) = 3$.

Case 2 It is easy to see that Γ has a subdiagram of the form $\Gamma_{2,1}$ or $\Gamma_{2,2}$ as in figure 10. Therefore we can choose a spanning tree of the form $T_{2,1}$ or $T_{2,2}$ and in both cases we are reduced to case 1 and the result holds.

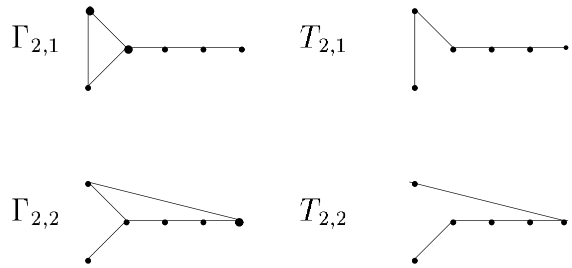


Figure 10. Subgraphs of Γ and alternative spanning trees

Case 3 It is easy to see that Γ has a subdiagram of the form $\Gamma_{3,1}$ or $\Gamma_{3,2}$ as in figure 11. Therefore we can choose a spanning tree of the form $T_{3,1}$ or $T_{3,2}$ which reduces us to case 2 or 1 and the result holds.

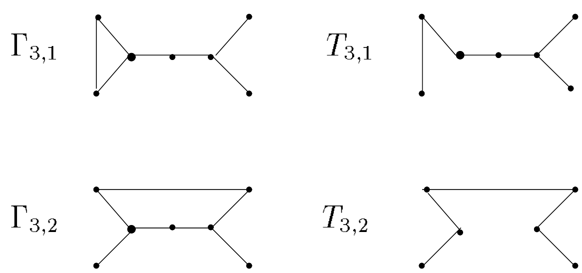


Figure 11. Subgraphs of Γ and alternative spanning trees

Case 4 It is easy to see that Γ has a subdiagram of the form $\Gamma_{4,1}$ or $\Gamma_{4,2}$ as in figure 12. Therefore we can choose a spanning tree of the form $T_{4,1}$ or $T_{4,2}$ which reduces us to case 1 or 2 and the result holds.

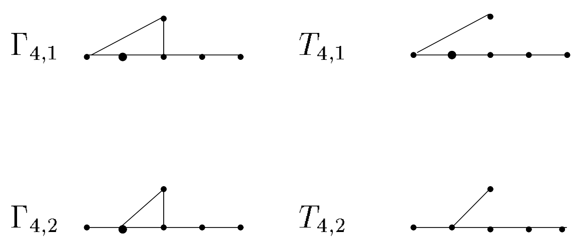


Figure 12. Subgraphs of Γ and alternative spanning trees

The proof of lemma 6.3 is complete. $\square$

Lemma 6.4. *Claim* 6.2 *holds for* $m = 1, 2$.

Proof. The proof is analogous to lemma 6.3. $m = 1$ is easy. If $m = 2$ and $\Gamma \in G[8]$ with $v_1(\Gamma) = 1$, then $v_3(\Gamma) = 5$ and a spanning tree T has 5 edges. The possibilities are shown in figure 9 and the cases are shown in figures 10, 11, 13. The proof of lemma 6.4 is complete. □

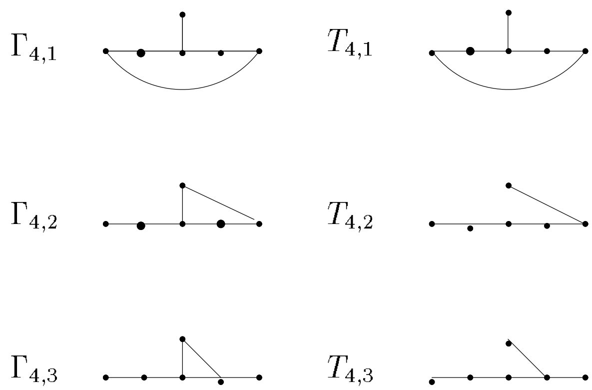


Figure 13. Subgraphs of Γ and alternative spanning trees

We can now prove the second part of theorem 3.

Proof. (of part (2) of theorem 3)

For $m = 1$ it is obvious. For $m = 2$, let $\lambda \in \mathcal{F}_2\mathcal{L} \hookrightarrow \mathcal{F}_6^{Oh}\mathcal{L}$, let $\psi_\lambda \in \mathcal{F}_5\mathcal{V}$ be the associated knot invariant and let W_λ be the associated degree 5 weight system, (see remark 3.2). Using remark 3.2 we see that W_λ vanishes on the chord diagrams of figure 5, since the associated 5 component links of remark 3.2 have boundary sublinks of 3 components. Therefore by lemma 2.3 we see that $W_\lambda = 0$, i.e., $\psi_\lambda \in \mathcal{F}_4\mathcal{V}$. Now letting W_λ be the associated degree 4 weight system, arguing as above, we see that it vanishes on the chord diagram of figure 4 and therefore by lemma 2.2 we see that the image of ψ_λ is in the space $\mathcal{F}_4^{Special}\mathcal{V}$ of special type 4 knot invariants. □

6.2. A new set of restrictions for Ohtsuki's invariants. In this section we prove theorem 4 and proposition 1.4 by introducing a new set of restrictions that the finite type invariants of Ohtsuki have to satisfy. The main idea is to study the map ψ of 1.3.

Proof. (of theorem 4) Let us assume that $\lambda \in \mathcal{G}_4^{Oh}\mathcal{L}$. Let ψ_λ be the associated type 3 knot invariant as in remark 3.2 and in proposition 1.3. It follows from theorem 6 of Ohtsuki that $\lambda \in \mathcal{G}_4^{Oh}\mathcal{L}$ is determined by its value on the graph (with the two possible vertex orientations). Choosing the counter clock-wise orientation on each vertex of it, and recalling the discussion of section 3.3 (in particular, uni-tri valent graphs correspond to algebraically split links in S^3 which correspong to linear combinations of $\mathbb{Z}HS$, using $+1$ framing on each component) we have the following equality:

$$\lambda(\quad) = \lambda(\chi(K_0^{1/1}; S^3)) - 2\lambda(\chi(T_+^{1/1}; S^3)) + \lambda(S^3) \tag{20}$$

$$= \psi_\lambda(K_0) - 2\psi_\lambda(T_+) \tag{21}$$

where K_0 denotes the knot (in S^3) obtained by blowing down the three components of and T_+ denotes the knot in S^3 obtained by blowing down (with $+1$ framing) any two components of . (indeed, each component of is unknotted (with linking numbers zero with the other components and with framing $+1$) and remains unknotted after blowing down the other components. This shows that T_+ exists). In fact, T_+ is the right handed trefoil.

In other words, λ is determined by the type 3 knot invariant ψ_λ. We can now finish the proof of theorem 4 as follows: A basis for type 3 knot invariants is $1, J^{(2)}, J^{(3)}$ (where $J^{(m)}(K) := \frac{d^m}{dh_m}|_{h=0} J(K)(e^h)$ is the m^{th} derivative of the Jones polynomial). A calculation shows that $J^{(m)}(K_0) = 2J^{(m)}(T_+)$ for $m = 2, 3$.[3]

Therefore, $\psi_\lambda(\quad) = 0$. Similarly, had we chosen a different vertex orientation of the graph , $K_0 and T_+$ would be replaced by their mirror image and still $\psi_\lambda(\quad) = 0$. Therefore, $\mathcal{G}_4\mathcal{L}^{Oh} = 0$. $\square$

Proof. (of proposition 1.4) Let $V = aJ^{(2)} + bJ^{(3)} \in \mathcal{F}_3\mathcal{V}$ be a type 3 knot invariant satisfying the assumptions of proposition 1.4. Figure 15 shows two knots K_3 and K_4 with the property that -1 surgery on them gives diffeomorphic $\mathbb{Z}HS$. The knots appear in [Li] as an example of distinct knots in S^3 whose -1 surgery gives diffeomorphic $\mathbb{Z}$ homology spheres. We are indebted to R. Kirby for pointing out this reference to us. Since $\chi(K_3^{-1}; S^3)$ and $\chi(K_4^{-1}; S^3)$ are diffeomorphic $\mathbb{Z}HS$, after a change of the orientation we obtain that $\chi(\tau K_3^{+1}; S^3)$ and $\chi(\tau K_4^{+1}; S^3)$ are diffeomorphic $\mathbb{Z}HS$, where τK is the mirror image of a knot K in S^3. Therefore we have that

$$aJ^{(2)}(\tau K_3) + bJ^{(3)}(\tau K_3) = aJ^{(2)}(\tau K_4) + bJ^{(3)}(\tau K_4) \tag{22}$$

The Jones polynomials of them are given as follows:

$$J(\tau K_3)(q) = -q^{-3} + 2q^{-2} - 2q^{-1} + 3 - 2q + 2q^2 - q^3 \tag{23}$$

$$J(\tau K_4)(q) = q^{-5} - q^{-4} - q^{-1} + 1 + q^2 + q^3 - q^4 + q^5 - q^6 \tag{24}$$

from which we can deduce that $J^{(2)}(K_3) = J^{(2)}(K_4) = -6$ (this is not a surprise, since the Casson invariant exists!) but $J^{(3)}(K_3) = 0 \neq J^{(3)}(K_4) = -180$. Therefore,

[3]There are various programs [B-N2], [EM], [Och] that calculate the Jones polynomial of knots. As a check, we used all of the above mentioned and got the same results. We thank D. Dar-Natan, L. Kauffman, K. Millet and M. Ochiai for their help in distibuting and running the programs.

$b = 0$ and V is a type 2 knot invariant. Needless to say, we do not understand why this happens. □

Remark 6.5. Note that proposition 1.4 implies in order to show theorem 4 it suffices to check that $J^{(2)}(K_0) = 2J^{(2)}(T_+)$.

Exercise 6.6. (after a conversation with L. Kauffman) Show that $K_3 = \tau K_3$, which actually explains why $J(\tau K_3)(q) = J(\tau K_3)(q^{-1})$ and therefore that $J^{(3)}(\tau K_3) = 0$.

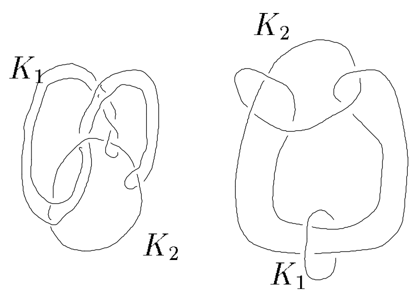


Figure 14. Two different views of the same two component link

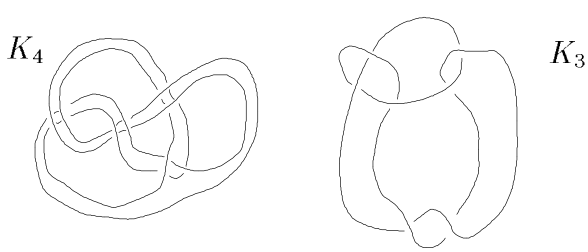


Figure 15. The result of figure 14 after blowing down K_2 (left) or K_1 (right)

7. Uniqueness of the Casson invariant

In this section we collect our results to show theorem 5. The first statement (1) is equivalent to (2) because of theorem 1. (2) implies (3) (by theorem 3) which implies (4). (4) implies (5) (by theorem 4) and finally (5) implies (1) by Casson, [AM].

Remark 7.1. It is surprising that we only used *nonintersecting,* bounding, simply closed curves in surfaces to characterize the Casson invariant.

References

[AM] S. Akbulut, J. C. McCarthy, *Casson's invariant for oriented homology 3-spheres: an exposition*, Princeton Math Notes, Princeton, 1990.

[At] M. F. Atiyah, *Topological Quantum Field Theories*, I.H.E.S. Publ. Math. **68** (1988) 175-186.

[AxS1] S. Axelrod, I. M. Singer, *Chern-Simons pertrubation theory*, Proc. XXth DGM Conference (New York, 1991) World Scientific, (1992) 3-45.

[AxS2] S. Axelrod, I. M. Singer, *Perturbative aspects of Chern-Simons topological quantum field theory II*, Jour. Diff. Geom. **39** (1994) 173-213.

[B-N1] D. Bar-Natan, *On the Vassiliev knot invariants*, preprint 1992, (to appear in Topology).

[B-N2] *Computer data files* available via anonymous file transfer from `math.harvard.edu`, user name `ftp`, subdirectory `dror`. Read the file `README` first.

[B-N3] ———, *Non-associative tangles*, to appear in the Georgia Inter. Topology Conference proceedings.

[B-NG] D. Bar-Natan, S. Garoufalidis, *On the Melvin-Morton-Rozansky conjecture*, preprint 1994, submitted to Inventiones.

[BL] J. Birman, X-S-Lin, *Knot polynomials and Vassiliev's invariants*, Invent. Math. **111** (1993) 225-270.

[BT] R. Bott, C. Taubes, *On the self linking of knots*, preprint 1994.

[Dr] V. G. Drinfeld, *On quasitriangular quasi Hopf algebras and a group closely connected with* $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$, Leningrad Math. J. **2** (1991) 829-860.

[EM] B. Ewing, K. Millett, *poly.c*, available at `millett@math.ucsb.edu`.

[Ka] C. Kassel, *Quantum groups*, to appear in GTM, Springer-Verlag, Heidelberg 1994.

[KR] L. Kauffman, *Invariants of 3-manifolds derived from finite dimensional Hopf algebras*, `hep-th` preprint 9406065.

[KM1] R. Kirby, P. Melvin *The 3-manifold invariants of Reshetikhin-Turaev for* $\mathfrak{sl}(2,\mathbb{C})$, Inventiones, **105** (1991) 473-545.

[Koh] Kohno, *Topological invariants of 3-manifolds using representations of mapping class groups I*, Topology, **31** (1992) 203-230.

[Ko1] M. Kontsevich, *Vassiliev's knot invariants*, Adv. in Sov. Math., **16(2)** (1993), 137–150.

[Ko2] ———, *Graphs, homotopical algebra and low dimensional topology*, preprint 1992.

[Ku] G. Kuperberg, *Non-involutory Hopf algebras and 3-manifold invariants*, preprint 1994.

[LM] T. Q. T. Le J. Murakami, *The universal Vassiliev-Kontsevich invariant for framed oriented links*, Max-Plank-Institut Bonn preprint, 1993.

[Li] W.B.R. Lickorish *Surgery on knots*, Proc. Amer. Math. Society, **60** (1976) 296-298.

[Mu1] H. Murakami, *Quantum* $SU(2)$ *invariants dominate Casson's* $SU(2)$ *invariant*, preprint 1993.

[Mu2] ——— *Quantum* $SO(3)$ *invariants dominate* $SU(2)$ *invariants of Casson and Walker*, preprint 1993.

[Och] M. Ochiai, *Knot theory by computer*, available at `ochai@ics.nara-wu.ac.jp`.

[Oh1] T. Ohtsuki, *A polynomial invariant of integral homology spheres*, preprint 1993.

[Oh2] ——— *Finite type invariants of integral homology 3-spheres*, preprint 1994.

[Po] M. Polyak, *A graphical approach to the 3-manifold invariants of Turaev and Viro*, J. Knot Theory **1** (1992) 219-240.

[RT1] N. Reshetikhin, V. Turaev, *Ribbon graphs and their invariants derived from quantum groups*, Commun. Math. Phys. **127** (1990) 1-26.

[RT2] ———, *Invariants of 3-manifolds via link polynomials and quantum groups*, Invent. Math.

103 (1991) 547-597.

[Ro] D. Rolfsen, *Knots and links*, Publish or Perish, 1976.

[TW] V. Turaev, H. Wenzl, *Quantum invariants of 3-manifolds associated with classical simple Lie algebras*, Inter. J. Math. **4** (1993) 323-358.

[Va] V. A. Vassiliev, *Complements of discriminants of smooth maps*, Trans. of Math. Mono. **98** Amer. Math. Society., Providence, 1992.

[Wa] K. Walker, *An extension of Casson's invariant*, Princeton Univ. Press 1993.

[Wi2] E. Witten, *Quantum field theory and the Jones polynomial*, Commun. Math. Phys. **121** (1989) 360-376.

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