

# ON FINITE TYPE 3-MANIFOLD INVARIANTS II

STAVROS GAROUFALIDIS AND JEROME LEVINE

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Fax number: (401) 863 9013, (617) 736 3085

Email: stavros@math.mit.edu Email: levine@max.math.brandeis.edu

ABSTRACT. The purpose of the present paper is, among other things, to relate the seemingly unrelated notions of surgical equivalence of links in  $S^3$  ([Le1]) and the notion of finite type invariants of oriented integral homology 3-spheres, due to T. Ohtsuki [Oh]. The paper consists of two parts. In the first part we classify pure braids and string links modulo the relation of surgical equivalence. We prove that the group of surgical equivalence classes of pure braids is isomorphic to the corresponding group of string links (Theorem 2). We also give two alternative descriptions of the above mentioned group  $P^{\text{SE}}(n)$  of surgical equivalence classes of  $n$  strand pure braids: one as a semidirect product of  $P^{\text{SE}}(n-1)$  together with an explicit quotient of the free group, and another description (Theorem 3) as a group of automorphisms of a nilpotent quotient of a free group. In the second part we apply these results to study the finite type invariants of  $\mathbb{Z}HS$ , originally introduced by Ohtsuki [Oh] and partially answer questions 1 and 2 from [Ga]. We reprove, in a more algebraic context, Ohtsuki's fundamental result which states that the space of type  $m$  invariants of  $\mathbb{Z}HS$  is finite dimensional for every  $m$ . Our proof allows us to show (Corollary 3.8) that the graded space of degree  $m$  invariants of  $\mathbb{Z}HS$  is zero dimensional unless  $m$  is divisible by 3. This partially answers question 1 of [Ga]. Furthermore, we study a map from knots (in  $S^3$ ) to  $\mathbb{Z}HS$ , and show that type  $5m+1$  invariants of  $\mathbb{Z}HS$  map to type  $4m$  invariants of knots, thus making progress towards question 2 of [Ga].

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## 1. INTRODUCTION

**1.1. History.** The present paper is a continuation of [Oh] and [Ga], devoted to the study of finite type invariants of *oriented integral homology 3-spheres* ( $\mathbb{Z}HS$  for short). Our purpose is, among other things, to relate the seemingly unrelated notions of surgical equivalence of links in  $S^3$  ([Le1]) and the notion of finite type invariants of oriented integral homology 3-spheres, due to T. Ohtsuki [Oh]. Finite type invariants of  $\mathbb{Z}HS$  were originally introduced by Ohtsuki [Oh] in his seminal paper (for a precise definition, see Definition 3.2). There are at least two sources of motivation/analogy/inspiration: the finite type knot invariants (see [B-N1], [BL], [Va]) and Chern-Simons theory in 3 dimensions (see [Wi2], [Rz1], [Rz2]).

Recall that a type  $m$  (otherwise called Vassiliev) invariant of knots in  $S^3$  (for a precise definition see the references above) satisfies a difference formula with respect to cutting along  $m + 1$  spheres, twisting and gluing back. Vassiliev invariants of knots in  $S^3$  form a unifying way of thinking about the Alexander, Jones, HOMFLY, Kauffman (and others) polynomials of knots.

Similarly, a type  $m$  invariant of  $\mathbb{Z}HS$  satisfies a “difference formula” with respect to cutting  $m + 1$  solid torii from a  $\mathbb{Z}HS$ , twisting them, and gluing them back. For a precise statement, see Definition 3.2. It is hoped that finite type invariants of  $\mathbb{Z}HS$  will provide a unifying way of dealing with 3-manifold invariants and with Chern-Simons theory in a way analogous to the case of Vassiliev invariants.

In either of the above sources of inspiration, we can think of type  $m$  invariants of  $\mathbb{Z}HS$  as “polynomials of degree  $m$ ” on the infinite dimensional vector space  $\mathcal{M}$ . Here and in the rest of this paper  $\mathcal{M}$  is the vector space (over  $\mathbb{Q}$ ) with basis the set of  $\mathbb{Z}HS$ . It is an important question to ask whether finite type invariants of  $\mathbb{Z}HS$  separate points in  $\mathcal{M}$ .

**1.2. Statement of results; plan of the proof.** This paper consists of two parts. In the first part, which consists of Section 2 we classify pure braids and string links modulo the relation of surgical equivalence. In Section 2.1 we recall some definitions of string links and pure braids. In Section 2.2 we study a subgroup  $A(F^4(n))$  of the automorphism group of the nilpotent quotient  $F^4(n)$  of a free group and show in Theorem 1 how to express  $A(F^4(n))$  as a semidirect product of  $A(F^4(n-1))$  together with a nilpotent quotient of a free group. In Section 2.3 we prove that the group of surgical equivalence classes of pure braids is isomorphic to the corresponding group of string links (Theorem descriptions of the above mentioned group  $P^{\text{SE}}(n)$  of surgical equivalence classes of  $n$  strand pure braids: one as a semidirect product of  $P^{\text{SE}}(n-1)$  together with an explicit quotient of the free group and another description (Theorem 3) as a group of automorphisms of a nilpotent quotient of a free group.

In the second part which consists of Section 3 we apply our results on surgery equivalence to study the finite type invariants of  $\mathbb{Z}HS$ , originally introduced by Ohtsuki [Oh] and partially answer questions 1 and 2 from [Ga]. In Section 3.2 we reprove Ohtsuki's fundamental result which states that the space of type  $m$  invariants of  $\mathbb{Z}HS$  is finite dimensional for every  $m$ . Our proof allows us to show a vanishing theorem (Corollary 3.8), namely that the graded space of degree  $m$  invariants of  $\mathbb{Z}HS$  is zero dimensional unless  $m$  is divisible by 3. This partially answers question 1 of [Ga]. In Section 3.4 we study the map from knots (in  $S^3$ ) to  $\mathbb{Z}HS$  defined by mapping a knot  $K$  in  $S^3$  to the  $\mathbb{Z}HS$   $S_{K,+1}^3$  obtained by  $+1$  surgery on  $K$  (see [Ro], and Section 3.1). We show (proposition 3.12) that type  $5m+1$  invariants of  $\mathbb{Z}HS$  map to type  $4m$  invariants of knots, thus making progress towards answering question 2 of [Ga]. We conclude in Section 4 with a philosophical comment about the appearance of trivalent graphs in this paper. It turns out that both the notion of surgical equivalence of links in  $S^3$  and the notion of finite type invariants of integral homology 3-spheres are ultimately related to invariants of vertex oriented trivalent graphs.

After the present work was completed we received a copy of [GrLi] in which they prove a very special case of Corollary 3.8 as well as a weak form of proposition 3.12.

**1.3. Acknowledgment.** We wish to thank D. Bar-Natan for for many useful conversations. Especially we wish to thank T. Ohtsuki for enlightening electronic e-mail conversations and helpful comments, and the **Internet** for providing the support for the relevant communications.

## 2. SURGICAL EQUIVALENCE OF STRING LINKS AND PURE BRAIDS

In this section we give two different classification theorems of the group of surgical equivalence classes of string links and pure braids. As in the case of string link homotopy (see [HL]) these groups are isomorphic. These classification theorems are entirely analogous to the classical theorems of Artin [Ar] on isotopy classification of braids and the more recent results of Habegger-Lin [HL] on homotopy classification

of string links. The first theorem is a recursive description of the  $n$  strand pure braid group in terms of the  $n - 1$  strand group and a certain nilpotent quotient of a free group. The second theorem gives an isomorphism with a certain group of automorphisms of another nilpotent quotient of a free group.

The proof of the first theorem is similar to that of [HL] while the second is rather different (*warning*: Lemma 1.9 of [HL] is not correct as stated and the proof of Theorem 1.7 of that paper seems to require some modification).

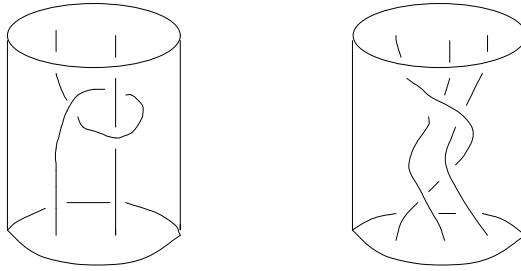
**2.1. Preliminaries on surgical equivalence.** We begin by recalling some definitions.

**Definition 2.1.** (1) Let  $n$  be a positive integer,  $I$  be the unit interval  $[0, 1]$  and  $D^2$  be the unit disc in the plane. An  $n$ -component string link is a disjoint union of  $n$  (smooth) arcs  $\sigma_1, \dots, \sigma_n$  in the solid cylinder  $I \times D^2$  so that the boundary of the  $i$ -th arc is  $\partial I \times p_i$ , where  $\{p_i\}$  are  $n$  distinct points in  $D^2$ .

(2) An  $n$ -strand pure braid is an  $n$ -component string link with the extra property that the tangent vector at any point of any  $\sigma_i$  is never horizontal. See figure 1.

(3) A surgery on a string link  $\sigma$  produces another string link  $\sigma'$  as follows. Let  $\gamma$  be an unknotted closed curve in  $I \times D^2 - \sigma$  whose linking number with each component of  $\sigma$  is zero. If we remove a tubular neighborhood of  $\gamma$  and sew it back in so that the new meridian is identified with a former longitude which links  $\gamma$  once, then  $I \times D^2$  is converted into a new manifold  $\Delta$  which is diffeomorphic to  $I \times D^2$  again. We define  $(I \times D^2, \sigma') = (\Delta, \sigma)$ . A more concrete description of  $\sigma'$  is given by removing a tubular neighborhood of a disk bounded by  $\gamma$  and then reinserting it with a single clockwise or counterclockwise twist. We say two string links or pure braids are *surgically equivalent* if one can be obtained from the other by a sequence of surgeries. It is clear that homotopic string links or braids (i.e., string links for which there is a homotopy  $h_t$  such that distinct components remain disjoint for all  $t \in [0, 1]$ . See also [Mi1], [Mi2], [HL] and figure 4.) are surgically equivalent. If  $\sigma, \sigma'$  are two string links, then the product string link  $\sigma\sigma'$  is obtained by stacking  $\sigma$  on top of  $\sigma'$ . If  $\bar{\sigma}$  is the reflection of  $\sigma$  about  $\frac{1}{2} \times D^2$ , then  $\sigma\bar{\sigma}$  and  $\bar{\sigma}\sigma$  are homotopic and, therefore, surgically equivalent to the trivial string link (see [HL]). Thus we obtain groups  $P^{\text{SE}}(n)$  (respectively,  $SL^{\text{SE}}(n)$ ) of surgical equivalence classes of  $n$ -strand pure braids (respectively,  $n$ -component string links). There is an obvious homomorphism  $P^{\text{SE}}(n) \rightarrow SL^{\text{SE}}(n)$ .

Let  $F(m)$  denote the free group on generators  $\{x_1, \dots, x_m\}$ . Recall the homomorphism  $\tau : F(n - 1) \rightarrow P^{\text{I}}(n)$  defined by inserting the  $n^{\text{th}}$  strand into a trivial  $(n - 1)$ -strand braid. Here  $P^{\text{I}}(n)$  stands for the group of isotopy classes of pure braids in  $n$  strands. For any group  $G$  let  $G_q$  denote the subgroup generated by all



**Figure 1.** A string link of 2 components (on the left) and a braid of 3 strands on the right.

commutators of order  $q$ , with the convention that  $G_1$  is equal to  $G$  and  $G_2 = [G, G]$ . Let  $G^q$  denote the quotient group  $G/G_q$ .

**Claim 2.2.** *For any  $w \in F(n-1)$  the surgical equivalence class of  $\tau(w)$  depends only on the image of  $w$  in  $F^3(n-1)$ .*

This follows directly from [Lel1].

Thus we have a homomorphism  $\tau : F^3(n-1) \rightarrow P^{\text{SE}}(n)$ . We also recall the obvious split epimorphisms  $P^{\text{SE}}(n) \rightarrow P^{\text{SE}}(n-1)$  and  $SL^{\text{SE}}(n) \rightarrow SL^{\text{SE}}(n-1)$  defined by deleting the  $n^{\text{th}}$  strand. Let  $\tau' : F^3(n-1) \rightarrow SL^{\text{SE}}(n)$  be the composition  $F^3(n-1) \xrightarrow{\tau} P^{\text{SE}}(n) \longrightarrow SL^{\text{SE}}(n)$ .

**2.2. The study of  $A(F^4(n))$ .** In this section we study a subgroup  $A(F^4(n))$  of the group of automorphisms of  $F^4(n)$  and prove Theorem 1, the key ingredient in determining the structure of the groups  $P^{\text{SE}}(n)$  and  $SL^{\text{SE}}(n)$  in the next section.

First consider all automorphisms  $\alpha$  of  $F^q(n)$  (for  $q \geq 2$ ) which send each  $x_i$  to a conjugate of itself. In fact, for any sequence of elements  $g_1, \dots, g_n$  of  $F(n)$ , there is an automorphism  $\alpha$  of  $F^q(n)$  such that  $\alpha(x_i) = g_i x_i g_i^{-1}$ . These equations only define endomorphisms of  $F(n)$ , in general, but always define automorphisms of  $F^q(n)$  for any  $q \geq 2$ . It is easy to see that  $\alpha$  depends only on the class of the  $\{g_i\}$  in  $F^{q-1}(n)$ . If we demand that  $g_i$  is  $i$ -reduced, i.e. the exponent sum of  $x_i$  in  $g_i$  is zero, then the  $g_i \in F^{q-1}(n)$  are determined by  $\alpha$ . We define  $A(F^q(n))$  to be the subgroup of all such  $\alpha$  satisfying the additional property:

$$(1) \quad \alpha(x_1 \dots x_n) = x_1 \dots x_n$$

The following lemma will be useful in our study of the group  $A(F^4(n))$ .

**Lemma 2.3.** (a) *If  $\lambda \in F_2^4(n) \cap \langle x_n \rangle$ , then there exist unique elements  $\lambda_i \in F^3(n) \cap \langle x_n \rangle$  such that*

$$(2) \quad [x_1, \lambda_1] \cdots [x_n, \lambda_n] = \lambda$$

where  $\langle x_n \rangle$  denotes the normal closure of  $x_n$  in  $F_2^4(n) = F_2(n)/F_4(n)$ .

(b) Suppose  $\lambda_1, \dots, \lambda_n \in F^3(n-1)$  satisfy

$$(3) \quad \lambda_1 x_1 \lambda_1^{-1} \cdots \lambda_{n-1} x_{n-1} \lambda_{n-1}^{-1} = x_1 \cdots x_{n-1} \quad \text{in } F^4(n-1)$$

Then there exists unique  $\tilde{\lambda}_1, \dots, \tilde{\lambda}_n \in F^3(n)$  such that  
 $\tilde{\lambda}_i \equiv \lambda_i \pmod{\langle x_n \rangle}$ ,  $\tilde{\lambda}_n$  is  $n$ -reduced and

$$(4) \quad \tilde{\lambda}_1 x_1 \tilde{\lambda}_1^{-1} \cdots \tilde{\lambda}_n x_n \tilde{\lambda}_n^{-1} = x_1 \cdots x_n \quad \text{in } F^4(n)$$

*Remark 2.4.* This lemma is *false* if we replace  $F^4(n)$  and  $F^3(n)$  by  $F^q(n)$  and  $F^{q-1}(n)$  if  $q > 4$ . Also, note that, in (b),  $\tilde{\lambda}_i$  is  $i$ -reduced if and only if  $\lambda_i$  is  $i$ -reduced.

*Proof.* To prove part (a) recall the notion of a Hall basis (see [MKS]). In particular, any element of  $F^4(n)$  can be uniquely written in the following form:

$$(5) \quad x_1^{e_1} \cdots x_n^{e_n} \prod_{i < j} [x_i, x_j]^{e_{ij}} \prod_{\substack{i < j \\ k \leq j}} [x_k, [x_i, x_j]]^{e_{ijk}}$$

Thus we may uniquely write:

$$(6) \quad \lambda = \prod_{i, n} [x_i, x_n]^{e_i} \prod_{\substack{i < n \\ j \leq n}} [x_j, [x_i, x_n]]^{e_{ij}}$$

Therefore, equation (2) has a solution as follows:

$$(7) \quad \lambda_j = x_n^{e_j} \prod_{i < n} [x_i, x_n]^{e_{ij}}$$

We have set  $e_n = 0$ . Since any allowable solution  $\lambda_j$  has such a representation, uniqueness follows also.

In order to show part (b), we observe that any acceptable solution  $\{\tilde{\lambda}_i\}$  of equation (4) can be written in the form  $\tilde{\lambda}_i = \lambda_i \alpha_i x_n^{e_i}$  for some  $\alpha_i \in F_2^3(n)$ , with  $e_n = 0$ . Substituting this into equation (4) with a little bit of commutator manipulation and using the fact that  $F_3^4(n)$  is the center of  $F^4(n)$ , we obtain the following formula in  $F^4(n)$ :

$$(8) \quad \left( \prod_{i=1}^{n-1} [\lambda_i, [x_n^{e_i}, x_i]] \right) \left( \prod_{i=1}^n [\alpha_i, x_i] \right) \prod_{i=1}^n ([x_n^{e_i}, x_i] \lambda_i x_i \lambda_i^{-1}) = \prod_{i=1}^n x_i$$

Reducing to  $F^3(n)$  we obtain the much simpler formula:

$$(9) \quad \left( \prod_{i=1}^{n-1} [x_n, x_i]^{e_i} \right) \prod_{i=1}^n \lambda_i x_i \lambda_i^{-1} = \prod_{i=1}^n x_i$$

Substituting from equation (3) this becomes:

$$(10) \quad \prod_{i=1}^{n-1} [x_n, x_i]^{e_i} = [x_n, \lambda_n]$$

and so  $\lambda_n$  determines the  $\{e_i\}$ . Now that the  $\{e_i\}$  are known, equation (8) can be written as  $\prod_{i=1}^n [\alpha_i, x_i] = \tau$ , where  $\tau$  is a specified element of  $F^4(n) \cap \langle x_n \rangle$ . It follows from (a) that there is a unique solution for  $\alpha_i$  in  $F^3(n) \cap \langle x_n \rangle$ . The uniqueness implies that  $\alpha_i \in F_2^3(n)$  since, as can be easily checked,  $\tau \in F_3^4(n)$ .  $\square$

We can now use part (b) of Lemma 2.3 to define a homomorphism  $\bar{\tau} : F^3(n-1) \rightarrow A(F^4(n))$  as follows.  $\bar{\tau}(w)$  is the unique element  $\alpha \in A(F^4(n))$  satisfying  $\alpha(x_i) \equiv x_i \pmod{\langle x_n \rangle}$  for  $i < n$  and  $\alpha(x_n) = \tilde{w}x_n\tilde{w}^{-1}$  for some  $\tilde{w} \equiv w \pmod{\langle x_n \rangle}$ .

Combining this with the restriction  $A(F^4(n)) \rightarrow A(F^4(n-1))$  we have the following result

**Theorem 1.** *The following sequence is split exact:*

$$1 \longrightarrow F^3(n-1) \xrightarrow{\bar{\tau}} A(F^4(n)) \longrightarrow A(F^4(n-1)) \longrightarrow 1$$

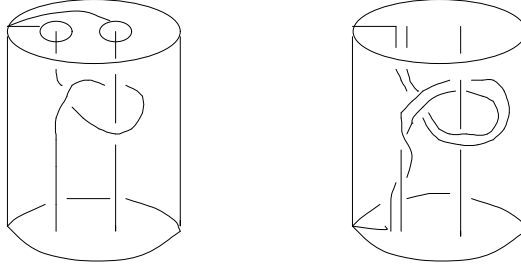
*Remark 2.5.* This theorem is *false* if we replace  $F^4(n), F^4(n-1)$  and  $F^3(n-1)$  by  $F^q(n), F^q(n-1)$  and  $F^{q-1}(n-1)$ , respectively, if  $q > 4$ . For example consider the automorphism  $\alpha$  of  $F^5(2)$  defined by:

$$(11) \quad \alpha(x_1) = [x_2, [x_1, x_2]]x_1[x_2, [x_1, x_2]]^{-1}$$

$$(12) \quad \alpha(x_2) = [x_1, [x_2, x_1]]x_2[x_1, [x_2, x_1]]^{-1}$$

*Proof of Theorem 1.* This is an immediate consequence of part (b) of Lemma 2.3.  $\square$

Our next goal is to define a homomorphism  $SL^{\mathbb{S}\mathbb{E}}(n) \rightarrow A(F^4(n))$ . First recall the definition of the  $\bar{\mu}$ -invariants of string links as formulated, for example, in [Le2]. Let  $\sigma$  be an  $n$ -string link; we define certain canonical *meridian* elements of  $\pi(\sigma) = \pi_1(I \times D^2 - \sigma)$ . Let  $m_i$  be a small circle in  $0 \times D^2$  with  $0 \times p_i$  as center. Let  $s_i$  be a straight line from a base point  $p$  in  $0 \times (D^2 - \{p_i\})$  to  $m_i$ . Now let  $\mu_i$  be the element of  $\pi(\sigma)$  represented by  $s_i m_i s_i^{-1}$ . Let  $\mu : F(n) \rightarrow \pi(\sigma)$  be defined by  $\mu(x_i) = \mu_i$ . It follows from a theorem of Stallings [Sta] that  $\mu$  induces an isomorphism  $F^q(n) \rightarrow \pi^q(\sigma)$  for each  $q \geq 3$ . We can also define canonical *longitude* elements of  $\pi(\sigma)$ . Let  $l_i$  be a curve on the boundary of a tubular neighborhood of  $\sigma_i$  which runs parallel from a point on  $m_i$  to the corresponding point on  $m'_i$ , where  $m'_i$  is the projection of  $m_i$  into  $1 \times D^2$ . Let  $s'_i$  be the projection of  $s_i$  into  $1 \times D^2$  and let  $u = I \times p$  oriented from 1 to 0. Define  $\tilde{\lambda}_i \in \pi$  to be represented by  $s'_i l_i (s'_i)^{-1} u$ . See figure 2. Requiring that  $\tilde{\lambda}_i$  have linking number 0 with  $\sigma_i$  determines  $l_i$  and, hence,  $\tilde{\lambda}_i$ . We now define  $\lambda_i(\sigma) \in F^q(\sigma)$  to be the element (which is  $i$ -reduced) corresponding to  $\tilde{\lambda}_i$  under the isomorphism induced by  $\mu$ . These elements are invariants of the concordance class of  $\sigma$ . For a reference on concordance class of string links see [Li] and references therein.



**Figure 2.** On the left are shown the elements  $m_i, s_i$  (for  $i = 1, 2$ ) and on the right is shown the element  $\tilde{\lambda}_i$  of a string link of two components.

**Proposition 2.6.** *The surgical equivalence class of  $\sigma$  determines the set  $\{\lambda_i(\sigma)\} \subset F^3(n)$ .*

*Proof.* This is essentially proved in [Le1] but for completeness we give a more direct argument here. Let  $\sigma$  be an  $n$ -strand string link and  $\gamma$  a simple closed curve in the complement of  $\sigma$ . We assume that the linking numbers of any two components of  $\sigma$  and any component of  $\sigma$  with  $\gamma$  is zero. Let  $(I \times D^2, \sigma') = (\Delta, \sigma)$  denote the surgically equivalent link obtained by Dehn surgery along  $\gamma$  as in Definition 2.1. Let  $X = (I \times D^2) - \sigma - \gamma$  and let

$$\pi = \pi_1(X), \quad \pi(\sigma) = \pi_1((I \times D^2) - \sigma), \quad \pi(\sigma') = \pi_1(\Delta - \sigma) = \pi_1((I \times D^2) - \sigma')$$

If  $m$  is a meridian element and  $l$  a longitude element for  $\gamma$  in  $\pi$ , then we have:

$$\pi(\sigma) \cong \pi / \langle m \rangle \quad \text{and} \quad \pi(\sigma') \cong \pi / \langle ml^{\pm 1} \rangle$$

We define homomorphisms:

$$F(n+1) \xrightarrow{\hat{\mu}_1} \pi, \quad F(n+1) \xrightarrow{\hat{\mu}_2} \pi, \quad F(n) \xrightarrow{\mu_1} \pi(\sigma), \quad F(n) \xrightarrow{\mu_2} \pi(\sigma')$$

all of which send  $x_i$  to a meridian of  $\sigma, \sigma'$ , for  $i \leq n$ , and, in addition,  $\hat{\mu}_1(x_{n+1}) = m$ ,  $\hat{\mu}_2(x_{n+1}) = ml^{\pm 1}$ . We have the following commutative diagram:

$$\begin{array}{ccccc} \pi & \xleftarrow{\hat{\mu}_2} & F(n+1) & \xrightarrow{\hat{\mu}_1} & \pi \\ \downarrow & & \downarrow & & \downarrow \\ \pi(\sigma') & \xleftarrow{\mu_2} & F(n) & \xrightarrow{\mu_1} & \pi(\sigma) \end{array}$$

Here  $F(n+1) \rightarrow F(n)$  is the obvious reduction defined by sending  $x_{n+1}$  to 1. Using results of Milnor [Mi1], [Mi2] and Stallings [Sta] we see that  $\mu_1$  and  $\mu_2$  induce isomorphisms  $\mu_1^3 : F^3(n) \xrightarrow{\cong} \pi^3(\sigma)$  and  $\mu_2^3 : F^3(n) \xrightarrow{\cong} \pi^3(\sigma')$ , respectively, while  $\hat{\mu}_1$  and  $\hat{\mu}_2$  are epimorphisms. Furthermore, there are elements  $\lambda$  and  $\lambda'$  in  $F^3(n+1)$  such that the kernels of  $\hat{\mu}_1$  and  $\hat{\mu}_2$  (in  $F^3(n+1)$ ) are normally generated, respectively, by  $[x_{n+1}, \lambda]$  and  $[x_{n+1}, \lambda']$ . In particular, since  $\lambda, \lambda' \in F_2(n+1)$ ,  $\mu_1$  and  $\mu_2$  induce



isomorphisms  $F^3(n) \xrightarrow{\cong} \pi^3$ . If  $\lambda_i, \lambda'_i \in F^3(n)$  are chosen to map to the longitude of  $\sigma_i$  and  $\sigma'_i$  in  $\pi^3(\sigma)$  and  $\pi^3(\sigma')$ , respectively, then they reduce in  $F^3(n)$  to  $\lambda_i(\sigma)$  and  $\lambda_i(\sigma')$ .

Since  $\lambda_i, \lambda'_i \in F^3(n)$ , we only need prove that  $\hat{\mu}_1$  and  $\hat{\mu}_2$  agree on  $F^3(n+1)$ . But this will follow if  $\hat{\mu}_2 \equiv \hat{\mu}_1 \pmod{\pi_2^3}$ , which is clear since  $l \in \pi_2$ .  $\square$

Now, we can introduce the following definition:

**Definition 2.7.** Let  $\Phi_n : SL^{\text{SE}}(n) \rightarrow A(F^4(n))$  be the homomorphism defined by

$$(13) \quad \Phi_n(\sigma)(x_i) = \lambda_i(\sigma)x_i\lambda_i(\sigma)^{-1}$$

To see that this is a homomorphism we give another interpretation of  $\Phi_n$ . Let  $X = I \times \partial D^2 \cup 1 \times (D^2 - \{p_i\}) \subseteq I \times D^2$ . Then we have the following homomorphisms induced by inclusion maps:

$$\pi_1(0 \times (D^2 - \{p_i\})) \xrightarrow{\mu} \pi_1((I \times D^2) - \sigma) \xleftarrow{\mu'} \pi_1(X)$$

We identify the first and last groups with  $F(n)$  equating  $x_i$  with the homotopy class of  $s_i m_i s_i^{-1}$  and  $u^{-1} s'_i m'_i (s'_i)^{-1} u$ , respectively. Now Stallings Theorem tells us that the maps in equation (2.7) become isomorphisms when we pass to any lower central series quotient. Thus the composition of the maps in (2.7) define an automorphism of  $F^3(n)$ ; we leave it as an exercise for the reader to show that this is  $\Phi_n(\sigma)$ . From this formulation, it is clear that  $\Phi_n$  is a homomorphism.

**2.3. Determining the groups  $P^{\text{SE}}(n)$  and  $SL^{\text{SE}}(n)$ .** In this section we determine the groups  $P^{\text{SE}}(n)$  and  $SL^{\text{SE}}(n)$ .

**Theorem 2.** (a)  $P^{\text{SE}}(n) \rightarrow SL^{\text{SE}}(n)$  is an isomorphism.

(b) The diagram:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & F^3(n-1) & \xrightarrow{\tau} & P^{\text{SE}}(n) & \longrightarrow & P^{\text{SE}}(n-1) & \longrightarrow & 1 \\ & & \parallel & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & F^3(n-1) & \xrightarrow{\tau'} & SL^{\text{SE}}(n) & \longrightarrow & SL^{\text{SE}}(n-1) & \longrightarrow & 1 \end{array}$$

is commutative and the rows are split short exact.

The exactness and commutativity in (b) is clear except for the injectivity of  $\tau$  and  $\tau'$ . This will be proved in Theorem 3 below. Clearly (a) will follow from (b).

**Theorem 3.**  $\Phi_n$  is an isomorphism.

*Proof of Theorems 2 and 3.* We can combine the statements of Theorems 1 2 and 3 into a single commutative diagram:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & F^3(n-1) & \xrightarrow{\tau} & P^{\mathbb{S}\mathbb{E}}(n) & \longrightarrow & P^{\mathbb{S}\mathbb{E}}(n-1) & \longrightarrow & 1 \\
& & \parallel & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & F^3(n-1) & \xrightarrow{\tau'} & SL^{\mathbb{S}\mathbb{E}}(n) & \longrightarrow & SL^{\mathbb{S}\mathbb{E}}(n-1) & \longrightarrow & 1 \\
& & \parallel & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & F^3(n-1) & \xrightarrow{\bar{\tau}} & A(F^4(n)) & \longrightarrow & A(F^4(n)) & \longrightarrow & 1
\end{array}$$

The commutativity of this diagram is clear. To establish that the first two rows are exact we need only confirm the injectivity of  $\tau$  and  $\tau'$ . But this now follows from the injectivity of  $\bar{\tau}$ . This concludes the proofs of Theorems 2 and 3.  $\square$

We can now deduce the following corollary to Theorem 3:

**Corollary 2.8.**  *$P^{\mathbb{S}\mathbb{E}}(n)$  is a nilpotent group of class two, i.e., with the notation of Section 2.1 we have that  $P_3^{\mathbb{S}\mathbb{E}}(n) = 1$ . Furthermore,  $P^{\mathbb{S}\mathbb{E}}(n)/P_2^{\mathbb{S}\mathbb{E}}(n)$  is a free abelian group in  $\binom{n}{2}$  generators and  $P_2^{\mathbb{S}\mathbb{E}}(n)$  is a free abelian group in  $\binom{n}{3}$  generators. In fact,  $P_2^{\mathbb{S}\mathbb{E}}(n)$  is a free abelian group on generators  $\alpha_I$  for  $I \in I_n := \{(i, j, k) | 1 \leq i, j, k \leq n\}$  (where all  $i, j, k$  are distinct) with identifications  $\alpha_I = \text{sgn}(\sigma)\alpha_{\sigma(I)}$ , where  $\sigma$  is any permutation of  $I$ . The element  $\alpha_I$  can be represented, if  $i < j < k$ , by the braid  $B_I$  in figure 3.*

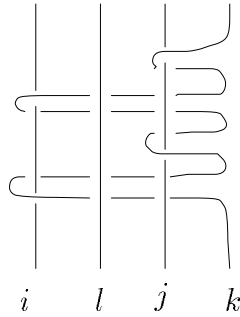
*Remark 2.9.* The relation of surgical equivalence on string links and pure braids is generated by the local moves of figures 4 and 5. This is proved in [Le1] for closed links but the proof is the same for string links.

### 3. RELATIONS WITH FINITE TYPE INVARIANTS OF $\mathbb{Z}HS$

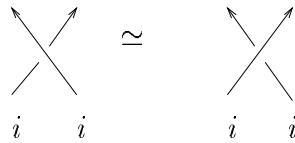
**3.1. Preliminaries on finite type 3-manifold invariants.** As a motivation to the notion of finite type 3-manifold invariants, let us recall the definition of type  $m$  Vassiliev invariants of (oriented) knots in (oriented)  $S^3$  (after [B-N1], [BL], [Va]):  $V$  is *type  $m$*  invariant of knots if for every knot  $K$  in  $S^3$  and every choice  $B_1, \dots, B_{m+1}$  of embedded 3-balls that intersect the knot as in figure 6 (the balls appear in the form of solid cylinders) we have that

$$(14) \quad \sum_{\epsilon_i \in \{0,1\}} \prod_{i=1}^{m+1} (-1)^{\epsilon_i} V(K_{\epsilon_1, \dots, \epsilon_{m+1}}) = 0$$

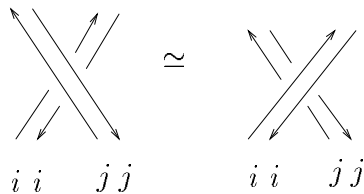
where  $K_{\epsilon_1, \dots, \epsilon_{m+1}}$  is the knot obtained by removing  $B_1, \dots, B_{m+1}$  from  $S^3$ , twisting every  $B_i$   $\epsilon_i$  times as in figure 6 and gluing back.



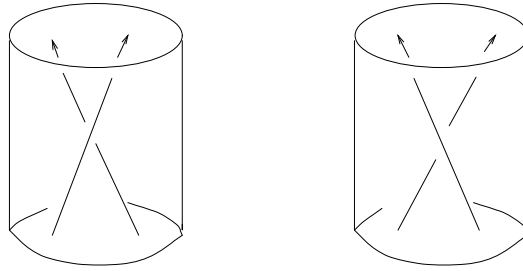
**Figure 3.** A 4 strand braid  $B_{i,j,k} = B_1 \cup B_2$  representing  $\alpha_{i,j,k}$ . Here  $B_1$  consists of the 3 strands  $i, j, k$  and  $B_2$  consists of all the rest strands (here only one). Note that the strands of  $B_2$  are on top of the  $i, j, k$  strands, and that the closure of the 3 strand braid  $B_2$  is a Borromean link of 3 components and that the link represented by the closure of the  $i, j$  strands is trivial. Furthermore, the longitude  $l_k$  of the  $k^{th}$  strand represents the element  $[\mu_i, \mu_j]$  in the fundamental group of the complement of the link consisting of the  $i, j$  strands, where  $\{\mu_j\}$  are the canonical free generators of the fundamental group of the complement of the  $i, j$  strands.



**Figure 4.** A local move that generates the equivalence relation of string link homotopy. Here arcs labeled by the same letter ( $i$  in this figure and  $i, j$  in the next figure) belong to the same link component.



**Figure 5.** A local move that is implied by the relation of surgical equivalence of string links.



**Figure 6.** An embedded 3-ball in  $S^3$  intersecting a knot as in the left part of the figure. On the right is the result of reinserting the ball with a full counterclockwise twist about the vertical axis.

Let  $\mathcal{F}_m \mathcal{V}$  be the space of type  $m$  knot invariants, and let  $\mathcal{F}_* \mathcal{V} := \cup_{m \geq 0} \mathcal{F}_m \mathcal{V}$  be the space of finite type knot invariants. It is easy to show that  $\mathcal{F}_* \mathcal{V}$  is a filtered commutative algebra (with pointwise multiplication). Let  $\mathcal{G}_* \mathcal{V}$  denote the associated graded algebra (and more generally, let  $\mathcal{G}_*(obj)$  denote the associated graded object of the filtered object  $\mathcal{F}_*(obj)$ ) and let  $\mathcal{W}_m$  denote the space of weight systems of degree  $m$  (i.e. linear functionals on the space of chord diagrams with  $m$  chords modulo  $4T$  relations and framing independence relation, see references above). Then it is easy to show that there is a map  $\mathcal{F}_m \mathcal{V} \rightarrow \mathcal{W}_m$  with kernel  $\mathcal{F}_{m-1} \mathcal{V}$ . Since  $\mathcal{W}_m$  is *a priori* finite dimensional, so is  $\mathcal{F}_m \mathcal{V}$ .

Before we talk about 3-manifold invariants, let us establish some useful notation: Let  $\mathcal{M}$  denote the vector space (over  $\mathbb{Q}$ ) on the set of *oriented integral homology 3-spheres* ( $\mathbb{Z}HS$  for short). A link  $L \subseteq M$  in a  $\mathbb{Z}HS$  is called *algebraically split* if the linking numbers between any two components vanish. A framing  $\mathbf{f} = (f_1, \dots, f_n)$  for an  $n$  component link is a sequence of integers associated to each component.

*Remark 3.1.* A framing for a link  $L$  corresponds to a choice of *longitudes* for each component. If  $N_i$  is a tubular neighborhood of the component  $L_i$ , then the associated longitude  $\gamma_i$  on  $\partial N_i$  is required to be homologous to  $L_i$  in  $N_i$  and to have linking number  $f_i$  with  $L_i$ . To make sense of this we must impose an orientation on  $L_i$ , but then it is easy to see that  $\gamma_i$  is independent of this choice. Note that linking numbers make sense in any  $\mathbb{Z}HS$ .

A framed link  $(L, \mathbf{f})$  in a  $\mathbb{Z}HS$   $M$  is called *unimodular* if  $f_i = \pm 1$  for all  $i$ . A framed link  $(L, \mathbf{f})$  is called *AS-admissible* if it is algebraically split and unimodular. For every framed link  $(L, \mathbf{f})$  in  $M$  we denote by  $M_{L, \mathbf{f}}$  the result of doing *Dehn surgery* on  $(L, \mathbf{f})$  in  $M$  [Ro], i.e. remove a tubular neighborhood  $N_i$  of each  $L_i$  and sew it back in so that  $\gamma_i$  now bounds a disk in  $N_i$ . For a framed link  $(L, \mathbf{f})$  in  $M$  we denote

$$(15) \quad [M, L, \mathbf{f}] := \sum_{L' \subseteq L} (-1)^{|L'|} M_{L', \mathbf{f}|_{L'}} \in \mathcal{M}$$

Recall that  $\mathcal{M}$  is the  $\mathbb{Q}$  vector space on the set of  $\mathbb{Z}HS$ , and that if  $(L, \mathbf{f})$  is a framed

link in a  $\mathbb{Z}HS$   $M$ , then  $(L, f)$  is  $AS$ -admissible if and only if  $M_{L', f|_{L'}}$  is a  $\mathbb{Z}HS$  for every sublink  $L'$  of  $L$ .

In case  $L$  is an  $AS$ -admissible link in  $S^3$  with framing  $+1$  on each component, we will denote

$$(16) \quad [L] = [S^3, L, \{+1, \dots, +1\}]$$

Let us define a decreasing filtration  $\mathcal{F}_*^{Oh}$  on the vector space  $\mathcal{M}$  as follows:  $\mathcal{F}_m^{Oh} \mathcal{M}$  is the subspace spanned by  $[M, L, \mathbf{f}]$  for all  $AS$ -admissible  $m$  component links in any  $\mathbb{Z}HS$   $M$ .

It is an immediate consequence of equation (15) that  $\mathcal{M}$  is the vector space generated by all triples  $[M, L, f]$  subject to the *fundamental relation*:

$$(17) \quad [M, L, f] = [M, L', f|_{L'}] - [M_{(l, f|_l)}, L', f|_{L'}]$$

where  $L'$  is any sublink of  $L$  obtained by removing a component  $l$ . If  $l$  bounds a disk  $D$  in  $M$ , then  $M_{(l, f|_l)} \cong M$  and we may construct the link in  $M$  corresponding to  $L$  in  $M_{(l, f|_l)}$  from  $L$  by just giving the bundle of strands of  $L$  which pass through  $D$  a full clockwise twist if  $f = +1$  or counterclockwise twist if  $f = -1$ .

We can now recall the following definition from [Oh]:

**Definition 3.2.**  $\lambda$  is a *type  $m$*  invariant of  $\mathbb{Z}HS$  (with values in  $\mathbb{Q}$ ) if  $\lambda(\mathcal{F}_{m+1}^{Oh} \mathcal{M}) = 0$  i.e., if for every  $AS$ -admissible link  $L$  of  $m+1$  components in a  $\mathbb{Z}HS$   $M$  we have that

$$(18) \quad \sum_{L' \subseteq L} (-1)^{|L'|} \lambda(M_{L', f|_{L'}}) = 0$$

Let  $\mathcal{F}_m \mathcal{O}$  denote the vector space of type  $m$  invariants of  $\mathbb{Z}HS$ , and let  $\mathcal{F}_* \mathcal{O}$  denote the union  $\cup_{m \geq 0} \mathcal{F}_m \mathcal{O}$ . It is easy to see that  $\mathcal{F}_* \mathcal{O}$  is a filtered commutative algebra with pointwise multiplication.

**3.2. Surgical equivalence and 3-manifold invariants.** In this section we link the results from Section 2 with the notion of finite type invariants of integral homology 3-spheres. In particular, we reprove Ohtsuki's fundamental result (Theorem 5) which states that the space of type  $m$  invariants of  $\mathbb{Z}HS$  is finite dimensional for every  $m$ .

We begin by observing that every  $\sigma \in P^{\mathbb{I}}(n)$  can be closed to a link  $\hat{\sigma}$  of  $n$  components in  $S^3$ . Furthermore, with the notation of Corollary 2.8 we have that  $\sigma \in P_2^{\mathbb{I}}(n)$  if and only if  $\hat{\sigma}$  is an algebraically split link. Let us now consider the map  $P_2^{\mathbb{I}}(n) \rightarrow \mathcal{M}$  defined as  $\sigma \rightarrow [\hat{\sigma}]$  with the notation of equation (16). We claim that this map descends to a well defined map ( *not* a group homomorphism)

$$(19) \quad P_2^{\mathbb{S}\mathbb{E}}(n) \longrightarrow \mathcal{G}_n^{Oh} \mathcal{M}$$

Indeed, it follows from the definition of  $\mathbb{S}\mathbb{E}$  equivalence of pure braids. This extends to a linear map from the rational group ring  $\mathbb{Q}[P_2^{\mathbb{S}\mathbb{E}}(n)] \rightarrow \mathcal{G}_n^{Oh} \mathcal{M}$ . By Corollary 2.8 we

can regard  $P_2^{\text{SE}}(n)$  as a free abelian group on generators  $\alpha_I$  for  $I \in I_n := \{(i, j, k) | 1 \leq i, j, k \leq n\}$  (where all  $i, j, k$  are distinct) with identifications  $\alpha_I = \text{sgn}(\sigma)\alpha_{\sigma(I)}$ , where  $\sigma$  is any permutation of  $I$ . Therefore,  $\mathbb{Q}[P_2^{\text{SE}}(n)]$  is the Laurent polynomial ring in the commuting variables  $\alpha_I$ .

*Remark 3.3.* Using the fundamental relation (17) and the notation of equation (16), we see that if an  $n$ -component  $AS$ -admissible link  $L'$  is obtained by twisting along an admissible unknot  $\gamma$  an  $AS$ -admissible  $n$  component link  $L$ , then we have that

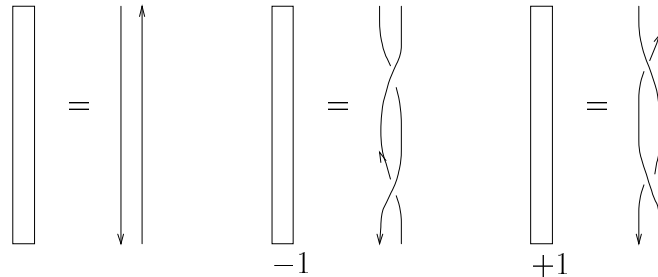
$$(20) \quad [L \cup \gamma] = [L] - [L']$$

Therefore, surgical equivalence relates  $n$  component  $AS$ -admissible links in  $S^3$  modulo  $n+1$  component  $AS$ -admissible ones. Since  $P_2^{\text{SE}}(n)$  is not a finite group (instead, it is a finitely generated free abelian group) at this point it is not clear why  $\mathcal{G}_n^{\text{Oh}}\mathcal{M}$  is a finite dimensional  $\mathbb{Q}$  vector space.

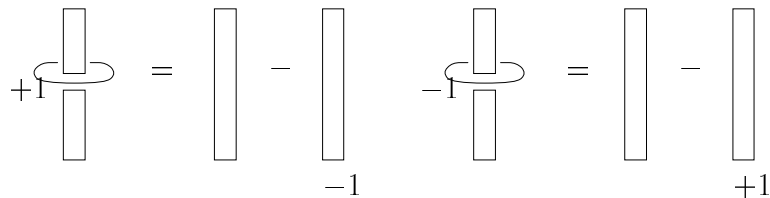
Our next task is to introduce a finite dimensional quotient of  $\mathbb{Q}[P_2^{\text{SE}}(n)]$  that maps onto  $\mathcal{G}_n^{\text{Oh}}\mathcal{M}$ . This will involve looking at  $n$  and  $n-1$  component  $AS$ -admissible links in  $S^3$ .

We begin by translating equation (17) into a *graphical form*. In figure 7 we give the drawing conventions for pieces of  $AS$ -admissible links in  $S^3$ .

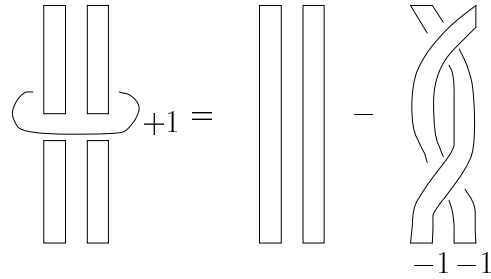
It is easy to see that figures 8, 9 and 10 are special cases of the fundamental equation (17).



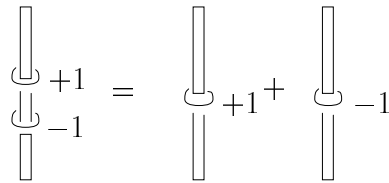
**Figure 7.** Some drawing conventions for bands. Shown here are ribbon parts of  $AS$ -admissible links that represent (linear combinations of)  $\mathbb{Z}HS$  under the map (15). The numbers in the bottom of each band indicate the number of twists that we put in the band.



**Figure 8.** A special case of equation (17) in a pictorial way.



**Figure 9.** Another special special case of equation (17) in a pictorial way.



**Figure 10.** Yet another special case of equation (17) in a pictorial way.

Let us now prepare some notation that will be used in Proposition 3.4 below. Let  $L$  be a  $(m - 1)$ -component link in  $S^3$  and  $\alpha$  be an element of  $\pi = \pi_1(S^3 - L)$ . Then we can consider the  $m$ -component link  $L(\alpha)$  defined by adding to  $L$  a new component which represents  $\alpha$ . Note that  $L(\alpha)$  is well-defined up to homotopy and, therefore, up to surgical equivalence. Note also that  $L(\alpha)$  is algebraically split if and only if  $L$  is algebraically split and  $\alpha \in \pi_2$ , the commutator subgroup of  $\pi$ .

**Proposition 3.4.** *Suppose that  $L$  is an algebraically split  $(k - 1)$ - component link and  $\alpha_1, \dots, \alpha_n \in \pi_2$  and  $n \geq 3$ . Then, with the notation of equation (16), we have the following identities in  $\mathcal{G}_k^{Oh} \mathcal{M}$ :*

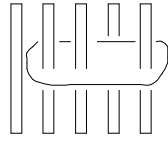
$$(21) \quad [L(\alpha_1 \dots \alpha_n)] = \sum_{1 \leq i < j \leq n} [L(\alpha_i \alpha_j)] - (n - 2) \sum_{1 \leq i \leq n} [L(\alpha_i)]$$

$$(22) \quad [L(\alpha^{-1})] = [L(\alpha)]$$

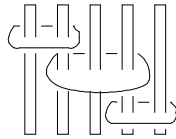
*Proof.* Equation (22) follows from the fact that surgery along a simple closed curve is independent of the orientation of the curve. Equation (21) will follow from a more general relation in  $\mathcal{M}$  (Theorem 4) stated below.  $\square$

Before we state Theorem 4, we need to fix some notation: Suppose  $M$  is  $\mathbb{Z}HS$  containing an  $AS$ -admissible link  $L$  with a  $+1$ -framing on each component. Let  $D$  be a 3-ball imbedded in  $M$  such that  $D \cap M$  consists of  $n$  parallel bundles of strands of  $L$ , such that, among the strands of any one component of  $L$  in any of the bundles, there are an equal number going in each of the two directions. For any sequence  $1 \leq i_1 \leq i_2 \leq \dots \leq i_k \leq n$ , let  $x_{i_1 \dots i_k}$  denote the circle in  $D$ , with a  $+1$ -framing, which encloses the  $i_1 \dots i_k$  bundles and passes *above* the other bundles (see Figure

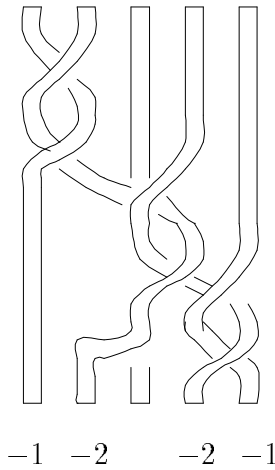
11) A *monomial*  $x_{i_1 \dots i_k} x_{j_1 \dots j_l} \cdots x_{r_1 \dots r_s}$  will denote the union of the circles given by the terms of the monomial placed in *descending* order as read from left to right (see Figure 12). Note that this multiplication is not commutative in general but that  $x_i$  commutes with any monomial. For any monomial  $m$ ,  $L(m)$  will denote the link obtained by adjoining to  $L$  the circles denoted by  $m$ .  $L(m)$  will denote the link  $L \subseteq M_m$  but since  $M_m \cong M$  this can be alternatively described as the link in  $M$  obtained by giving a full clockwise twist to the bundles which pass through each of the components of  $m$  (see Figure 13).



**Figure 11.** A graphical representation of the monomial  $x_{235}$ .



**Figure 12.** A graphical representation of the product  $x_{12} \cdot x_{234} \cdot x_{45}$ .



**Figure 13.** The bands have  $-1, -2, 0, -2, -1$  twists.

**Lemma 3.5.** *With the above notation we have the following identity in  $\mathcal{G}_k^{Oh} \mathcal{M}$ :*

$$L((x_1 \cdots x_n)^{n-2} x_{1..n}) = L\left(\prod_{1 \leq i < j \leq n} x_{ij}\right)$$

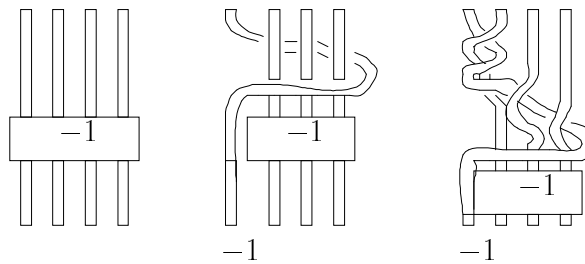


where the  $x_{ij}$  appear in lexicographic order from left to right, i.e.  $x_{ij}$  is to the left of  $x_{rs}$  if  $i < r$  or  $i = r$  and  $j < s$ .

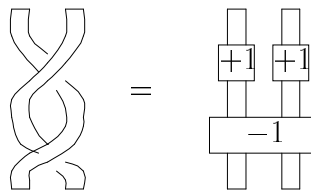
*Proof.* The proof is a refinement and generalization of Ohtsuki's idea of resolving a full twist of three bands into a sequence of twists of pairs and individual bands. We proceed by induction on  $n$ . The result is obvious for  $n = 2$ . The inductive step is the equation:

$$(23) \quad L(x_1^{n-2}x_2 \cdots x_n x_{1\dots n}) = L(x_{12} \cdots x_{1n} x_{2\dots n})$$

To prove this we observe that a full clockwise twist of  $L$  in  $D$  can be decomposed into a full twist of the first bundle followed by winding the first bundle around the other bundles once in a clockwise direction and finally by a full twist of the other bundles. This is illustrated in Figure 14. By using the relation given in Figure 15, we obtain a product of simple twists of two bundles which however force us to insert counterclockwise twists of each bundle to compensate. The result is pictured in Figure 16. Premultiplying  $x_{1\dots n}$  by  $x_1^{n-2}x_2 \cdots x_n$  eliminates the counterclockwise twists of the individual bundles and results in the desired element  $L(x_{12} \cdots x_{1n} x_{2\dots n})$ .  $\square$



**Figure 14.** A full clockwise twist on 4 bands.

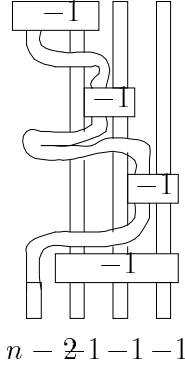


**Figure 15.** A full clockwise twist on 2 bands.

We want to use Lemma 3.5 to "solve" for  $[M; L, x_{1\dots n}]$ . We first need another lemma.

**Lemma 3.6.** *Let  $L' \cup L$  be an AS link in  $M$ . Then:*

$$(24) \quad [M_L, L'] = \sum_{L'' \subseteq L} (-1)^{|L''|} [M, L'' \cup L']$$



**Figure 16.** More twists. Note that the number of individual twists in the bands are  $n - 2$ ,  $-1$ ,  $-1$ ,  $-1$ .

*Proof.* This is essentially an inversion of equation (15). According to (15)

$$[M_L, L'] = \sum_{K \subseteq L'} (-1)^{|K|} M_{K \cup L}$$

while the right hand side of equation (24) is:

$$\sum_{\substack{K' \subseteq L'' \subseteq L \\ \bar{K} \subseteq L'}} (-1)^{|L''| + |K \cup K'|} M_{K \cup K'}$$

Thus it suffices to show that, for any *non-empty* link  $l \subseteq M$ ,  $\sum_{l' \subseteq l} (-1)^{|l'|} = 0$ . But, if  $|l| = p$ , then the left-hand side of this equation is just  $\sum_{i=0}^p \binom{p}{i} (-1)^{p-i}$ , which is zero by the binomial theorem.  $\square$

We now use the following notation. If  $m$  is any monomial, then let  $\bar{m} = [M; L, m]$ . Now it follows from Lemma 3.6 that:

$$(25) \quad [M, L(m)] = \sum_{m' \leq m} (-1)^{d(m')} \bar{m}'$$

where  $m'$  ranges over all submonomials of  $m$ , i.e. those obtained by deleting zero or more terms from  $m$ , and  $d(m')$  is the number of terms in  $m'$ . If  $m = y_1 \cdots y_k$ , then the right side of equation (25) is just  $(1 - \bar{y}_1) \cdots (1 - \bar{y}_k)$ . So when  $m = (x_1 \cdots x_n)^{n-2} x_{1 \dots n}$  the right side of (25) is:

$$\prod_{i=1}^n (1 - \bar{x}_i)^{n-2} (1 - \bar{x}_{1 \dots n})$$

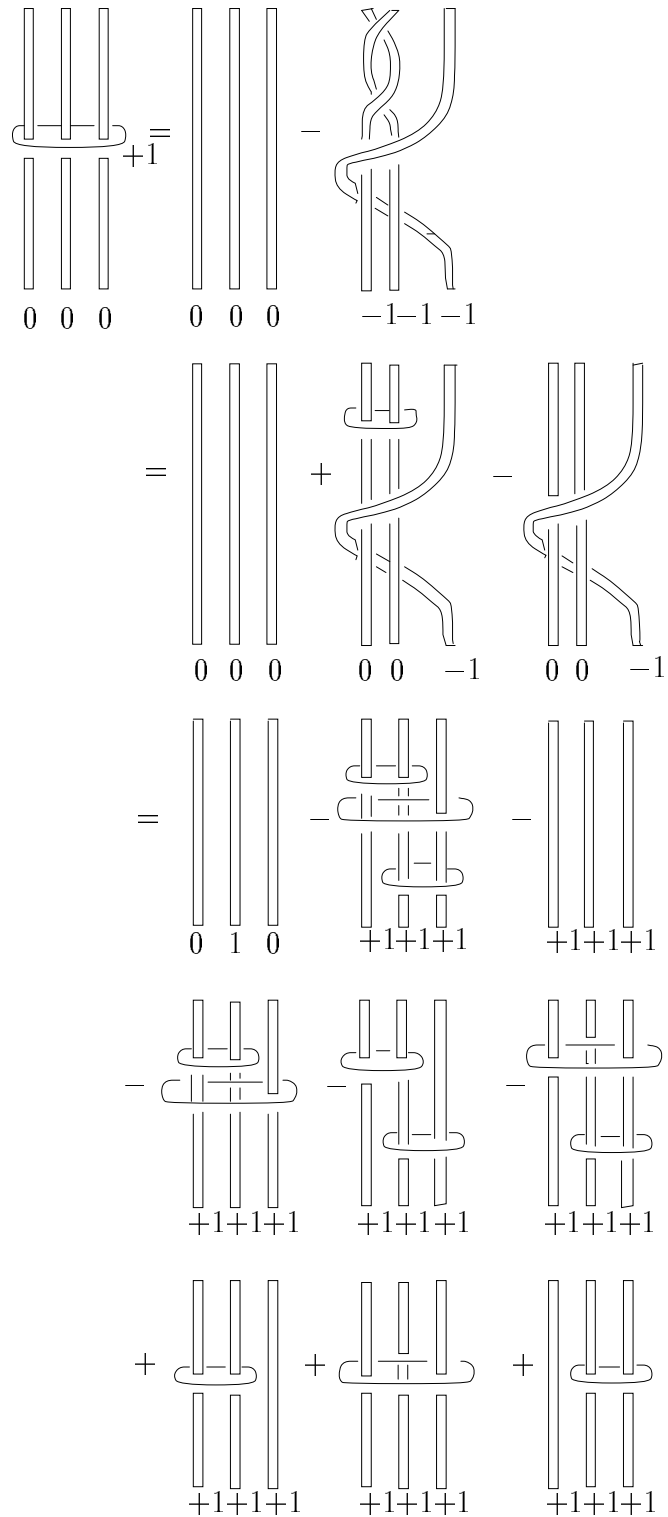
and when  $m = \prod_{1 \leq i < j \leq n} x_{ij}$  the right side is  $\prod_{1 \leq i < j \leq n} (1 - \bar{x}_{ij})$ . Combining these observations with Lemma 3.5 proves:

**Theorem 4.**

$$(26) \quad 1 - \frac{1}{x_{1\dots n}} = \frac{\prod_{1 \leq i < j \leq n} (1 - \overline{x_{ij}})}{\prod_{i=1}^n (1 - \overline{x_i})^{n-2}}$$

To correctly interpret this theorem recall that  $x_i$  commutes with every monomial and then equation (26) should be viewed as an equation in the completion of  $\mathcal{M}$  or in  $\mathcal{M}/\mathcal{F}_k^{Oh} \mathcal{M}$  for any  $k$ . Note that the denominator is invertible.

The proof of Theorem 4, and therefore of Proposition 3.4, is now complete.



**Figure 17.** The proof of Theorem 4 if  $n = 3$ . The two equalities here follow from repeated applications of figures 7 and 9. The framings in all horizontal components is +1.

$$\left. \begin{array}{c} \left| \right| \\ 0 \end{array} - \begin{array}{c} \left| \right| \\ 1 \end{array} = \sum_{n=0}^{\infty} \left. \begin{array}{c} \text{Diagram} \\ \text{Diagram} \\ \text{Diagram} \end{array} \right\} n$$

**Figure 18.** The end of the proof of Theorem 4. The identity in the present figure follows from figure 10. The framings of the horizontal components represented by unknots is +1.

Let us now come back to the problem (stated in the beginning of the present section) of showing that  $\mathcal{G}_n^{Oh} \mathcal{M}$  is a finite dimensional  $\mathbb{Q}$  vector space. Motivated by Proposition 3.4 and Theorem 4 we define, for each  $n$  a quotient  $\mathbb{Q}[P_2^{\text{SE}}(n)]/\mathcal{R}_n$ . Using the notation of Corollary 2.8 we define a monomial  $a = \prod_{I \in I_n} \alpha_I^{a(I)} \in P_2^{\text{SE}}(n)$  to be *i-trivial* (for  $1 \leq i \leq n$ ) if  $a(I) \neq 0$  implies  $i \in I$  and to be *i-disjoint* if  $a(I) \neq 0$  implies  $i \notin I$ . (If we look at the canonical braid representative of an *i-trivial* monomial, it has the property that removing the *i*-th strand leaves a trivial braid. If we look at the canonical braid representative of an *i-disjoint* monomial, it has the property that the *i*-th strand separates from the remaining strands.) Now we define our set of relations to be all those obtained by taking linear combinations with rational coefficients of the following elementary relations:

$$(27) \quad a = 0 \quad \text{if } a \text{ is } i\text{-disjoint for some } i$$

$$(28) \quad d(a-1)(b-1)(c-1) = -d \quad \text{if } a, b, c \text{ are } i\text{-trivial and } d \text{ is } i\text{-disjoint for some } i$$

$$(29) \quad d(a - a^{-1}) = 0 \quad \text{if } a \text{ is } i\text{-trivial and } d \text{ is } i\text{-disjoint for some } i$$

We will denote by  $\mathcal{R}_n$  the subspace of  $\mathbb{Q}[P_2^{\text{SE}}(n)]$  generated by equations (27), (28) and (30).

*Remark 3.7.* A few comments are in order: recall that Proposition 3.4 and Theorem 4 are given in *graphical notation*, whereas our relations  $\mathcal{R}_n$  are given in *algebraic notation*. For a precise comparison of the relations  $\mathcal{R}_n$  and the statements of Proposition 3.4 and Theorem 4 see the proof of Theorem 5.

We record here some useful particular consequences of our algebraic relations:

$$\begin{aligned}
(30) \quad & 1 = 0 \\
(31) \quad & da^2 = d(4a - 2) \quad \text{if } a \text{ is } i\text{-trivial} \\
& \quad \quad \quad \text{and } d \text{ is } i\text{-disjoint for some } i \\
(32) \quad & d(ab^{-1} + ab) = d(2a + 2b - 1) \quad \text{if } a, b \text{ are } i\text{-trivial and} \\
& \quad \quad \quad d \text{ is } i\text{-disjoint for some } i
\end{aligned}$$

Indeed, setting  $a = 1$  in (27) gives (30). Setting  $b = a, c = a^{-1}$  in (28) and using (27) gives (31). Setting  $c = b^{-1}$  in (28) and using (27) gives (32).  $\square$

**Theorem 5.** [Oh] *The map  $\mathbb{Q}[P_2^{\text{SE}}(n)] \rightarrow \mathcal{G}_n^{Oh} \mathcal{M}$  is onto. Furthermore, it factors through a (necessarily onto) map  $\mathbb{Q}[P_2^{\text{SE}}(n)]/\mathcal{R}_n \rightarrow \mathcal{G}_n^{Oh} \mathcal{M}$ .*

*Proof of Theorem 5.* We first show ontoeness. Recall that  $\mathcal{G}_n^{Oh} \mathcal{M}$  is generated by triples  $[M, L, f]$  where  $L$  is an  $AS$ -admissible link of  $n$  components in a  $\mathbb{Z}HS$   $M$ . Every 3-manifold can be obtained from  $S^3$  by surgery on a framed link and, if the manifold is a  $\mathbb{Z}HS$ , this means that the matrix of linking numbers of the link is unimodular. By a sequence of handle slides we can diagonalize the linking matrix and so we can assume that  $M$  is obtained by surgery on an  $AS$ -admissible link in  $S^3$ . Using our fundamental relation (17) we can assume that  $M = S^3$ . Ontoeness now follows once we can show that we can assume that the framing  $f_i$  is equal to  $+1$ , for all  $1 \leq i \leq n$ . This follows by an argument due to Ohtsuki [Oh]. Let  $L$  be any  $AS$ -admissible  $n$  component link with two framings  $f$  and  $f'$  so that, if  $L'$  is the sublink obtained by deleting the  $n$ -th component, then  $f|L' = f'|L'$  and  $f_n = -f'_n$ . Let  $(\tilde{L}, \tilde{f})$  be obtained from  $(L, f)$  by replacing  $L_n$  by two parallel non-linking copies of  $L_n$ , and with  $\tilde{f}|L' = f|L', \tilde{f}_n = f_n$  and  $\tilde{f}_{n+1} = -f_n$ . Then we have  $[M, \tilde{L}, \tilde{f}] = [M, L, f] - [M^-, L, f]$  where  $M_\epsilon = M_{(L_n, \epsilon f_n)}$ . So it suffices to show that  $[M^-, L, f] = -[M, L, f']$ . But we have  $[M, L, f'] = [M, L', f'|L'] - [M^-, L', f'|L']$  and  $[M^-, L, f] = [M^-, L', f'|L'] - [M^{-+}, L', f'|L']$ . Since  $M^{-+} = M$  and  $f|L' = f'|L'$ , we are done. The same proof is represented in figure 10.

To prove that the map  $\mathbb{Q}[P_2^{\text{SE}}(n)] \rightarrow \mathcal{G}_n^{Oh} \mathcal{M}$  factors through a map  $\mathbb{Q}[P_2^{\text{SE}}(n)]/\mathcal{R}_n \rightarrow \mathcal{G}_n^{Oh} \mathcal{M}$  we recast the relations in  $\mathcal{R}_n$  in terms of links.

To prove equation (27), we map  $a$  to  $(S^3, L, f)$  where the  $i$ -th component  $l$  of  $L$  bounds a disk disjoint from the remaining components  $L'$ . Since surgery on  $l$  does not change  $L'$  this relation follows from the fundamental relation (17).

To prove equation (28), after closing up the braid  $abcd$  to a link  $L$ , the  $i$ -th component  $l$  represents the element  $\alpha\beta\gamma$  in the fundamental group of the complement of the remaining components  $L'$  and the surgical equivalence class of  $L$  only depends on this element. Furthermore we can assume that  $l$  is unknotted since this can be achieved by crossing changes which do not change the surgical equivalence class of  $L$ . In this way, with the notation preceding Proposition 3.4, we have that  $dabc \simeq_s L'(\alpha\beta\gamma)$ ,

$dab \simeq_s L'(\alpha\beta)$ ,  $da \simeq_s L'(\alpha)$ , where  $\simeq_s$  denotes surgically equivalent links. Therefore, (28) and (30) follow (and in fact are equivalent) by Proposition 3.4. This completes the proof of Theorem 5.  $\square$

**3.3. A vanishing theorem for finite type invariants of  $\mathbb{Z}HS$ .** In this section we prove a vanishing theorem (Theorem 6 and Corollary 3.8) for the graded vector space of finite type invariants of  $\mathbb{Z}HS$ . The proof will exploit the algebraic form of the relations  $\mathcal{R}_n$ . Thus, for the results in the present section we do not need to draw any links, bands or braids. With the notation of the introduction (see Sections 1.1, 1.2), recall that  $\mathcal{M}$  is the vector space (over  $\mathbb{Q}$ ) on the set of oriented  $\mathbb{Z}HS$ , and elements in  $\mathcal{F}_m\mathcal{O}$  are thought of as polynomial functions of degree  $m$  on  $\mathcal{M}$ .

Consider the following three properties of a graph  $G$ :

- (a) Every edge has two distinct vertices.
- (b) Every vertex is either trivalent and oriented (i.e., one of the two possible cyclic orders for the edges emanating from it has been chosen), or univalent.
- (c) Every component of  $G$  is either a  $Y$ -graph (i.e., has exactly one trivalent vertex, three univalent vertices and three edges) or has every vertex trivalent.
- (d)  $G$  contains no  $\Theta$  component, i.e. a trivalent graph with 3 edges and 2 vertices.

We will say  $G$  is a *UT graph* if it satisfies (a) and (b), and a *Chinese manifold character* if it also satisfies (c) and (d). The *degree* of  $G$  is the number of edges. It is easy to see that any Chinese manifold character has degree a multiple of 3.

We can now state the following theorem:<sup>1</sup>

**Theorem 6.** *For every  $n \geq 0$ , the vector space  $\mathbb{Q}[P_2^{\text{SE}}(n)]/\mathcal{R}_n$  has a basis in one-one correspondence with the set of Ohtsuki graphs of order  $n$ .*

*Proof of Theorem 6.* First, we define a correspondence between Ohtsuki graphs of order  $n$  and a set of elements that generates (over  $\mathbb{Q}$ ) the vector space  $\mathbb{Q}[P_2^{\text{SE}}(n)]/\mathcal{R}_n$ . For a monomial  $a = \prod_{I \in I_n} \alpha_I^{a(I)}$  and for  $1 \leq i \leq n$  we define the  *$i$ -order* of  $a$ ,  $o_i(a)$ , to be  $\sum_{i \in I} |a(I)|$  and the *total order* of  $a$  to be  $\sum_I |a(I)|$ . We show that  $\mathbb{Q}[P_2^{\text{SE}}(n)]/\mathcal{R}_n$  is generated by those  $a$  which satisfy:  $o_i(a) \leq 2$  for every  $i$ , and  $a(I) = \pm 1$  for every  $I$ — we call such  $a$  *admissible*. If  $o_i(a) > 2$  then we may write  $a = \alpha_{I_1}^{\epsilon_1} \alpha_{I_2}^{\epsilon_2} bc$ , for some  $i \in I_1 \cap I_2$ , where  $b$  is  $i$ -trivial and  $c$  is  $i$ -disjoint. By equation (28) we see that  $a$  can be written as a linear combination of monomials with strictly smaller  $i$ -order and no larger total order. So we can assume  $o_i(a) \leq 2$  for every  $i$ . Now suppose  $a(I) = \pm 2$  for some  $I$ . Then  $a = \alpha_I^{a(I)} b$  where  $b$  is  $i$ -disjoint and  $i \in I$ . We can apply equation (30) and (31) to express  $a$  as a linear combination of monomials with strictly smaller  $i$ -order and no larger total order.

We can, following Ohtsuki, associate to any admissible monomial  $a$  a graph  $G(a)$  as follows. For every  $i = 1, \dots, n$  we associate an edge  $e(i)$  with two distinct vertices.

<sup>1</sup>A more refined version of Theorem 6 will appear in [GO].

For every  $I$  with  $a(I) \neq 0$  we define a trivalent vertex  $v(I)$  incident to each  $e(i)$  with  $i \in I$ . We can also give each trivalent vertex  $v(I)$  an *orientation*, i.e. a cyclic ordering to its incident edges, by choosing the ordering given by those  $I$  for which  $a(I) = +1$ . It is easy to see that  $G(a)$  is a  $UT$  graph with no  $\Theta$  components. Furthermore,  $G(a)$  determines  $a$ . We can use this graphical interpretation to describe further reductions to the generating set of monomials. We will say an edge is interior if both its vertices are trivalent and exterior otherwise.

We make the following observations:

- (a) If  $G(a)$  has an edge with both vertices univalent, which will be true if the order of  $a$  is small enough, then  $a = 0$ ; this follows from (27).
- (b) Suppose  $e$  is any edge in  $G(a)$ . Define  $a'$  so that  $G(a) = G(a')$  except that the trivalent vertices of  $e$  have orientations reversed. Then (30) implies  $a = a'$ .
- (c) If  $e$  is an interior edge and we define  $a''$  by changing the orientation of only one of the vertices, then (32) says that  $a + a''$  is a linear combination of monomials with smaller total order.

Suppose now that  $G(a)$  has two edges, one interior and the other exterior, which share a common vertex. Then (b) and (c) imply that  $2a$  is a linear combination of monomials with smaller total order. But if  $G(a)$  contains no such edges it is easy to see that it must be an Ohtsuki graph. Thus the set of all such graphs *with oriented vertices* form a generating set. To complete the proof we choose some preferred orientation for each Ohtsuki graph and show that, if  $a$  is any monomial corresponding to this graph with *another* orientation then it is equal to  $\pm$  the preferred one plus monomials of lower total order. To see this let  $G(a)$  be the associated graph and let  $G(a')$  be produced by changing the orientation of one of the trivalent vertices  $v$ . Then  $a' = \pm a +$  monomials of lower total order. In fact, if  $v$  lies in a Y component then, by (a),  $a = a'$  while if  $v$  lies in a completely trivalent component then, by (b),  $a' = -a +$  monomials of lower total order.

Second, we prove the linear independence of the set of admissible monomials  $\{G(a)\}$  where  $G$  lies in the set of Ohtsuki graphs of order  $n$ . To prove linear independence we reinterpret  $\mathbb{Q}[P_2^{\text{SE}}(n)]/\mathcal{R}_n$  as the quotient of the vector space spanned by *unreduced* but commuting monomials in the  $\{\alpha_I\}$ , i.e.  $\alpha_I$  and  $\alpha_I^{-1}$  can both occur in a monomial, by the relations defining  $\mathcal{R}_n$  and the cancellation relation:

$$(33) \quad \begin{aligned} aa^{-1}bd = d & \quad \text{if } a, b \text{ are } i\text{-trivial and} \\ & \quad d \text{ is } i\text{-disjoint for some } i \end{aligned}$$

We write an unreduced monomial uniquely in the form  $\alpha = \alpha_{I_1}^{e_1} \cdots \alpha_{I_n}^{e_n}$ , where each  $I_r = ijk$  with  $i < j < k$  and  $e_r = \pm 1$ . Then we define the *multiplicity*  $m_i = \sum_{i \in I_r} |e_r|$ , for each  $i = 1, \dots, n$ . We say  $i$  is *multiple* if  $m_i > 1$  and *singular* if  $m_i = 1$ . We now define a canonical reduction of  $\alpha$  in five steps.



Step 1. Write:

$$\alpha = \sum \alpha_{I_1}^{e_{i_1}} \cdots \alpha_{I_k}^{e_{i_k}} - \sum c_{i_1 \dots i_k} \alpha_{I_1}^{e_{i_1}} \cdots \alpha_{I_k}^{e_{i_k}}$$

where the first summation is over all subsequences of  $I_1, \dots, I_n$  such that the multiplicities  $m'_i$  of  $\alpha_{I_1}^{e_{i_1}} \cdots \alpha_{I_k}^{e_{i_k}}$  satisfy  $m'_i = \min\{m_i, 2\}$  and the second sum is over all remaining subsequences such that  $m'_i = m_i$  if  $m_i \leq 1$ ,  $m'_i = 2$  if  $m_i = 2$  and  $m'_i = 1$  or  $2$  if  $m_i > 2$ . The coefficients are defined by the formula:  $c_{i_1 \dots i_k} = \prod_r (m_{i_r} - 2)$ , where the product is over all  $r$  such that  $m_{i_r} > 2$  and  $m'_{i_r} = 1$ . Note that there must be at least one such  $r$ .

Step 2. After Step 1 we can assume every  $m_i \leq 2$ . Now we replace every occurrence of  $\alpha_I^{\pm 2}$  by  $4\alpha_I^{\pm 1}$  and any occurrence of  $\alpha_I \alpha_I^{-1}$  by 1.

Step 3. After Step 2 we can assume, in addition, that  $I_i = I_j$  if and only if  $i = j$ . At this point any such monomial is determined by its corresponding *UT* graph. We describe the next two reductions in terms of these graphs and linear combinations of them. First we describe a preliminary modification. A *connected UT* graph  $G$  will be called *even* if every cycle has an even number of edges. In this case the vertices of  $G$  can be divided into two classes: any two vertices in the same class are connected by a path with an even number of edges. If we choose one of these classes and remove each trivalent vertex in the class from the graph, replacing it by three univalent vertices, we obtain a new graph which is a union of *Y* graphs and isolated edges. We carry over the orientations of  $G$  to the new graph. Now define  $G'$  to be the sum  $G_1 + G_2$  of these new graphs, if  $G$  is even, and 0 otherwise. We can now describe Step 3 by replacing every component  $G$  with at least one univalent vertex, in a *UT* graph, by  $G'$ .

Step 4. After Step 3 we have a linear combination of Chinese manifold characters (with, perhaps, some additional isolated edges). This step will reduce it to a linear combination of chinese manifold characters. For example, we can always prefer the orientation given by  $i, j, k$  where  $i < j < k$ . We describe Step 4 as follows. Suppose  $G$  is a component all of whose vertices are trivalent and  $k$  of its vertices have the wrong orientation. Then we replace  $G$  by  $(-1)^k \tilde{G} + 2kG'$ , where  $\tilde{G} = G$  except that the orientations are now all the preferred ones. If  $G$  is a *Y* graph, then we replace  $G$  by  $\tilde{G}$ .

Step 5. The final step is to eliminate any graph which has an isolated edge.

We leave as a straightforward, if, perhaps, lengthy exercise to show that these five steps can be achieved by using the relations given by equations 27-32 and 33. The important thing is to show that the result of this reduction depends only on the class of the monomial in  $\mathbb{Q}[P_2^{\text{SE}}(n)]/\mathcal{R}_n$ . But this can be achieved by checking that, for each of the equations 27-30 and 33, reducing both sides of the equation gives the same result.  $\square$

**Corollary 3.8.**  $\mathcal{G}_m\mathcal{O}$  is a finite dimensional vector space and is nonzero only if  $m$  is a multiple of 3.

*Remark 3.9.* This partially answers question 1 of [Ga].

*Remark 3.10.* In [GO] the notion of manifold weight systems for finite type invariants of  $\mathbb{Z}HS$  will be introduced.

**3.4. From knots to 3-manifolds.** In this section we prove a vanishing theorem (Theorem 7) for type  $5m + 1$  invariants of  $\mathbb{Z}HS$ . The proof exploits the graphical, as well as the algebraic notation from the previous chapters. As a corollary, in Proposition 3.12 we make some progress on question 2 of [Ga].

With the notation of (16), we can state the following theorem:

**Theorem 7.** *If  $L$  is an AS-admissible link in  $S^3$  with a  $(4m + 1)$ -component trivial sublink, then  $[L] \in \mathcal{F}_{5m+2}\mathcal{M}$ .*

*Proof.* If  $L$  has  $4m + 1 + r$  components, we proceed by downward induction on  $r$ . Obviously if  $r > m$ , there is nothing to prove. We record the following consequence of the defining relations in  $\mathcal{M}$ .

**Lemma 3.11.** *If  $\tilde{L}$  is obtained from  $L$  by changing a crossing of two bands, then  $[\tilde{L}] - [L]$  is a linear combination of  $[L_i]$ , where each  $L_i$  contains  $L$  as a proper sublink.*

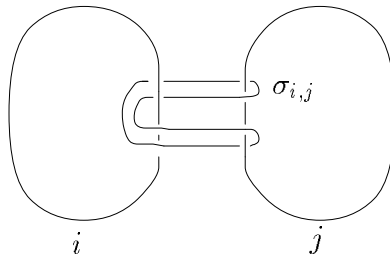
A *band* means a collection of parallel subarcs of components of  $L$ , and we assume, after choosing some orientation for the components of  $L$  and a direction for each band, that each component of  $L$  has an equal number of subarcs travelling in the positive direction in the two bands as in the negative direction. (Note that this condition is independent of the orientations chosen but does depend on the directions of the bands.) See figures 4 and 5.

*Proof.* This follows from figures 4, 5 and equations (15) and (20).  $\square$

Suppose we write  $L = L_0 \cup L'$ , where  $L_0$  is the trivial sublink. If we perform a crossing change, as in Lemma 3.11 where at most one of the two bands contains arcs from  $L_0$ , then  $\tilde{L}$  and each  $L_i$  also contains  $L_0$  as a sublink and so, by the inductive assumption,  $[L] \in \mathcal{F}_{5m+2}\mathcal{M}$  if and only if  $[\tilde{L}] \in \mathcal{F}_{5m+2}\mathcal{M}$ . So, for example, we are free to change any component of  $L'$  within its homotopy class in the complement of  $L_0$ . Furthermore, if  $\pi = \pi_1(S^3 - L_0)$  then we can change a component  $l$  by any element of  $\pi_3$ , by the argument in [Le1, pages 58-9], using band crossings in which one band consists of arcs from  $L_0$  and the other consists of arcs from  $l$ .

As a consequence of these observations, we may assume that  $L'$  consists of components  $l_k$  so that each  $l_k$  represents a product of commutators of degree two  $\prod_{i < j} [x_i, x_j]^{e_{ij}^k}$ . Now each commutator  $[x_i, x_j]$  can be represented by a curve  $\sigma_{ij}$  which intersects only two of the disjoint disks  $D_j$  bounded by the components of  $L_0$ . See figure 19. We may therefore assume that each  $l_k$  is a band sum of a number of copies of the  $\sigma_{ij}$ ,

slightly translated so that they are all disjoint. We now want to apply Theorem 4. This tells us that  $[L]$  is a linear combination of  $[L_i]$ , where  $L_i$  coincides with  $L$  except that each  $l_k$  has been replaced by either one of the  $\sigma_{ij}$  or a band sum of two of the  $\sigma_{ij}$ , or several of these. We can ignore any terms in which any  $l_k$  has been replaced by more than one component, by the inductive assumption. In each of the  $L_i$  that remain, each new  $l_k$  intersects at most four of the  $D_j$  and so  $L'_i$  intersects at most  $4r$  of the  $D_j$ . Therefore if  $r \leq m$ , there will be at least one of the  $D_j$  not intersected by any of the component of  $L$ . But it then follows from equation (27) that  $[L] = 0$ .  $\square$



**Figure 19.** A curve  $\sigma_{ij}$  that represents a commutator  $[x_i, x_j]$  in the fundamental group of the complement of a trivial 2 component link. Compare this to Figure 3

As an application of Theorem 7, we study the map from knots to (linear combinations of) 3-manifolds defined by  $K \rightarrow [S^3, K, +1] := S^3_{(K,+1)} - S^3$ . Dually this map induces a map from 3-manifold invariants to knot invariants. In [Ga] it was shown that this descends to a map

$$(34) \quad \mathcal{F}_n \mathcal{O} \longrightarrow \mathcal{F}_{n-1} \mathcal{V}$$

where  $\mathcal{F}_n \mathcal{V}$  is the space of type  $n$  knot invariants [B-N1], [BL], [Va]. In [Ga] it is conjectured that the above map actually descends to a map:

$$(35) \quad \mathcal{F}_{3m} \mathcal{O} \longrightarrow \mathcal{F}_{2m} \mathcal{V}$$

**Proposition 3.12.** *The above map descends to a map:*

$$(36) \quad \mathcal{F}_{5m+1} \mathcal{O} \longrightarrow \mathcal{F}_{4m} \mathcal{V}$$

*Proof.* Let  $\lambda \in \mathcal{F}_{5m+1} \mathcal{O}$  be a type  $5m$  invariant of  $\mathbb{Z}HS$ , and let  $\psi_\lambda$  be the associated knot invariant of equation (34). Let  $K$  be an immersed knot in  $S^3$  with  $4m + 1$  double points. Let  $K \cup L_0$  denote the  $AS$ -admissible link in  $S^3$  of  $4m + 2$  components obtained by replacing each double point with the left hand of figure 20. Note that  $K \cup L_0$  contains an unlink  $L_0$  of  $4m + 1$  components. Using the equality of figure 20 and the definition of the associated knot invariant  $\psi_\lambda$  we obtain the following equality

$$(37) \quad \psi_\lambda(K) = \lambda([K \cup L_0])$$

Using Theorem 7 we have that  $[K \cup L_0] \in \mathcal{F}_{5m+2}\mathcal{M}$  therefore  $\lambda([K \cup L_0]) = 0$ . This shows that  $\psi_\lambda$  is a type  $4m$  invariant of knots in  $S^3$ .  $\square$

$$\begin{array}{c} \diagup \\ \diagdown \\ \text{loop} \end{array} = \begin{array}{c} \diagup \\ \diagdown \end{array} - \begin{array}{c} \diagdown \\ \diagup \end{array}$$

**Figure 20.** Another special case of equation (17) in graphical notation.

*Remark 3.13.* Proposition 3.12 is a  $\frac{4m}{5m+1} \simeq \frac{4}{5}$  result, whereas question 2 of [Ga] asks for a  $\frac{2}{3} < \frac{4}{5}$  result. In a recent preprint M. Greenwood and Xiao-Song Lin [GrLi] have shown a  $\frac{n-2}{n} \simeq 1$  result, which is a weaker statement than proposition 3.12 if  $n > 6$ .

*Addendum 3.14.* N. Habegger [Ha] has recently given a proof of the  $\frac{2}{3}$  result:

★: If  $L$  is an  $AS$ -admissible link in  $S^3$  with a  $(2m - 1)$ -component trivial sublink, then  $[L] \in \mathcal{F}_{3m}\mathcal{M}$ .

A corollary of this is that the map  $\mathcal{F}_n\mathcal{O} \rightarrow \mathcal{F}_{n-1}\mathcal{V}$  in Equation (30) descends to a map  $\mathcal{F}_{3m}\mathcal{O} \rightarrow \mathcal{F}_{2m}\mathcal{V}$ , as conjectured by Garoufalidis in [Ga]. We thank Habegger for sending us a copy of [Ha]. After reading Habegger's proof we saw how to refine our proof of Theorem 7 to give another proof of Theorem ★. Our proof of Theorem ★ is rather different from Habegger's and will be included in a future work [GaLe].

#### 4. A PHILOSOPHICAL COMMENT

We feel that we owe a word on the appearance of trivalent graphs. Trivalent graphs appear both in Section 2 and in Section 3. As mentioned in the introduction, a motivation for the notion of finite type invariants of  $\mathbb{Z}HS$  is Chern-Simons theory, exploited by Witten [Wi2]. Chern-Simons theory is a *topological quantum field theory* with a topological Lagrangian containing a quadratic and a cubic term. The asymptotic expansion of the associated path integral (over the space of connections) as the coupling parameter goes to infinity can be approximated by a power series sum, each term of which is a finite sum over trivalent graphs (Feynman diagrams). This is the reason that trivalent graphs appear in Chern-Simons theory.

In the theory of finite type knot invariants, trivalent graphs appear either as triple point degenerations of knots, or as Feynman diagrams of an associated conformal field theory (governed by the  $KZ$  equation), see [Dr], [B-N1] and [Ko].

In the theory of finite type invariants of  $\mathbb{Z}HS$  trivalent graphs appear because of the presence of the Kirby moves, an intrinsic 3-dimensional property of space.

Finally, in the notion of surgical equivalence of links trivalent graphs appear because of the association of figure 5 with triple commutators  $[a, [b, c]]$  in fundamental groups.

In motivic cohomology, trivalent graphs appear because of the algebraic fundamental group  $\pi_1^{alg}(\mathbb{P}_{\mathbb{C}}^1 - \{0, 1, \infty\})$ .

The use of trivalent graphs, whether they come from the topology of 3-dimensional space, or the algebra (commutator groups, or cubic interaction terms in path integrals) is a unifying approach, and as such, it can be a source of inspiration, or confusion. We will let the reader decide which.

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DEPARTMENT OF MATHEMATICS, MASSACHUSETTS INSTITUTE OF TECHNOLOGY, CAMBRIDGE, MA 02139

*E-mail address:* stavros@math.mit.edu

DEPARTMENT OF MATHEMATICS, BRANDEIS UNIVERSITY, WALTHAM, MA 02254-9110

*E-mail address:* levine@max.math.brandeis.edu