

# MODULAR $q$ -HOLONOMIC MODULES

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ABSTRACT. We introduce the notion of modular  $q$ -holonomic modules whose fundamental matrices define a cocycle with improved analyticity properties and show that the generalised  $q$ -hypergeometric equation, as well as three key  $q$ -holonomic modules of complex Chern–Simons theory are modular. This notion explains conceptually recent structural properties of quantum invariants of knots and 3-manifolds, and of exact and perturbative Chern–Simons theory [26, 27, 28, 37], and in addition provides an effective method to solve the corresponding linear  $q$ -difference equations. An alternative title of our paper, emphasising the equations rather than the modules, is:

Modular linear  $q$ -difference equations

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## 1. INTRODUCTION

1.1. **Summary.** We introduce a new class of linear  $q$ -difference equations and (corresponding  $q$ -holonomic modules) which we call *modular*, with several key features.

- Their fundamental solutions at 0 and infinity have explicitly computable monodromy.
- A natural  $\mathrm{SL}_2(\mathbb{Z})$ -cocycle constructed from a fundamental solution extends from  $\mathbb{C} \setminus \mathbb{R}$  to the complex plane minus a ray in the real numbers.
- Their fundamental solutions are meromorphic and their residues are expressed in terms of the solutions themselves (kind of 'resurgence').

Modular  $q$ -holonomic modules are abundant. We show that the generalised  $q$ -hypergeometric equation (11) (and in particular the  $q$ -hypergeometric equation) is modular and self-dual; see Theorem 1.4 below. Among other things, this implies an improved analyticity for the  $q$ -hypergeometric function

$${}_2\phi_1(a, b; c; q, t) = \sum_{k=0}^{\infty} \frac{(a; q)_k (b; q)_k}{(c; q)_k (q; q)_k} t^k, \quad (1)$$

namely, that the bilinear combination

$$\begin{aligned} & \frac{(q; q)_{\infty} (c; q)_{\infty}}{(a; q)_{\infty} (b; q)_{\infty}} \frac{(\tilde{q}\tilde{a}; \tilde{q})_{\infty} (\tilde{q}\tilde{b}; \tilde{q})_{\infty}}{(\tilde{q}; \tilde{q})_{\infty} (\tilde{q}\tilde{c}; \tilde{q})_{\infty}} {}_2\phi_1(a, b; c; q, t) {}_2\phi_1(\tilde{a}^{-1}, \tilde{b}^{-1}; \tilde{c}^{-1}; \tilde{q}, \tilde{q}\tilde{a}\tilde{b}\tilde{c}^{-1}\tilde{t}) \\ & + \tau \frac{\theta(q^{-1}ct; q)(q; q)_{\infty}^3}{\theta(t; q)\theta(q^{-1}c; q)} \frac{(q^2c^{-1}; q)_{\infty} (q; q)_{\infty}}{(qac^{-1}; q)_{\infty} (qbc^{-1}; q)_{\infty}} \frac{\theta(\tilde{q}\tilde{c}; \tilde{q})\theta(\tilde{t}; \tilde{q})}{\theta(\tilde{q}\tilde{c}\tilde{t}; \tilde{q})(\tilde{q}; \tilde{q})_{\infty}^3} \frac{(\tilde{a}\tilde{c}^{-1}; \tilde{q})_{\infty} (\tilde{b}\tilde{c}^{-1}; \tilde{q})_{\infty}}{(\tilde{q}^{-1}\tilde{c}^{-1}; \tilde{q})_{\infty} (\tilde{q}; \tilde{q})_{\infty}} \\ & \times {}_2\phi_1(qac^{-1}, qbc^{-1}; q^2c^{-1}; q, t) {}_2\phi_1(\tilde{q}\tilde{a}^{-1}\tilde{c}, \tilde{q}\tilde{b}^{-1}\tilde{c}; \tilde{q}^2\tilde{c}; \tilde{q}, \tilde{q}\tilde{a}\tilde{b}\tilde{c}^{-1}\tilde{t}) \end{aligned} \quad (2)$$

(which a priori is a meromorphic function of  $\tau \in \mathbb{C} \setminus \mathbb{R}$ ) extends to  $\tau \in \mathbb{C} \setminus (-\infty, 0]$ . We also show that modular linear  $q$ -difference equations and the corresponding  $q$ -holonomic modules appear naturally in complex Chern-Simons theory. We illustrate with three examples (Theorems 1.5, 1.6 and 1.7 below) whose corresponding cocycles are the Faddeev quantum dilogarithm, the Appell-Lerch sums, and the Andersen–Kashaev state integrals of the  $4_1$  knot.

We expect that all proper (i.e., basic)  $q$ -hypergeometric modules are modular, and in particular the ones that appear in the quantum differential equation in Quantum Cohomology, or the linear  $q$ -difference equation for the small J-function of Quantum K-Theory.

**1.2. Motivation.** In a recent talk [59, 60], Okounkov asked the question:

*What does it take to solve a  $q$ -difference equation?*

Solving a linear equation usually means giving a basis of solutions, say at  $t = 0$  and  $t = \infty$  (the only two canonical points, fixed under the shift transformation  $t \mapsto qt$ ), and to compute the monodromy (i.e., the connection) between  $t = 0$  and  $t = \infty$  in terms of “known” functions. This problem has a rich history with an interesting balance between concrete special functions and the abstract, which the reader may consult in Okounkov’s talks and their references.

A key to the solution is to choose linear  $q$ -difference equations of natural origin, for instance the ones appearing in quantum cohomology [61], in Kontsevich’s talks on resurgence [48], or in quantum topology and Chern–Simons theory [25, 34].

Recently, it was observed that the solutions of some linear  $q$ -difference equations concerning quantum knot invariants [36, 37] or resurgence and Borel resummation of perturbative Chern–Simons theory [26, 27, 28], have an improved analyticity property. Roughly speaking, this means that some holomorphic functions on  $\mathbb{C} \setminus \mathbb{R}$  extend on the cut plane  $\mathbb{C}' = \mathbb{C} \setminus (-\infty, 0]$ . This extension property has been formalised recently by Zagier as holomorphic quantum modular forms [70]. This extends previous ideas around quantum modularity [72] that appeared in Chern–Simons theory [11, 14, 43, 50].

Our attempt to understand the mechanism behind this property abstractly led us to the notion of a modular  $q$ -holonomic module. Quite by accident, this suggests an answer to the question posed above, namely:

*Modularity can solve effectively a  $q$ -difference equation.*

**1.3. Definition and properties.** To explain the new concept, consider the linear  $q$ -difference equation

$$\sigma X = AX \tag{3}$$

for a vector-valued function  $X = X(t, q)$ , where  $(\sigma X)(t, q) = X(qt, q)$  denotes the shift operator,  $A(t, q) \in \mathrm{GL}_r(\mathbb{Q}(t, q))$  and  $q$  is a nonzero complex number with  $|q| \neq 1$ . Fix a fundamental matrix solution  $U$  to (3) at  $t = 0$ , which we assume is filtration-preserving and of weight  $\kappa_U = \mathrm{diag}(\tau^{\kappa_{U,1}}, \dots, \tau^{\kappa_{U,r}})$  and define

$$\Omega_{U,\gamma} = (U|_{\kappa_U} \gamma)U^{-1}, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \tag{4}$$

where the slash operator  $|_{\kappa_U} \gamma$  is defined in Section 2.4 below. The map  $\gamma \mapsto \Omega_{U,\gamma}$  is a cocycle of  $\mathrm{SL}_2(\mathbb{Z})$ , i.e., it satisfies

$$\Omega_{U,\gamma\gamma'}(z, \tau) = \Omega_{U,\gamma}(\gamma'(z, \tau))\Omega_{U,\gamma'}(z, \tau) \tag{5}$$

for all  $\gamma, \gamma' \in \mathrm{SL}_2(\mathbb{Z})$ . Likewise, let  $V$  denote a filtration-preserving fundamental solution at  $t = \infty$ , and  $\vec{M} = V^{-1}U$  denote the monodromy matrix (often called “connection matrix”). The monodromy is always an elliptic function (i.e.,  $\sigma$ -invariant). If it satisfies the equation

$$\vec{M} = \Delta_{\kappa_V,\gamma}(\vec{M}|_{\kappa_U} \gamma), \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \tag{6}$$

for some  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$  then the cocycle matrices associated to  $U$  and  $V$  are equal:

$$\Omega_{U,\gamma} = \Omega_{V,\gamma} \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}). \quad (7)$$

(The converse holds, too, and the equivalence of (7) and (6) holds for each fixed  $\gamma$ ). A priori,  $\Omega_\gamma = \Omega_{U,\gamma} = \Omega_{V,\gamma}$  is a holomorphic function of  $\tau \in \mathbb{C} \setminus \mathbb{R}$  and a meromorphic function of  $z$  (with  $t = \mathbf{e}(z)$ ) with poles in a finite union of translates of the lattice  $\mathbb{Z} + \tau\mathbb{Z}$ . For an element  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ , we let  $\mathbb{C}_\gamma$  denote the cut plane  $\mathbb{C} \setminus (-\infty, -d/c]$  (if  $c > 0$ ),  $\mathbb{C} \setminus [-d/c, \infty)$  (if  $c < 0$ ) and  $\mathbb{C} \setminus \mathbb{R}$  when  $c = 0$ . The next definition concerns an improved analytic property of  $\Omega_\gamma$ .

**Definition 1.1.** We say that a  $q$ -difference equation is *modular* if for all  $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ , its cocycle  $\Omega_\gamma$  extends to a meromorphic function of  $(z, \tau) \in \mathbb{C} \times \mathbb{C}_\gamma$  with potential poles at  $z \in S_\gamma + \mathbb{Z} + \tau\mathbb{Z}$  for a finite set  $S_\gamma$ .

We make several comments concerning this definition.

1. For a modular  $q$ -difference equation, the two sides of  $\tau \in \mathbb{C} \setminus \mathbb{R}$  communicate: the restriction of  $\Omega_\gamma$  on the one side of the plane  $\mathrm{Im}(\tau) > 0$  uniquely determines (and is determined by) the function on the other side  $\mathrm{Im}(\tau) < 0$ .
2. In a sense, our notion of modular linear  $q$ -difference equation is a  $q$ -analogue of the second, third and fourth order modular linear differential equations studied by Kaneko, Nagatomo, Zagier and others [4, 45, 46] in relation to Conformal Field Theory and Vertex Operator Algebras. The characters of a VOA (under a mild  $C_2$ -condition) are solutions to a modular linear differential equation [19, 73]. An alternative title of our paper could be “Modular linear  $q$ -difference equations” in direct analogy with the modular linear differential equations of CFTs and VOAs. On the other hand, the current title emphasises the “module” aspect as opposed to the “equation aspect”.
3. Our notion explains conceptually the improved analyticity properties conjectured in relation to the structure of exact and perturbative invariants of Chern–Simons theory [26, 27, 28, 36, 37], as well as the work of Gukov et al relating logarithmic CFTs and VOAs to Chern–Simons theory [13].
4. Our notion leads to a different perspective on quantum Jacobi forms [10]. While the matrix valued aspect is more general, the analytic conditions are much stricter. However, many examples, with some refinements, are expected to fit into our definition.
5. Modular linear differential equations seem rare. On the other hand, modular linear  $q$ -difference equations appear abundant: we conjecture (see the end of Section 1) that the  $\hat{A}$ -polynomial of a knot (i.e., the linear  $q$ -difference equation satisfied by the colored Jones polynomial of a knot [34]) is a modular linear  $q$ -difference equation. This brings a new perspective to the Jones polynomial of a knot.
6. The modularity of the monodromy for modular  $q$ -holonomic modules is a restricted condition, which leads to the complete determination of the monodromy even when a fundamental solution is defined by  $q$ -Borel resummation. This in turn completely determines the so-called  $q$ -Stokes phenomenon coming from the change of the ray of the  $q$ -Laplace transform.
7. The cocycle  $\Omega_{U,\gamma}$  as a function of a fundamental solution of a linear  $q$ -difference equation appears new and different from the matrices considered in Etingof [22] that generate the Galois group of the equation.

8. The above definition involves all elements of  $\mathrm{SL}_2(\mathbb{Z})$ , although it imposes no improved extension when  $\gamma_{2,1} = 0$ . The third part of the next theorem (under the hypothesis that  $\Omega_T = I$  which is satisfied in all of our examples), rephrases the modularity of Definition 1.1 in terms of  $\Omega_S$  alone, where  $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$  and  $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$  of  $\mathrm{SL}_2(\mathbb{Z})$  are the standard generators of  $\mathrm{SL}_2(\mathbb{Z})$ . Recall that  $\mathbb{C}' = \mathbb{C}_S = \mathbb{C} \setminus (-\infty, 0]$ .

**Theorem 1.2.** (a) *If Equation (6) for the monodromy holds for  $\gamma = S$  and  $\gamma = T$ , then it holds for all  $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ .*

(b) *If  $\Omega$  is an  $\mathrm{SL}_2(\mathbb{Z})$ -cocycle with  $\Omega_T = I$ , then  $\Omega_S$  satisfies the 4-term and the 3-term functional equations*

$$1 = \Omega_S \left( -\frac{z}{\tau}, -\frac{1}{\tau} \right) \Omega_S(-z, \tau) \Omega_S \left( \frac{z}{\tau}, -\frac{1}{\tau} \right) \Omega_S(z, \tau) \quad (8a)$$

$$\Omega_S(z, \tau) = \Omega_S \left( \frac{z}{\tau+1}, \frac{\tau}{\tau+1} \right) \Omega_S(z, \tau+1). \quad (8b)$$

*Conversely, given  $\Omega_T = I$  and  $\Omega_S$  that satisfies the above functional equations, there is a  $\mathrm{SL}_2(\mathbb{Z})$ -cocycle with those values.*

(c) *If  $\Omega$  is a cocycle such that  $\Omega_T = I$  and  $\Omega_S$  extends as a meromorphic function of  $(z, \tau) \in \mathbb{C} \times \mathbb{C}'$ , then  $\Omega_\gamma$  extends as a meromorphic function of  $(z, \tau) \in \mathbb{C} \times \mathbb{C}'_\gamma$  for all  $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ .*

Note that with the assumptions of part (c) above, we have

$$\Omega_{-I}(z, \tau) = \Omega_S \left( \frac{z}{\tau}, -\frac{1}{\tau} \right) \Omega_S(z, \tau) \quad (9)$$

which is meromorphic on  $\mathbb{C} \times (\mathbb{C} \setminus \mathbb{R})$ , in general does not extend any further; see for instance the cocycle of Theorem 1.5 below.

9. Our final comment concerns the distinction between a linear  $q$ -difference equation to a  $q$ -holonomic module. A linear  $q$ -difference equation leads to a pair  $(M, e)$  of a  $q$ -holonomic module  $M$  over the  $q$ -Weyl algebra  $\mathcal{W} = \mathbb{Q}(q, t)\langle \sigma \rangle / (\sigma t - qt\sigma)$  and a cyclic vector  $e$  of  $M$ . The category  $\mathcal{M}$  of  $q$ -holonomic modules is an abelian category, which admits a multivariable extension closed under the usual operations on sheaves. For a detailed introduction, see [16] and [65], and also [33]. Whereas  $D$ -modules over the Weyl algebra have only one dual (see for example, [67, Sec.2.2]), modules over the  $q$ -Weyl algebra have two duals  $M^\wedge$  (the usual Hom-dual as in the case of  $D$ -modules) and a dual  $M^\vee$  coming from the involution  $q \mapsto q^{-1}$ , discussed in Section 2.5 below. The next lemma summarises some categorical properties of modular  $q$ -holonomic modules.

**Lemma 1.3.** (a) *Modularity of a  $q$ -holonomic module is independent of the choice of a cyclic vector.*

(b) *If  $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$  is a short exact sequence of  $q$ -holonomic modules and  $M$  is modular, then  $M$  and  $M'$  are also modular.*

(c) *If  $M$  is a modular, then  $M^\wedge$  and  $M^\vee$  are modular with cocycles given by*

$$\Omega^\wedge(z, \tau) := \Omega(z, -\tau), \quad \text{and} \quad \Omega^\vee = (\Omega^{-1})^t \quad (10)$$

where  $\Omega$  is a cocycle of  $M$ .

**1.4. The generalised  $q$ -hypergeometric equation is modular.** Contrary to what one might expect, modular  $q$ -holonomic modules are abundant. This section concerns the generalised  $q$ -hypergeometric equation

$$\left( \prod_{j=0}^{r-1} (1 - q^{-1}b_j\sigma) - t \prod_{j=1}^r (1 - a_j\sigma) \right) f = 0 \quad (11)$$

for a function  $f = f(t, a, b, q)$  where  $a = (a_1, \dots, a_r)$  and  $b = (b_0, \dots, b_{r-1})$  with  $b_0 = q$ , where  $\sigma$  is the operator that shifts  $t$  to  $qt$ . This equation is a  $q$ -deformation of the hypergeometric equation (a regular singular linear differential equations with singularities at  $0, 1, \infty$ ), itself a generalisation of the Gauss hypergeometric equation. The hypergeometric equation has a rich history related to, among other things, periods in algebraic geometry and in the Gauss–Manin connection of the middle cohomology of the Dwork family of smooth hypersurfaces; see for example [7, 47] and references therein.

The generalised  $q$ -hypergeometric equation (11) was introduced and studied by Heine, who shows that this equation has a solution, analytic at  $t = 0$ , given by the  $q$ -hypergeometric series

$${}_r\phi_{r-1}(a; b; q, t) = \sum_{k=0}^{\infty} \frac{(a_1; q)_k \cdots (a_r; q)_k}{(b_1; q)_k \cdots (b_{r-1}; q)_k (q; q)_k} t^k. \quad (12)$$

The generalised  $q$ -hypergeometric equation and the  $q$ -hypergeometric series are classical examples of functions that appear in many areas of research, and for a comprehensive treatment, we refer the reader to the book of Gasper–Rahman [38] and references therein.

Consider the matrices

$$U = W(f^{(1)}, f^{(qb_1^{-1})}, \dots, f^{(qb_{r-1}^{-1})}), \quad V = W(g^{(a_1^{-1})}, \dots, f^{(a_r^{-1})}) \quad (13)$$

where  $W$  denotes the Wronskian of  $r$  functions  $f_1, \dots, f_r$ , which is an  $r \times r$  matrix defined by  $W(f_1, \dots, f_r)_{ij} = (\sigma^{i-1} f_j)$  for  $i, j = 1, \dots, r$  and

$$f^{(q^{-1}b_j)}(t, a, b, q) = B_j(a, b, q) \frac{\theta(q^{-1}b_j t; q)}{\theta(t; q)} {}_r\phi_{r-1}(qa/b_j; qb/b_j; q, t) \quad (14)$$

for  $j = 0, \dots, r-1$  with  $B_0 = 1$  and, for  $j > 0$ ,

$$B_j(a, b, q) = \left( \prod_{i=1}^r \frac{(a_i; q)_{\infty} (qb_{i-1}/b_j; q)_{\infty}}{(qa_i/b_j; q)_{\infty} (b_{i-1}; q)_{\infty}} \right) \frac{(q; q)_{\infty}^3}{\theta(q^{-1}b_j; q)}, \quad (15)$$

and

$$g^{(a_j^{-1})}(t, a, b, q) = A_j(a, b, q) \frac{\theta(q^{-1}a_j t; q)}{\theta(q^{-1}t; q)} {}_r\phi_{r-1}(qa_i/b; qa_i/a; q, qb_1 \dots b_{r-1} a_1^{-1} \dots a_r^{-1} t^{-1}) \quad (16)$$

for  $j = 1, \dots, r$  with

$$A_j(a, b, q) = \frac{(q; q)_{\infty}^2}{\theta(q^{-1}a_j; q) \prod_{i=1, i \neq j}^r (a_i/a_j; q)_{\infty}} \prod_{i=1}^r \frac{(a_i; q)_{\infty} (b_{i-1}/a_j; q)_{\infty}}{(b_{i-1}; q)_{\infty}}. \quad (17)$$

The calculation of the monodromy is a classical result and follows from [38, Eq.(4.5.2)]. The monodromy  $\vec{M}(t, a, b, q) = V(t, a, b, q)^{-1}U(t, a, b, q)$  is given explicitly by

$$\vec{M}(t, a, b, q) = \begin{pmatrix} 1 & \frac{(q; q)_\infty^3 \theta(b_1 t; q) \theta(a_1 / b_1 t; q) \theta(a_1; q)}{\theta(t; q) \theta(a_1 t; q) \theta(b_1; q) \theta(a_1 / b_1; q)} & \cdots & \frac{(q; q)_\infty^3 \theta(b_{r-1} t; q) \theta(a_1 / b_{r-1} t; q) \theta(a_1; q)}{\theta(t; q) \theta(a_1 t; q) \theta(b_{r-1}; q) \theta(a_1 / b_{r-1}; q)} \\ 1 & \frac{(q; q)_\infty^3 \theta(b_1 t; q) \theta(a_2 / b_1 t; q) \theta(a_2; q)}{\theta(t; q) \theta(a_2 t; q) \theta(b_1; q) \theta(a_2 / b_1; q)} & \cdots & \frac{(q; q)_\infty^3 \theta(b_{r-1} t; q) \theta(a_2 / b_{r-1} t; q) \theta(a_2; q)}{\theta(t; q) \theta(a_2 t; q) \theta(b_{r-1}; q) \theta(a_2 / b_{r-1}; q)} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & \frac{(q; q)_\infty^3 \theta(b_1 t; q) \theta(a_r / b_1 t; q) \theta(a_r; q)}{\theta(t; q) \theta(a_r t; q) \theta(b_1; q) \theta(a_r / b_1; q)} & \cdots & \frac{(q; q)_\infty^3 \theta(b_{r-1} t; q) \theta(a_r / b_{r-1} t; q) \theta(a_r; q)}{\theta(t; q) \theta(a_r t; q) \theta(b_{r-1}; q) \theta(a_r / b_{r-1}; q)} \end{pmatrix}. \quad (18)$$

Then from the modularity of the Dedekind  $\eta$ -function and the Jacobi  $\theta$ -function the matrix  $\vec{M}$  satisfies the modular transformation (6) with weights  $\kappa_U = (0, 1, \dots, 1)$  and  $\kappa_V = (0, \dots, 0)$ .

When  $a, b$  are specialised at points where there are singularities, one can take coefficients of the expansion around these points to define  $U, V$ .

Theorem 1.4 states that the generalised  $q$ -hypergeometric equation is modular. The theorem will also hold when we specialise the values of  $a, b$ . This is because the integral representations of the cocycle will be regular at these specialisations and the factorisation of the integral will similarly involve the expansion around these points as it will involve computation of residues of higher order.

**Theorem 1.4.** *The two cocycles (4)  $\Omega_U$  and  $\Omega_V$  are equal and modular.*

The proof of the above theorem is given in Section 3 and uses an integral representation of the solutions of Equation (11) in terms of a special function, the Faddeev quantum dilogarithm [23, 24], the factorisation of the corresponding integrals (so-called state integrals) as a bilinear combination functions of  $z, \tau$  and  $(z/\tau, -1/\tau)$  along the lines of [30], combined with an explicit description of the self-duality of the corresponding  $q$ -holonomic module. What is more, the  $U$  and the  $V$  cocycles are obtained by moving the contour of integration of the state integral upwards or downwards, respectively, and this is one explanation of their equality. We also remark that the above theorem is valid for  $r = 1$ , where the corresponding cocycle is none other than a ratio of two Faddeev quantum dilogarithms.

**1.5. Modular  $q$ -holonomic modules in Chern–Simons theory.** In this section we present three  $q$ -holonomic modules whose cocycles play a key role in complex Chern–Simons theory and prove their modularity. We will not need a detailed knowledge of Chern–Simons theory, but focus on the fact that it is a gauge theory with gauge group a complex Lie group, whose partition function can be identified by a finite dimensional integrals of products of the Faddeev quantum dilogarithm. A detailed exposition of Chern–Simons theory is given in the work of Andersen–Kashaev and Beem, Dimofte and Pasquetti [3, 18, 5, 62]. The quasi-periodicity of the Faddeev quantum dilogarithm implies that the partition function of complex Chern–Simons theory satisfies a  $q$ -holonomic module of linear  $q$ -difference equations.

In the modular  $q$ -holonomic modules discussed below, we prove their modularity *and* at the same time define and compute fundamental matrices  $U$  and  $V$  of solutions algorithmically, and give explicit formula for their monodromy. This is possible because of two key features, namely:

- (a) The fundamental matrices  $U$  and  $V$  are meromorphic functions with explicit poles and with residues expressed linearly in terms of  $U$  and  $V$ . This is some kind of resurgence property and it is ultimately responsible for determining the monodromy.

- (b) The monodromy is uniquely determined by its elliptic property, the explicit poles and principle parts, and its limiting values at  $t = 0$  or  $t = \infty$ .

Our first module, the linear  $q$ -difference equation for the infinite Pochhammer symbol

$$(1 - qt)f(qt, q) - f(t, q) = 0, \quad (19)$$

has trivial monodromy but its cocycle is a very interesting function, the Faddeev quantum dilogarithm function which is the building block of the partition function of complex Chern–Simons theory. With the definition of  $f^{(0)}$  and  $g^{(1)}$  given in Equations (156) and (161) below, we define the fundamental matrices  $U(t, q)$  and  $V(t, q)$  of solutions at  $t = 0$  and  $t = \infty$  by

$$U(t, q) = W(f^{(0)}(t, q)), \quad V(t, q) = W(g^{(1)}(t, q)), \quad (20)$$

with equal weights  $\kappa = 0$ . In our theorems below, we will use the terms “a  $q$ -holonomic module is modular” and “its cocycle is modular” interchangeably.

**Theorem 1.5.** *The monodromy matrix is  $\vec{M}(t, q) = (1)$ . The two cocycles (4)  $\Omega_U$  and  $\Omega_V$  are equal, modular and when  $\gamma = S$ , they are given explicitly by*

$$\Omega_S(z, \tau) = \Phi_{\mathbf{b}} \left( \frac{iz}{\mathbf{b}} + \frac{i\mathbf{b}}{2} + \frac{1}{2i\mathbf{b}} \right)^{-1} \quad (21)$$

where  $\Phi$  is the Faddeev quantum dilogarithm function [23, 24].

In forthcoming work [31], the cocycle  $\Omega_\gamma$  will be identified with an  $\mathrm{SL}_2(\mathbb{Z})$ -extension of the Faddeev quantum dilogarithm, and its modularity will be deduced independently, using properties of the odd Eisenstein series.

Our second module is the  $q$ -difference equation of the Appell-Lerch sums studied in [9, 74] and given by

$$f(q^2t, q) + (qt - 1)f(qt, q) - tf(t, q) = 0. \quad (22)$$

The cocycle of (22) is the Appell-Lerch sum, which is the building block for the extension of the partition function of complex Chern–Simons theory that detects the trivial flat connection, see [28]. In the above equation, some formal power series solutions are divergent, and their  $q$ -Laplace resummation is expressed in terms of an Appell-Lerch sum that depends on an additional elliptic variable. The monodromy of this equation is an explicit product of theta functions that depends on two elliptic variables, but the cocycle is independent of them, and is expressed explicitly in terms of the Mordell integral. Our results give a new interpretation of the Mordell integral and Appell-Lerch sums emphasising the role of the linear  $q$ -difference equations as opposed to the aspects of modular forms advocated by Zwegers in his thesis [74] and by Dabholkar–Murthy–Zagier [17] in the study of mock modular forms and their incarnations in the mathematical physics of black holes.

With the definition of  $f^{(-1)}$ ,  $f^{(0)}$ ,  $g^{(0)}$  and  $g^{(1)}$  given in Equations (175), (170), (177) and (178) below, we define the fundamental matrices  $U(t, \lambda, q)$  and  $V(t, \mu, q)$  of solutions at  $t = 0$  and  $t = \infty$  by

$$U(t, \lambda, q) = W(f^{(0)}(t, \lambda, q), f^{(-1)}(t, q)), \quad V(t, \mu, q) = W(g^{(0)}(t, \mu, q), g^{(-1)}(t, q)), \quad (23)$$

with equal weights  $\kappa = (0, 1)$ .



**Theorem 1.6.** (a) *The monodromy matrix  $\vec{M}(t, \lambda, \mu, q) = V(t, \mu, q)^{-1}U(t, \lambda, q)$  is given by*

$$\vec{M}(t, \lambda, \mu, q) = \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix} \quad (24)$$

where

$$\vec{M}_{2,1}(t, \lambda, \mu, q) = -\frac{(q; q)_\infty^3 \theta(q^{-1}t; q) \theta(\lambda^{-1}\mu; q) \theta(\lambda^{-1}\mu^{-1}t^{-1}; q)}{\theta(\lambda^{-1}; q) \theta(\mu; q) \theta(\lambda^{-1}t^{-1}; q) \theta(\mu^{-1}t^{-1}; q)}. \quad (25)$$

The matrix  $\vec{M}$  satisfies the modular transformation (6).

(b) *The two cocycles (4)  $\Omega_U$  and  $\Omega_V$  are equal, modular, independent of  $\lambda$  and  $\mu$  and when  $\gamma = S$ , are given explicitly by*

$$\Omega_S(z, \tau) = \begin{pmatrix} 0 & 1 \\ 1 & -\tilde{t} \end{pmatrix} \begin{pmatrix} \tau^{-1} & 0 \\ \frac{e(\frac{1}{8})}{\sqrt{\tau}} \tilde{t}^{-\frac{1}{2}} \tilde{q}^{\frac{1}{8}} e\left(\frac{z^2}{2\tau}\right) h(z, \tau) & \frac{e(\frac{1}{8})}{\sqrt{\tau}} e\left(-\frac{(z+1/2-\tau/2)^2}{2\tau}\right) \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & -t \end{pmatrix}^{-1} \quad (26)$$

where

$$h(z, \tau) = \int_{\mathbb{R}} \frac{e^{\pi i(\tau x^2 + 2izx)}}{2 \cosh(\pi x)} dx. \quad (27)$$

is the Mordell integral.

Our third module is a linear  $q$ -difference equation associated to quantum invariants of the simplest hyperbolic knot  $4_1$ ,

$$tqf(qt; q) + (1 - 2t)f(t; q) + tq^{-1}f(q^{-1}t; q) = 0. \quad (28)$$

This equation, which is also related to the  $q$ -Hahn Bessel function, appeared in homogeneous form in [27, Eqn.(11)]. With the definition of  $f^{(-1)}$ ,  $f^{(1)}$  and  $g^{(0,0)}$  and  $g^{(0,1)}$  given in Equations (201), (205) and (209) below, we define the fundamental matrices  $U(t, \lambda, q)$  and  $V(t, q)$  of solutions at  $t = 0$  and  $t = \infty$  by

$$U(t, \lambda, q) = W(f^{(1)}(t, \lambda, q), f^{(-1)}(t, q)), \quad V(t, q) = W(g^{(0,0)}(t, q), g^{(0,1)}(t, q)), \quad (29)$$

and equal weights  $\kappa = (1, 0)$ .

**Theorem 1.7.** (a) *The monodromy matrix  $\vec{M}(t, \lambda, q) = V^{-1}(t, q)U(t, \lambda, q)$  is given by*

$$\vec{M}(t, \lambda, q) = \begin{pmatrix} -1 & 0 \\ \vec{M}_{2,1}(t, \lambda, q) & 1 \end{pmatrix} \quad (30)$$

where

$$\vec{M}_{2,1}(t, \lambda, q) = \frac{\theta(qt; q) \theta(t\lambda; q) \theta(t\lambda q^{-1/2}; q) \theta(t\lambda^2 q^{-1/2}; q)}{\theta(t\lambda q^{1/4}; q) \theta(-t\lambda q^{1/4}; q) \theta(t\lambda q^{-1/4}; q) \theta(-t\lambda q^{-1/4}; q) \theta(q^{-1}\lambda; q) \theta(q^{-3/2}\lambda; q)}. \quad (31)$$

The matrix  $M$  satisfies the modular transformations

$$\Delta_{T^2, \kappa} \vec{M}|_{\kappa} T^2 = \vec{M}, \quad \Delta_{STS, \kappa} \vec{M}|_{\kappa} STS = \vec{M}, \quad \Delta_{TST, \kappa} \vec{M}|_{\kappa} TST = \vec{M}. \quad (32)$$

It follows that the  $\mathrm{SL}_2(\mathbb{Z})$ -orbit of  $M$  consists of three functions  $M, \Delta_{T, \kappa} \vec{M}|_{\kappa} T, \Delta_{S, \kappa} \vec{M}|_{\kappa} S$ .

(b) *The two cocycles (4)  $\Omega_{\mathrm{Av}(U)}$  and  $\Omega_V$ , defined by the averaging*

$$\mathrm{Av}(U) = V \mathrm{Av}(\vec{M}), \quad \mathrm{Av}(\vec{M}) = \frac{1}{3} \left( \vec{M} + \Delta_{T, \kappa} \vec{M}|_{\kappa} T + \Delta_{S, \kappa} \vec{M}|_{\kappa} S \right), \quad (33)$$

are equal, modular, independent of  $\lambda$  and when  $\gamma = S$ , they are given by elementary functions times the state integrals

$$\int_{\mathbb{R}+i\varepsilon} \Phi_{\mathbf{b}}(x)^2 \exp(-\pi i x^2 - 2\pi \mathbf{b}^{-1} z x) dx. \quad (34)$$

The above can be phrased by saying that  $M$  is modular on an index three subgroup  $\Gamma'$  of  $\mathrm{SL}_2(\mathbb{Z})$  (conjugate to the  $\theta$  subgroup and to the congruence subgroup  $\Gamma_0(2)$  of  $\mathrm{SL}_2(\mathbb{Z})$ ) and if  $\gamma \in \Gamma'$  then  $\Omega_{U,\gamma} = \Omega_{V,\gamma}$  is independent of  $\lambda$  and modular. The corresponding state integrals are defined in forthcoming work [31]. However, the modularity of the module follows from Theorem 1.2.

The equation (28) has two important extensions, each containing important information about the knot. The first extension is an inhomogeneous version, which for the  $4_1$  knot, takes the form (see [37, Eqn.(98)] and [28, Sec.2.2])

$$tqf(qt; q) + (1 - 2t)f(t; q) + tq^{-1}f(q^{-1}t; q) = 1. \quad (35)$$

The solution  $f^{(0)}$  at  $t = 0$  given in Equation (255) below is a resummation of the Kashaev invariant of the  $4_1$  knot, whereas the solution  $g^{(0,2)}$  given in Equation (258) below is a  $q$ -series that appeared recently in [28] in relation to the asymptotic expansion of the Kashaev invariant at the trivial representation. With those two solutions, we define the fundamental matrices

$$\begin{aligned} U(t, \lambda_1, \lambda_2, q) &= W(f^{(1)}(t, \lambda_1, q), f^{(-1)}(t, q), f^{(0)}(t, \lambda_2, q)) \\ V(t, q) &= W(g^{(0,0)}(t, q), g^{(0,1)}(t, q), g^{(0,2)}(t, q)), \end{aligned} \quad (36)$$

with equal weights  $\kappa = (2, 1, 0)$ . The monodromy matrix involves the Weierstrass elliptic function  $\wp$ , a well-known function discussed in detail for example in [2].

**Theorem 1.8.** (a) *The monodromy matrix  $\vec{M}(t, \lambda_1, \lambda_2, q) = V(t, q)^{-1}U(t, \lambda_1, \lambda_2, q)$  is given by*

$$\vec{M}(t, \lambda_1, \lambda_2, q) = \begin{pmatrix} -1 & 0 & \wp(t, q) \\ \vec{M}_{2,1}(t, \lambda_1, q) & 1 & \frac{1}{2} \frac{\wp'(t, q) - \wp'(\lambda_2, q)}{\wp(t, q) - \wp(\lambda_2, q)} \\ 0 & 0 & 1 \end{pmatrix} \quad (37)$$

where  $\vec{M}_{2,1}$  is given by (31). The matrix  $M$  satisfies the modular transformations (32) with weight  $\kappa = (2, 1, 0)$ .

(b) *The two cocycles (4)  $\Omega_{\mathrm{Av}(U)}$  and  $\Omega_V$  are equal, modular, independent of  $\lambda_1, \lambda_2$  and when  $\gamma = S$ , they are given by combinations of elementary functions times the state integrals*

$$\int_{\mathbb{R}+i\varepsilon} \Phi_{\mathbf{b}}(x)^2 \frac{\exp(-\pi i x^2 - 2\pi \mathbf{b}^{-1} z x)}{1 + \tilde{q}^{1/2} \exp(-2\pi \mathbf{b}^{-1} x)} dx. \quad (38)$$

The second and last extension of equation (35) is the addition of an  $x$ -variable in  $\mathbb{C}^\times$ , which topologically measures the holonomy of the meridian of the knot, or the color of the colored Jones polynomial, and behaves like a Jacobi variable. The new equations are now a two variable holonomic system and take the form

$$tqf(qt, x, q) + (1 - (x^{-1} + x)t)f(t, x, q) + tq^{-1}f(q^{-1}t, x, q) = 1 \quad (39a)$$

$$\begin{aligned}
 (1 - qx)(1 - q^{-1}x^2)f(t, qx, q) \\
 - (x - 1)^2(x + 1)(x^2t - x - (q^{-1} + q)t - x^{-1} + x^{-2}t)f(t, x, q) \\
 + (1 - qx^2)(1 - q^{-1}x)f(t, q^{-1}x, q) = (1 + x^{-1})(1 - qx^2)(1 - q^{-1}x^2),
 \end{aligned} \tag{39b}$$

$$(1 - xq)(f(t, qx, q) - x^{-1}f(qt, qx, q)) = (1 - x^{-1})(f(t, x, q) - qxf(qt, x, q)). \tag{39c}$$

This is not a random system of equations, instead they are the defining equations of the descendant colored Jones polynomial of the  $4_1$  knot, and appeared explicitly in [28, Eqn.(97)]. This is a  $q$ -holonomic module of rank 3, with fundamental solutions  $f^{(j)}(t, x, q)$  for  $j = -1, 0, 1$  at  $t = 0$  and  $g^{(0, x^j)}(t, x, q)$  for  $j = -1, 0, 1$  at  $t = \infty$  defined in Equations (283), (290) and (291). With these solutions we can define the fundamental matrices with respect to the shift in  $t$  as

$$\begin{aligned}
 U(t, x, \lambda_1, \lambda_2, q) &= W(f^{(1)}(t, x, \lambda_1, q), f^{(-1)}(t, x, q), f^{(0)}(t, x, \lambda_2, q)), \\
 V(t, x, q) &= W(g^{(0, x^{-1})}(t, x, q), g^{(0, x)}(t, x, q), g^{(0, 1)}(t, x, q)),
 \end{aligned} \tag{40}$$

with weights  $\kappa_U = (0, 1, 0)$  and  $\kappa_V = (1, 1, 0)$ . The next theorem gives the properties of this monodromy.

**Theorem 1.9.** (a) *The monodromy matrix is given by*

$$\begin{aligned}
 \vec{M}(t, x, \lambda_1, \lambda_2, q) &= \begin{pmatrix} 1 & -1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \times \\
 &\left( \begin{array}{ccc} \frac{\theta(x^{-2}; q)\theta(t; q)^2(q; q)_\infty^3}{2\theta(q^{-1}x; q)^2\theta(tx; q)\theta(tx^{-1}; q)} & 0 & \frac{\theta(x^{-2}; q)\theta(t; q)^2(q; q)_\infty^3}{2\theta(q^{-1}x; q)^2\theta(tx; q)\theta(tx^{-1}; q)} \\ m_{2,1}(t, x, \lambda_1, q) & 1 & \left( \frac{\theta'(t\lambda_2)}{\theta(t\lambda_2)} - \frac{\theta'(tx)}{2\theta(tx)} - \frac{\theta'(tx^{-1})}{2\theta(tx^{-1})} - \frac{\theta'(\lambda_2)}{\theta(\lambda_2)} - \frac{1}{2} \right) \\ 0 & 0 & 1 \end{array} \right)
 \end{aligned} \tag{41}$$

where  $m_{2,1}(t, x, \lambda_1, q)$  is the unique elliptic function in  $t$  satisfying  $m_{2,1}(1, x, \lambda_1, q) = 0$  with simple poles at  $t_0 \in \{x^\pm q^{\mathbb{Z}}, \pm \lambda^{-1} q^{\frac{1}{2}\mathbb{Z}}\}$  and residues  $\rho_{t_0} := \text{Res}_{t=t_0} m_{2,1}(t, x, \lambda_1, q) \frac{dt}{2\pi i t}$  given by

$$\begin{aligned}
 \rho_{x^\pm q^{\mathbb{Z}}} &= \frac{-1}{2} \\
 \rho_{\pm \lambda_1^{-1} q^{-1/4 - \mathbb{Z}}} &= \frac{\theta(\pm q^{3/4} \lambda_1^{-1}; q)\theta(\pm q^{-1/4} x; q)\theta(\pm q^{-3/4} x; q)\theta(\pm q^{-3/4} \lambda_1; q)}{2\theta(x; q)\theta(x^{-1}; q)\theta(q^{-1} \lambda_1; q)\theta(q^{-3/2} \lambda_1; q)} \\
 \rho_{\pm \lambda_1^{-1} q^{-3/4 - \mathbb{Z}}} &= \frac{\theta(\pm q^{1/4} \lambda_1^{-1}; q)\theta(\pm q^{-1/4} x; q)\theta(\pm q^{-3/4} x; q)\theta(\pm q^{-1/4} \lambda_1; q)}{2\theta(x; q)\theta(x^{-1}; q)\theta(q^{-1/2} \lambda_1; q)\theta(q^{-1} \lambda_1; q)}.
 \end{aligned} \tag{42}$$

The monodromy satisfies the following modular transformations

$$\Delta_{T^2, \kappa_U} \vec{M}|_{\kappa_U} T^2 = \vec{M}, \quad \Delta_{STS, \kappa_V} \vec{M}|_{\kappa_V} STS = \vec{M}, \quad \Delta_{TST, \kappa_V} \vec{M}|_{\kappa_V} TST = \vec{M}. \tag{43}$$

(b) *The two cocycles (4)  $\Omega_{\text{Av}(U)}$  and  $\Omega_V$  are equal, modular, independent of  $\lambda_1, \lambda_2$  and when  $\gamma = S$ , they are given by combinations of elementary functions times the state integrals*

$$\int_C \Phi_b(x + ibu)\Phi_b(x - ibu) \frac{\exp(-\pi i x^2 - 2\pi \mathbf{b}^{-1} z x)}{1 + \tilde{q}^{1/2} \exp(-2\pi \mathbf{b}^{-1} x)} dx. \tag{44}$$

An explicit formula for the function  $m_{2,1}$  is given in Equation (296) below.

The next theorem identifies the function  $f^{(0)}(t, x, \lambda_1, q)$  with the  $q$ -Borel resummation of the descendant of the colored Jones polynomial, and provides a lift of the colored Jones polynomial (as a function of  $N$  and  $q$ ) to an analytic function of  $x$  and  $q$ . Its extension to all knots will be discussed in forthcoming work.

**Theorem 1.10.** *The  $N$ -th colored Jones polynomial  $J_N(q)$  of  $4_1$  for  $|q| < 1$  is given by*

$$J_N(q) = f^{(0)}(1, q^N, \lambda_2, q). \quad (45)$$

Note that there is no  $\lambda_2$  dependence on the left hand side of (45) and that  $\lambda_2$  parametrises a family of analytic continuations of the colored Jones polynomial as we vary  $x$  away from  $q^N$ .

The cocycles given in the previous theorems reveal the relation of the following three special functions (all being entries of a matrix-valued  $S$ -cocycle of a modular  $q$ -difference equation)

- (a) the Fadeev quantum dilogarithm
- (b) the Mordell integral
- (c) the Andersen-Kashaev state integral.

We end this section with some comments regarding modular  $q$ -holonomic modules. It is not obvious that  $q$ -holonomic modules exist or occur naturally. Yet, they are abundant, for instance all proper  $q$ -hypergeometric multidimensional sums are  $q$ -holonomic; see Zeilberger–Wilf [69]. A detailed introduction of  $q$ -holonomic modules and their functional and closure properties can be found in [33, 49, 63].

Regarding the occurrence of  $q$ -holonomic modules, they are often given in the form  $M_f = \mathcal{W}f$  where  $f : \mathbb{Z}^r \rightarrow \mathbb{Q}(q)$  is a quantum invariant. For instance, if  $f$  denotes the colored Jones polynomial of a link (colored with an arbitrary representation of a fixed simple Lie group), or the colored HOMFLY-PT polynomial of a link (colored by arbitrary partitions with a fixed number of rows or columns), the corresponding modules  $M_f$  are  $q$ -holonomic; see [34] and [32]. In addition, the special  $q$ -hypergeometric sums (so-called Nahm sums) studied in [37, Sec.4.7] that depend on the upper-half of a symplectic matrix, are  $q$ -holonomic. Moreover, the linear  $q$ -difference equations in quantum cohomology or in quantum  $K$ -theory are often (and perhaps always?) specialisations of  $q$ -hypergeometric series; see [61, 35, 44, 68]. Finally, the  $q$ -GKZ modules of Gelfand–Kapranov–Zelevinsky [39, 40, 41] which are constructed by combinatorial data (a matrix of integers) together with some “charge vectors” are  $q$ -holonomic.

The above discussion leads naturally to the following conjecture.

**Conjecture 1.11.** Every  $q$ -holonomic module associated to a proper  $q$ -hypergeometric multi-dimensional sum is modular.

## 2. A REVIEW OF LINEAR $q$ -DIFFERENCE EQUATIONS

**2.1. Preliminaries.** It is well-known that formal power series solutions to linear difference equations with a small parameter are typically factorially divergent series, which lead to

analytic solutions to the difference equation after applying the process of a Borel transformation, followed by a Laplace transformation. This subject is classical and well-studied, see for example [6, 15, 52, 53].

A corresponding theory for linear  $q$ -difference equations was developed recently by di Vizio, Sauloy, Ramis and others [20, 42, 64], with particular emphasis given on the arithmetic and the Galois theoretic aspects of the theory. A  $q$ -holonomic module has two canonical filtrations, one from  $t = 0$  and another from  $t = \infty$ . These filtrations can be computed concretely choosing a cyclic vector which converts the  $q$ -holonomic module into a linear  $q$ -difference equation.

To explain the solutions of linear  $q$ -difference equations, let us recall the Jacobi theta function which is given by a one dimensional lattice sum, and by an infinite product (known as the Jacobi triple product identity) by

$$\theta(t; q) = \sum_{k \in \mathbb{Z}} (-1)^k q^{k(k+1)/2} t^k = (qt; q)_\infty (t^{-1}; q)_\infty (q; q)_\infty. \quad (46)$$

From the above representation, it is easy to see that it satisfies the functional equations

$$\theta(t^{-1}; q) = \theta(q^{-1}t; q) = -t\theta(t; q) \quad (47)$$

which imply that

$$\theta(q^\ell t; q) = (-1)^\ell q^{-\ell(\ell+1)/2} t^{-\ell} \theta(t; q), \quad (\ell \in \mathbb{Z}). \quad (48)$$

The Jacobi theta function is modular. Its transformation under the element  $S$  of  $\mathrm{SL}_2(\mathbb{Z})$  is given by

$$\theta(\tilde{t}; \tilde{q}) = \mathbf{e}(-3/8) \mathbf{e}\left(\frac{z^2}{2\tau}\right) t^{\frac{1}{2}} \tilde{t}^{-\frac{1}{2}} q^{\frac{1}{8}} \tilde{q}^{-\frac{1}{8}} \sqrt{\tau} \theta(t; q). \quad (49)$$

The derivative of  $\theta$  with respect to  $td/dt = 1/(2\pi i)d/dz$ ,

$$\theta'(t; q) = \sum_k (-1)^k k q^{k(k+1)/2} t^k \quad (50)$$

satisfies the  $q$ -difference equation

$$\frac{\theta'(q^\ell t; q)}{\theta(q^\ell t; q)} = \frac{\theta'(t; q)}{\theta(t; q)} - \ell. \quad (51)$$

It has transformation under the element  $S$  of  $\mathrm{SL}_2(\mathbb{Z})$  given by

$$\frac{\theta'(t; q)}{\theta(t; q)} \Big|_1 S := \frac{1}{\tau} \frac{\theta'(\tilde{t}; \tilde{q})}{\theta(\tilde{t}; \tilde{q})} = \frac{\theta'(t; q)}{\theta(t; q)} + \frac{z}{\tau} + \frac{1}{2} - \frac{1}{2\tau}. \quad (52)$$

**2.2. An algorithm for a fundamental matrix.** In this section we review an algorithm to obtain a fundamental matrix solution to a  $q$ -holonomic module given by Dreyfus [20] using the  $q$ -Borel and the  $q$ -Laplace transform. Let us describe the main steps here.

- Choose a cyclic vector to present a  $q$ -holonomic module in the form of a linear  $q$ -difference equation

$$\sum_{j=0}^r a_j \sigma^j f = 0 \quad (53)$$

where  $f = f(t, q)$ ,  $(\sigma f)(t, q) = f(qt, q)$  and  $a_j(t, q) \in \mathbb{Q}(q)[t^{\pm 1}]$ , with  $a_r \neq 0$ . The rest of the algorithm depends on (53) alone.

- The lower Newton polygon of (53) is the lower convex hull of the points  $(j, v_0(a_j))$  for  $j = 0, \dots, r$ , where  $v_0(p)$  denotes the minimum  $t$ -exponent of a Laurent polynomial  $p(t)$  (with the convention that  $v_0(0) = \infty$ ). The boundary of the lower Newton polygon is a finite sequence of edges with increasing slopes.
- For each edge of the lower Newton polygon, we replace our function by a ratio of theta functions times a new function, so that the corresponding edge is now horizontal, and then apply the Frobenius method to get  $t$ -formal power series solutions, once for each root of the indicial polynomial of the edge. If  $\kappa$  is the slope of the edge and an integer, the solutions have the form

$$\theta(t; q)^\kappa \sum_{k=0}^{\infty} \alpha_k(q) t^k \frac{\theta(\rho^{-1}t; q)}{\theta(t; q)} \quad (54)$$

where  $\rho$  is determined by the roots of the indicial polynomial of the edge. A special feature in the equations that we study is that the roots of the indicial polynomials are roots of unity times a fractional power of  $q$  times a monomial in any additional variables.

- If a solution is not convergent at  $t = 0$ , apply a  $q$ -Borel transform, followed by an iterated  $q$ -Laplace transform (defined below), to construct a fundamental matrix  $U(t, q)$  of analytic solutions at  $t = 0$ . The  $q$ -Borel and  $q$ -Laplace transforms preserve linear  $q$ -difference equations and change their Newton polygons by an affine linear transformation.
- Repeat the above steps using the upper Newton polygon of (53), that is the upper convex hull of the points  $(j, v_\infty(a_j))$  for  $j = 0, \dots, r$ , where  $v_\infty(p)$  denotes the maximum  $t$ -exponent of a Laurent polynomial  $p(t)$  (with the convention that  $v_\infty(0) = -\infty$ ). Call the corresponding fundamental matrix  $V(t, q)$ .

The ratio  $M = V^{-1}U$  is a matrix of elliptic functions. These functions depend on additional elliptic parameters that come from the  $q$ -Laplace transform, a feature of  $q$ -difference equations which is absent in the world of linear differential equations. The connection problem, i.e., the determination of this matrix, is largely unsolved, with partial success for the case of many  $q$ -hypergeometric difference equations; see Ohyama, Morita [55, 56, 57, 58].

**2.3. The  $q$ -Borel and the  $q$ -Laplace transforms.** We now recall the  $q$ -Borel transform  $\mathcal{B}_\kappa$  (for a rational number  $\kappa$ ) defined by

$$\mathcal{B}_\kappa \left( \sum_{\ell=0}^{\infty} a_\ell t^\ell \right) (\xi, q) = \sum_{\ell=0}^{\infty} (-1)^\ell q^{\kappa\ell(\ell+1)/2} a_\ell \xi^\ell. \quad (55)$$

Its role is to convert divergent series, e.g., of the form  $\sum_{\ell=0}^{\infty} q^{-\kappa\ell(\ell+1)/2} a_\ell t^\ell$  (where  $a_\ell$  is bounded and  $\kappa > 0$ ) to convergent ones at  $\xi = 0$ .

An inverse of the  $q$ -Borel transform is the  $q$ -Laplace transform, whose role is to construct analytic solutions to the linear  $q$ -difference equation with prescribed asymptotics. It is defined for  $\kappa > 0$  by

$$\mathcal{L}_\kappa(f)(t, \lambda, q) = \frac{1}{\theta(\lambda; q^\kappa)} \sum_{\ell \in \mathbb{Z}} (-1)^\ell q^{\kappa\ell(\ell+1)/2} \lambda^\ell f(q^{\kappa\ell} \lambda t, q) = \sum_{\ell \in \mathbb{Z}} \frac{f(q^{\kappa\ell} \lambda t, q)}{\theta(q^{\kappa\ell} \lambda; q^\kappa)}, \quad (56)$$

and for  $\kappa < 0$ , by

$$\mathcal{L}_\kappa(f)(t, q) = \oint f(\xi, q) \theta(t/\xi; q^{-\kappa}) \frac{d\xi}{2\pi i \xi}. \quad (57)$$

The  $q$ -Laplace transform for positive  $\kappa$  is more common, however, in the computation of the monodromy of the  $4_1$  knot, we will use  $\kappa < 0$ , following analogous computations of Morita [54].

In some sense, the two transformations are inverse to each other. More precisely, for all  $\kappa$  and all natural numbers  $n$ , we have

$$\mathcal{L}_\kappa \mathcal{B}_\kappa(t^n)(t, \lambda, q) = t^n. \quad (58)$$

(Interestingly, the right hand side is independent of  $\lambda$ .) Indeed, for  $\kappa > 0$ , we have

$$\begin{aligned} \mathcal{L}_\kappa \mathcal{B}_\kappa(t^n) &= \frac{1}{\theta(\lambda; q^\kappa)} \sum_{\ell \in \mathbb{Z}} (-1)^\ell q^{\kappa \ell(\ell+1)/2} \lambda^\ell (-1)^n q^{\kappa n(n+1)/2} q^{\kappa \ell n} \lambda^n t^n \\ &= \frac{1}{\theta(\lambda; q^\kappa)} \sum_{\ell \in \mathbb{Z}} (-1)^{\ell+n} q^{\kappa(\ell+n)(\ell+n+1)/2} \lambda^{\ell+n} t^n \\ &= \frac{1}{\theta(\lambda; q^\kappa)} \sum_{\ell \in \mathbb{Z}} (-1)^\ell q^{\kappa \ell(\ell+1)/2} \lambda^\ell t^n = t^n. \end{aligned} \quad (59)$$

A similar calculation using the residue theorem shows (58) for  $\kappa < 0$ . More generally, the fact solutions constructed by Borel transforms followed by iterated Laplace transforms are asymptotic to the original formal power series is known in the literature as Watson's lemma, a modern proof of which may be found for instance in Miller [52, p. 53, Prop. 2.1]. An analogous lemma holds for the  $q$ -case, see [20, Prop.2.9].

By its very definition,  $\mathcal{L}_\kappa$  for  $\kappa > 0$  depends on the variable  $\lambda$  in an elliptic way

$$\mathcal{L}_\kappa(f)(t, q^\kappa \lambda, q) = \mathcal{L}_\kappa(f)(t, \lambda, q) \quad (60)$$

whereas  $\mathcal{L}_\kappa$  for  $\kappa < 0$  does not involve a variable  $\lambda$ . The next lemma, whose proof follows from an elementary application of the residue theorem, concerns the dependence of the  $q$ -Laplace transform on the auxiliary variable  $\lambda$ , and may be of independent interest.

**Lemma 2.1.** Assuming that

$$\lim_{\varepsilon \rightarrow 0} \oint_{|\xi|=\varepsilon^\pm} \frac{f(\xi t, q) \theta(\mu^{-1} \lambda^{-1} \xi; q)}{\theta(\xi \mu^{-1}; q) \theta(\xi \lambda^{-1}; q)} \frac{d\xi}{2\pi i \xi} = 0 \quad (61)$$

where  $\varepsilon$  avoids the poles of the integrand we have

$$\begin{aligned} &\mathcal{L}_1(f)(t, \lambda, q) - \mathcal{L}_1(f)(t, \mu, q) \\ &= \frac{\theta(\lambda^{-1} \mu; q) (q; q)_\infty^3}{\theta(\lambda^{-1}; q) \theta(\mu; q)} \sum_{x \in \text{poles of } f} \text{Res}_{\xi=x} \frac{f(\xi, q) \theta(\lambda^{-1} \mu^{-1} t^{-1} \xi; q)}{\theta(\xi \lambda^{-1} t^{-1}; q) \theta(\xi \mu^{-1} t^{-1}; q)}. \end{aligned} \quad (62)$$

Note that the assumption on  $f$  is mild since as  $t$  approaches 0 or  $\infty$  bounded away from the poles, the quotient of  $\theta$ 's approaches 0. Note also that the lemma can be extended to the case of  $\mathcal{L}_\kappa$  for  $\kappa > 0$  by substituting  $q \mapsto q^\kappa$  except in the argument of  $f$ .

*Proof.* We compute

$$\begin{aligned}
0 &= \sum_{\xi \in \text{poles}} \text{Res}_{\xi} \frac{f(\xi t, q) \theta(\mu^{-1} \lambda^{-1} \xi; q)}{\theta(\xi \mu^{-1}; q) \theta(\xi \lambda^{-1}; q)} \frac{d\xi}{2\pi i \xi} \\
&= \sum_k \text{Res}_{\xi=q^k \lambda} \frac{f(\xi t, q) \theta(\mu^{-1} \lambda^{-1} \xi; q)}{\theta(\xi \mu^{-1}; q) \theta(\xi \lambda^{-1}; q)} \frac{d\xi}{2\pi i \xi} + \sum_k \text{Res}_{\xi=q^k \mu} \frac{f(\xi t, q) \theta(\mu^{-1} \lambda^{-1} \xi; q)}{\theta(\xi \mu^{-1}; q) \theta(\xi \lambda^{-1}; q)} \frac{d\xi}{2\pi i \xi} \\
&\quad + \sum_{x \in \text{poles of } f} \text{Res}_{\xi=t^{-1}x} \frac{f(\xi t, q) \theta(\mu^{-1} \lambda^{-1} \xi; q)}{\theta(\xi \mu^{-1}; q) \theta(\xi \lambda^{-1}; q)} \frac{d\xi}{2\pi i \xi} \\
&= \sum_k (-1)^k q^{k(k+1)/2} \lambda^k \frac{f(q^k \lambda t, q) \theta(\mu^{-1}; q)}{\theta(\mu^{-1} \lambda; q) (q; q)_{\infty}^3} + \sum_k (-1)^k q^{k(k+1)/2} \mu^k \frac{f(q^k \mu t, q) \theta(\lambda^{-1}; q)}{\theta(\mu \lambda^{-1}; q) (q; q)_{\infty}^3} \quad (63) \\
&\quad + \sum_{x \in \text{poles of } f} \text{Res}_{\xi=t^{-1}x} \frac{f(\xi t, q) \theta(\mu^{-1} \lambda^{-1} \xi; q)}{\theta(\xi \mu^{-1}; q) \theta(\xi \lambda^{-1}; q)} \frac{d\xi}{2\pi i \xi} \\
&= \frac{\theta(\mu^{-1}; q) \theta(\lambda; q)}{\theta(\mu^{-1} \lambda; q) (q; q)_{\infty}^3} \mathcal{L}_1(f)(t, \lambda, q) + \frac{\theta(\mu; q) \theta(\lambda^{-1}; q)}{\theta(\mu \lambda^{-1}; q) (q; q)_{\infty}^3} \mathcal{L}_1(f)(t, \mu, q) \\
&\quad + \sum_{x \in \text{poles of } f} \text{Res}_{\xi=x} \frac{f(\xi, q) \theta(\mu^{-1} \lambda^{-1} t^{-1} \xi; q)}{\theta(\xi \mu^{-1} t^{-1}; q) \theta(\xi \lambda^{-1} t^{-1}; q)} \frac{d\xi}{2\pi i \xi}.
\end{aligned}$$

□

This type of residue formula for the Laplace transform is similar to the definition of the Laplace transform for  $\kappa < 0$ . We can find a similar expression for a single Laplace transform using a special function. The Appell-Lerch sum will be studied later in Section 4.2 but we will define it here.

$$L(t, \lambda, q) = \frac{1}{\theta(\lambda; q)} \sum_{k \in \mathbb{Z}} (-1)^k \frac{q^{k(k+1)/2} \lambda^k}{1 - q^k \lambda t}. \quad (64)$$

Using this we have the following integral expression for the Laplace transform for  $\kappa > 0$ .

**Lemma 2.2.** For  $\kappa > 0$ , we have

$$(\mathcal{L}_{\kappa} f)(t, \lambda, q) = \sum_{k \in \mathbb{Z}} \text{Res}_{\xi=q^{\kappa} k} L(\xi^{-1} \lambda^{-1}, \lambda, q^{\kappa}) f(\xi \lambda t, q) \frac{d\xi}{2\pi i \xi}. \quad (65)$$

Deforming the contour and the residue theorem give the following lemma.

**Lemma 2.3.** Assuming that

$$\lim_{\varepsilon \rightarrow 0} \oint_{|\xi|=\varepsilon^{\pm}} L(\xi^{-1} \lambda^{-1}, \lambda, q^{\kappa}) f(\xi \lambda t, q) \frac{d\xi}{2\pi i \xi} = 0 \quad (66)$$

where  $\varepsilon$  avoids the poles of the integrand we have

$$(\mathcal{L}_{\kappa} f)(t, \lambda, q) = - \sum_{x \in \text{poles of } f} L(x^{-1} t, \lambda, q) \text{Res}_{\xi=x} f(\xi, q) \frac{d\xi}{2\pi i \xi} \quad (67)$$



Notice that all the dependence on  $t$  and  $\lambda$  is now in the arguments of the Appell-Lerch sums. This illustrates the important role the residues of the Borel transform play in the resummation. An application of this lemma leads to the following remarkable formula.

**Corollary 2.4.**

$$f^{(1)}(t, \lambda, q) = -\theta(t; q) \sum_{k=0}^{\infty} (L(q^{1/4+k/2}t, \lambda, q^{1/2})R_+(k, q) + L(-q^{1/4+k/2}t, \lambda, q^{1/2})R_-(k, q)) \quad (68)$$

where  $f^{(1)}$  is given in Equation (205) and  $R_{\pm}$  is given in Equation (235).

**2.4. The slash operator and proof of Theorem 1.2.** In this section we recall the slash operator, an important ingredient to express modularity. We will use the usual conventions for the modular  $q = \mathbf{e}(\tau)$  and the Jacobi  $t = \mathbf{e}(z)$ ,  $\lambda_i = \mathbf{e}(u_i)$ ,  $\mu_i = \mathbf{e}(v_i)$ ,  $x = \mathbf{e}(w)$  variables, i.e.,  $\tilde{q} = \mathbf{e}(-1/\tau)$ ,  $\tilde{t} = \mathbf{e}(z/\tau)$  and  $\mathbf{e}(x) = e^{2\pi ix}$ . Now recall the slash operator  $f|_{\kappa}\gamma$  (see, eg. [12, p.13] and [21]) for  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$  acting on a function  $f(z, \tau)$  by

$$(f|_{\kappa}\gamma)(z, \tau) = (c\tau + d)^{-\kappa} f(\gamma(z, \tau)), \quad \gamma(z, \tau) = \left( \frac{z}{c\tau + d}, \frac{a\tau + b}{c\tau + d} \right). \quad (69)$$

This action can be extended to a matrix-valued function  $F(z, \tau)$  of weight  $\kappa = \mathrm{diag}(\kappa_1, \dots, \kappa_r)$  by

$$(F|_{\kappa}\gamma)(z, \tau) = F(\gamma(z, \tau)) \mathrm{diag}((c\tau + d)^{-\kappa}). \quad (70)$$

This extension satisfies

$$\begin{aligned} (F|_{\kappa}\gamma)|_{\kappa}\gamma' &= F|_{\kappa}\gamma\gamma' \\ (FG)|_{\kappa}\gamma &= (F|_{\kappa}\gamma)\Delta_{\gamma}(G|_{\kappa}\gamma) \end{aligned} \quad (71)$$

for matrix-valued functions  $F$  and  $G$  and for  $\gamma, \gamma' \in \mathrm{SL}_2(\mathbb{Z})$  where  $\Delta_{\gamma}(\tau) = \mathrm{diag}((c\tau + d)^{\kappa})$ . We can extend these definitions to include half integral weight using a multiplier system as done for the Dedekind  $\eta$ -function. In all our examples the relative weights  $\kappa_i - \kappa_j$  are integers and the absolute weights are either always integers, or always half-integers. Our choice of absolute weight depends on the normalisation of our solutions, and multiplying them by  $\eta$ -functions leads to a shift of the absolute weights by half-integers, but has no effect on the modularity of the linear  $q$ -difference equation.

Recall the cocycle  $\Omega_{U, \gamma}$  from Equation (4), and the corresponding cocycle  $\Omega_{V, \gamma}$ . If the monodromy matrix  $M = V^{-1}U$  satisfies Equation (6), it follows that the cocycle matrices associated to  $U$  and  $V$  are equal:

$$\begin{aligned} \Omega_{U, \gamma} &= (V\vec{M}|_{\kappa_U}\gamma)(V\vec{M})^{-1} = (V|_{\kappa_V}\gamma)\Delta_{\kappa_V, \gamma}(\vec{M}|_{\kappa_U}\gamma)(V\vec{M})^{-1} \\ &= (V|_{\kappa_V}\gamma)\Delta_{\kappa_V, \gamma}(\vec{M}|_{\kappa_U}\gamma)\vec{M}^{-1}V^{-1} = (V|_{\kappa_V}\gamma)V^{-1} = \Omega_{V, \gamma}. \end{aligned} \quad (72)$$

We remark that sometimes the monodromy matrix of the fundamental bases  $U$  and  $V$  that come from the algorithm of Section 2.2 satisfies Equation (6) on a finite index subgroup of  $\mathrm{SL}_2(\mathbb{Z})$  (this happens, e.g., in Theorem 1.7) which contains a conjugate of a congruence subgroup of  $\mathrm{SL}_2(\mathbb{Z})$ . In this case, an averaging of the monodromy leads to fundamental solutions whose cocycle extends for all  $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ .

In the rest of this subsection, we give a proof of Theorem 1.2

*Proof of Theorem 1.2.* Using the behaviour of the slash operator on the product of two matrices (71), it is easy to see that if Equation (6) holds for two elements of  $\mathrm{SL}_2(\mathbb{Z})$ , it also holds for their product. Since  $S$  and  $T$  generate  $\mathrm{SL}_2(\mathbb{Z})$ , part (a) follows.

Part (b) follows from the presentation of  $\mathrm{SL}_2(\mathbb{Z})$  given by

$$\mathrm{SL}_2(\mathbb{Z}) = \langle S, T \mid S^4 = I, TSTST = S \rangle \quad (73)$$

and from the cocycle property, which implies that if  $\gamma = T^{a_0}ST^{a_1}ST^{a_2} \dots ST^{a_r}ST^{a_{r+1}}$ , then

$$\Omega_\gamma(z, \tau) = \prod_{j=1}^r \Omega_S((T^{a_j}ST^{a_{j+1}} \dots T^{a_r}ST^{a_{r+1}})(z, \tau)). \quad (74)$$

The idea for part (c) is to use reduction theory, with attention paid to the domain of extension. Fix a cocycle  $\Omega$  that satisfies  $\Omega_T = I$  and  $\Omega_S$  extends as a meromorphic function to  $\mathbb{C} \times \mathbb{C}'$ . Below, when we say that  $\Omega_\gamma$  extends, we will mean that it extends to  $\mathbb{C} \times \mathbb{C}'$ . We will give the proof in several steps.

**Step 1.** The cocycle property implies that

$$\Omega_{\gamma^{-1}}(z, \tau) = (\Omega_\gamma(\gamma^{-1}(z, \tau)))^{-1}. \quad (75)$$

This, together with the fact that  $\gamma^{-1}(\tau) - \gamma^{-1}(\infty) = 1/(c(-c\tau + a))$  imply that  $\Omega_\gamma$  extends if and only if  $\Omega_{\gamma^{-1}}$  extends. In particular, applying it to  $\gamma = S$ , where  $\gamma^{-1} = -S$ , we obtain that

$$\Omega_{-S}(z, \tau) = (\Omega_S(-z/\tau, -1/\tau))^{-1}. \quad (76)$$

Using our assumptions on  $\Omega_S$ , it follows that  $\Omega_{-S}$  extends.

**Step 2.** Suppose now that  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$  with  $a > 0$  and  $c > 0$ . It follows by Gauss reduction theory that we can write

$$\gamma = T^{a_0}ST^{a_1}S \dots T^{a_r}ST^{a_{r+1}} \quad (77)$$

for integers  $a_i$  where  $a_0 \geq 0$  and  $a_1, \dots, a_r > 0$ . Moreover,  $a_0, \dots, a_r$  can be obtained from the negative continued fraction expansion (using nearest integers from above, rather than from below)

$$\frac{a}{c} = [a_0, a_1, \dots, a_r] := a_0 - 1/(a_1 - 1/(a_2 - \dots)). \quad (78)$$

The continued fraction expansion shows that the first column of  $\gamma$  agrees with that of the product  $T^{a_0}ST^{a_1}S \dots T^{a_r}S$ , and the last integer  $a_{r+1}$  is chosen so that Equation (77) holds. Equation (74) implies that  $\Omega_\gamma(z, \tau)$  is a product of  $r$  matrices  $\Omega_S$  matrices evaluated suitably, and  $\Omega_\gamma$  extends to real  $\tau$  that satisfy  $(T^{a_j}ST^{a_{j+1}} \dots T^{a_r}ST^{a_{r+1}})(\tau) > 0$  for all  $j = 1, \dots, r$ . The key property is that the first column of the matrices  $T^{a_j}ST^{a_{j+1}} \dots T^{a_r}ST^{a_{r+1}}$  for  $j = 1, \dots, r$  consists of positive integers (with the possible exception of  $j = r + 1$  where the  $(2, 1)$  entry may be zero). It follows that the system of inequalities cascades, and becomes equivalent to the single inequality  $\tau > -d/c$ . It follows that  $\Omega_\gamma$  extends in the case when  $\gamma = \begin{pmatrix} + & * \\ + & * \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ .

The above argument is best explained by an example. Consider the matrix  $\gamma = \begin{pmatrix} 17 & 29 \\ 7 & 12 \end{pmatrix}$ . We expand the rational number  $17/7$  of its first column in negative continued fractions

$$\frac{17}{7} = [3, 2, 4] = 3 - \frac{4}{7} = 3 - \frac{1}{7/4} = 3 - \frac{1}{2 - 1/4}$$

and obtain that the matrix  $T^3ST^2ST^4S = \begin{pmatrix} 17 & -5 \\ 7 & -2 \end{pmatrix}$ , which further adjusting it by multiplying it on the right by  $T^2$ , gives

$$\gamma = \begin{pmatrix} 17 & -5 \\ 7 & -2 \end{pmatrix} T^2 = T^3ST^2ST^4ST^2.$$

The cocycle property and the fact that

$$T^2ST^4ST^2 = \begin{pmatrix} 7 & 12 \\ 4 & 7 \end{pmatrix}, \quad T^4ST^2 = \begin{pmatrix} 4 & 7 \\ 1 & 2 \end{pmatrix}, \quad T^2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

implies that

$$\Omega_\gamma(z, \tau) = \Omega_S \left( \frac{z}{4\tau + 7}, \frac{7\tau + 12}{4\tau + 7} \right) \Omega_S \left( \frac{z}{\tau + 2}, \frac{4\tau + 7}{\tau + 2} \right) \Omega_S(z, \tau + 2). \quad (79)$$

The right hand side extends when  $\tau$  is real that satisfies

$$\frac{7\tau + 12}{4\tau + 7} > 0, \quad \frac{4\tau + 7}{\tau + 2} > 0, \quad \tau + 2 > 0$$

which (when reading the inequalities from last to first and simplifying) is equivalent to the system  $\tau + 2 > 0, 4\tau + 7 > 0, 7\tau + 12 > 0$ , which is equivalent to  $\tau > -12/7 = \gamma^{-1}(\infty)$ .

**Step 3.** We will now use the element  $\varepsilon = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z})$  of order 2, and observe that  $\varepsilon \begin{pmatrix} a & b \\ c & d \end{pmatrix} \varepsilon = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$  for all  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ . In particular,  $\varepsilon T \varepsilon = T^{-1}$  and  $\varepsilon S \varepsilon = -S$ . It follows that if  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$  with  $a > 0$  and  $c > 0$  is given by (77), then

$$\varepsilon \gamma \varepsilon = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix} = T^{-a_0}(-S)T^{-a_1}(-S) \dots T^{-a_r}(-S)T^{-a_{r+1}}. \quad (80)$$

The cocycle property implies that

$$\Omega_{\varepsilon \gamma \varepsilon}(z, \tau) = \prod_{j=1}^r \Omega_{-S}((T^{-a_j}(-S)T^{-a_{j+1}} \dots T^{-a_r}(-S)T^{-a_{r+1}})(z, \tau)) \quad (81)$$

where  $\Omega_{-S}(z, \tau)$  extends for  $\tau < 0$  by Step 1. Thus  $\Omega_{\varepsilon \gamma \varepsilon}$  extends when the inequalities  $(T^{-a_j}(-S)T^{-a_{j+1}} \dots T^{-a_r}(-S)T^{-a_{r+1}})(\tau) < 0$  for  $j = 1, \dots, r$ . The key point now is that these inequalities cascade to a single inequality, namely,  $\tau < d/c = (\varepsilon \gamma \varepsilon)^{-1}(\infty)$ . It follows that  $\Omega_\gamma$  extends when  $\gamma = \begin{pmatrix} + & * \\ - & * \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ . In our running example above  $\gamma = \begin{pmatrix} 17 & 29 \\ 7 & 12 \end{pmatrix}$ , we have

$$\begin{pmatrix} 17 & -20 \\ -7 & 12 \end{pmatrix} = \varepsilon \gamma \varepsilon = T^{-3}(-S)T^{-2}(-S)T^{-4}(-S)T^{-2} \quad (82)$$

and the cocycle property and the fact that

$$T^{-2}(-S)T^{-4}(-S)T^{-2} = \begin{pmatrix} 7 & -12 \\ -4 & 7 \end{pmatrix}, \quad T^{-4}(-S)T^{-2} = \begin{pmatrix} 4 & -7 \\ -1 & 2 \end{pmatrix}, \quad T^{-2} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$$

implies that

$$\Omega_{\varepsilon \gamma \varepsilon} = \Omega_{-S} \left( \frac{z}{-4\tau + 7}, \frac{7\tau - 12}{-4\tau + 7} \right) \Omega_{-S} \left( \frac{z}{-\tau + 2}, \frac{4\tau - 7}{-\tau + 2} \right) \Omega_{-S}(z, \tau - 2). \quad (83)$$

The right hand side extends when  $\tau$  is real that satisfies

$$\frac{7\tau - 12}{-4\tau + 7} < 0, \quad \frac{4\tau - 7}{-\tau + 2} < 0, \quad \tau - 2 < 0$$

which is equivalent to the system  $7\tau - 12 < 0, 4\tau - 7 < 0, \tau - 2 < 0$  which is equivalent to  $\tau < 12/7 = (\varepsilon\gamma\varepsilon)^{-1}(\infty)$ .

**Step 4.** The cocycle property and the triviality of  $\Omega_T = I$  implies that  $\Omega_{T\gamma} = \Omega_\gamma$ . Since  $T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+c & b+d \\ c & d \end{pmatrix}$ , it follows that  $\Omega_\gamma$  (and hence its extension) depends only on the bottom row  $(c, d)$  of  $\gamma$ . When  $\gamma_{2,1} \neq 0$ , the extension follows from either Step 2 or Step 3, depending on the sign of  $\gamma_{2,1}$ .

This concludes the proof of the theorem.  $\square$

**2.5. Duality.** In this section we review some elementary facts about duality of  $q$ -holonomic modules. Recall that we can write the linear  $q$ -difference equation

$$a_r(t, q)f(q^r t, q) + a_{r-1}(t, q)f(q^{r-1}t, q) + \cdots + a_0(t, q)f(t, q) = 0 \quad (84)$$

for a function  $f(t, q)$  in matrix form  $\sigma X = AX$  where  $X = X_f = (f, \sigma f, \dots, \sigma^{r-1}f)^t$  is a column vector and  $A = \text{comp}(-a_0/a_r, \dots, -a_{r-1}/a_r)$  is the companion matrix where

$$\text{comp}(c_0, c_1, \dots, c_{r-1}) = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ c_0 & c_1 & c_2 & \dots & c_{r-1} \end{pmatrix}. \quad (85)$$

We will also write (84) in operator form  $Lf = 0$  where  $L = \sum_{j=0}^r a_j \sigma^{r-j} \in \mathcal{W}$  and denote by  $M_f = \mathcal{W}f$  the corresponding module over the  $q$ -Weyl algebra  $\mathcal{W}$ . The module  $M_f$  has two duals. The first one is defined by

$$M_f^\wedge = M_{f^\wedge}, \quad f^\wedge(t, q) = f(t, q^{-1}) \quad (86)$$

which in matrix form is given by

$$\sigma X_{f^\wedge} = A^\wedge X_{f^\wedge}, \quad A^\wedge(t, q) = A(qt, q^{-1})^{-1}. \quad (87)$$

Indeed, we have

$$a_r(t, q^{-1})f^\wedge(q^{-r}t, q) + a_{r-1}(t, q^{-1})f^\wedge(q^{-r+1}t, q) + \cdots + a_0(t, q^{-1})f^\wedge(t, q) = 0$$

and inverting  $a_r$ , we obtain that

$$f^\wedge(qt, q) = -\frac{a_r(qt, q^{-1})}{a_0(qt, q^{-1})}f^\wedge(q^{1-r}t, q) - \frac{a_{r-1}(qt, q^{-1})}{a_0(qt, q^{-1})}f^\wedge(t, q) - \cdots - \frac{a_1(qt, q^{-1})}{a_0(qt, q^{-1})}f^\wedge(t, q)$$

which implies (87). The second dual module is defined by

$$M_f^\vee = \text{Hom}_{\mathbb{Q}(q)[t^{\pm 1}]}(M_f, \mathbb{Q}(q)[t^{\pm 1}]) \quad (88)$$

with a basis  $f_i^\vee$  for  $i = 0, \dots, r-1$  such that for  $j = 0, \dots, r-1$  we have  $f_i^\vee(\sigma^j f) = \delta_{i,j}$ . We claim that in matrix form, this dual module is given by

$$\sigma X_{f^\vee} = A^\vee X_{f^\vee}, \quad A^\vee(t, q) = (A(t, q)^{-1})^t. \quad (89)$$

Indeed, using the basis  $f^\vee$ , we can define an action of  $\mathcal{W}$  on  $M_f^\vee$  via conjugation with  $\sigma$ . By definition, the action satisfies  $\sigma(\lambda(v)) = (\sigma \cdot \lambda)(\sigma v)$  for  $\lambda \in M_f^\vee$  and  $v \in M_f$ . In particular we have  $\sigma \cdot f_i^\vee = \sigma f_i^\vee \sigma^{-1}$ . Notice that for  $i, j = 0, \dots, r-1$

$$\begin{aligned} (\sigma \cdot f_i^\vee)(\sigma^j f) &= (\sigma f_i^\vee)(\sigma^{j-1} f) = (\sigma f_i^\vee) \left( \sum_{k=0}^{r-1} (A(q^{-1}t, q)^{-1})_{jk} \sigma^k f \right) \\ &= \sigma \left( \sum_{k=0}^{r-1} (A(q^{-1}t, q)^{-1})_{jk} \delta_{ik} \right) = (A(t, q)^{-1})_{ji} = \left( \sum_{k=0}^{r-1} (A(t, q)^{-t})_{ik} f_k^\vee \right) (\sigma^j f) \end{aligned}$$

which implies (89).

Recall that if two modules with companion matrices  $A$  and  $B$  are isomorphic, there exists a change of basis  $P$  such that

$$B = (\sigma P) A P^{-1}. \quad (90)$$

Then taking

$$P^\vee(t, q) = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ a_1/a_0 & 1 & 0 & \dots & 0 \\ a_2/a_0 & \sigma(a_1/a_0) & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ a_{r-1}/a_0 & \sigma(a_{r-2}/a_0) & \sigma(a_{r-3}/a_0) & \dots & 1 \end{pmatrix}, \quad P^\wedge(t, q) = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 1 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{pmatrix} \quad (91)$$

we find that

$$\begin{aligned} (\sigma P^\vee)^{-1} A^\vee P^\vee &= \text{comp}(-a_r/a_0, \sigma(a_{r-1}/a_0), \dots, \sigma^r(a_1/a_0)) \\ (\sigma P^\wedge)^{-1} A^\wedge P^\wedge &= \text{comp} \left( -\frac{a_r(qt, q^{-1})}{a_0(qt, q^{-1})}, \frac{a_{r-1}(qt, q^{-1})}{a_0(qt, q^{-1})}, \dots, \frac{a_1(qt, q^{-1})}{a_0(qt, q^{-1})} \right). \end{aligned} \quad (92)$$

We now remark an elementary relation between fundamental solutions of inhomogeneous linear  $q$ -difference equations and their corresponding homogeneous ones. Consider the inhomogeneous equation

$$a_0(t, q)f(t, q) + \dots + a_{r-1}(t, q)f(q^{r-1}t, q) + f(q^r t, q) = c_0(q). \quad (93)$$

We can write it either in the form  $\sigma X_{f^{(in)}} = A^{(in)} X_{f^{(in)}}$  with

$$X_{f^{(in)}} = \begin{pmatrix} 1 \\ f \\ \vdots \\ \sigma^{r-1} f \end{pmatrix}, \quad A^{(in)} = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 \\ c_0 & -a_0 & -a_1 & -a_2 & \dots & -a_{r-1} \end{pmatrix}$$

or in the form  $\sigma X = AX$  with

$$X_f = (f, \sigma f, \dots, \sigma^r f)^t, \quad A = \text{comp}(-a_0, \sigma a_0 - a_1, \sigma a_1 - a_2, \dots, \sigma a_{r-2} - a_{r-1}, \sigma a_{r-1} - 1).$$

The two equations are related by  $X_f = P X_{f^{(in)}}$ ,  $A = \sigma P A^{(in)} P^{-1}$  where

$$P = \text{comp}(c_0, -a_0, -a_1, -a_2, \dots, -a_{r-1}).$$

We end this subsection by discussing the duality

$$M^\vee \cong M^\wedge \quad (94)$$

which follows from (but is not equivalent to) the existence of matrices  $P^\vee, P^\wedge \in GL_r(\mathbb{Q}(t, q))$  such that

$$\sigma P^\wedge A^\wedge (P^\wedge)^{-1} = \sigma P^\vee A^\vee (P^\vee)^{-1} \quad (95)$$

is a companion matrix. For the  $q$ -holonomic modules that come from Chern-Simons theory, the duality  $M \mapsto M^\wedge$  corresponds to orientation-reversal of the ambient 3-manifold. On the other hand, the factorisation of the Andersen–Kashaev state integrals into elements of  $M$  and  $M^\wedge$  suggests that in those examples, we have  $M^\wedge \cong M^\vee$ . The following proposition confirms this for the case of the  $4_1$  knot.

**Proposition 2.5.** The  $q$ -difference module  $M$  associated to Equation (28) satisfies

$$M \cong M^\wedge \cong M^\vee. \quad (96)$$

The fundamental matrices satisfy

$$\begin{aligned} U(t, \lambda, q) &= P^\wedge(t, q)U^\wedge(t, \lambda, q) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = P^\vee(t, q)U^\vee(t, \lambda, q) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \\ V(t, q) &= P^\wedge(t, q)V^\wedge(t, q) = P^\vee(t, q)V^\vee(t, q) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \end{aligned} \quad (97)$$

with

$$P^\wedge(t, q) = \begin{pmatrix} 1 & 0 \\ 2q^{-1} - t^{-1}q^{-1} & -q^{-2} \end{pmatrix}, \quad P^\vee(t, q) = \begin{pmatrix} 0 & -q^{-1}t^{-2} \\ q^{-1}t^{-2} & 0 \end{pmatrix} \quad (98)$$

and cocycle  $\Omega$  of  $M$  satisfies

$$\Omega = (P^\wedge|_{\kappa\gamma})\Omega^\wedge(P^\wedge)^{-1} = (P^\vee|_{\kappa\gamma})\Omega^\vee(P^\vee)^{-1}. \quad (99)$$

Equation (99) was called ‘‘quadratic relations’’ in Section 3.3 and equations (68)-(70) of [37]. An example of a self-dual module is the generalised  $q$ -hypergeometric equation (see Equation (119) below).

*Proof.* With  $A$  as in Equation (238),

$$A(t, q) = \begin{pmatrix} 0 & 1 \\ -q^{-2} & 2q^{-1} - q^{-2}t^{-1} \end{pmatrix} \quad (100)$$

and the gauge transformations (98), we have

$$A(t, q) = P^\wedge(qt, q)A^\wedge(t, q)P^\wedge(t, q)^{-1} = P^\vee(qt, q)A^\vee(t, q)P^\vee(t, q)^{-1}. \quad (101)$$

Then note that by definition

$$U^\wedge(t, q) = U(t, q^{-1}) \quad \text{and} \quad U^\vee(t, q) = (U(t, q)^{-1})^t \quad (102)$$

and similarly for  $V^\wedge$  and  $V^\vee$ . We then see that for example

$$U(t, q)^{-1}P^\wedge(t, q)U^\wedge(t, q) \quad (103)$$

is elliptic. It can then be shown to be holomorphic for  $t \in \mathbb{C}^\times$  and therefore constant in  $t$ . Finally checking the behaviour as  $t \rightarrow 0$  for example we can prove they satisfy equation (97). Similar arguments hold for the other functions.  $\square$

The extra symmetry with  $M \cong M^\wedge$  comes from the fact the  $4_1$  knot is amphichiral. However, this symmetry does not persist to the module associated to the inhomogeneous equation. This can again be seen from the state integrals introduced in [28] whose integrand lacks the symmetry the Andersen-Kashaev state integrals have.

**Proposition 2.6.** The  $q$ -difference module  $M^\vee$  associated to Equation (35) is not isomorphic to  $M^\wedge$ .

*Proof.* The companion matrices of  $M$  and  $M^\wedge$  are given by

$$A^\wedge(t, q) = \begin{pmatrix} 1 & 0 & 0 \\ q^{-1}t^{-1} & 2q^{-1} - t^{-1}q^{-1} & -q^{-2} \\ 0 & 1 & 0 \end{pmatrix}, \quad A^\vee(t, q) = \begin{pmatrix} 1 & t^{-1} & 0 \\ 0 & 2q - t^{-1} & 1 \\ 0 & -q^2 & 0 \end{pmatrix}. \quad (104)$$

If there was an isomorphism there would exist  $P(t, q) \in \mathrm{GL}_3(\mathbb{Q}(t, q))$  such that

$$P(qt, q)A^\vee(t, q) = A^\wedge(t, q)P(t, q). \quad (105)$$

It follows that

$$P_{1,1}(qt, q) = P_{1,1}(t, q), \quad P_{1,2}(qt, q) = P_{1,3}(t, q), \quad (106)$$

which then implies

$$tP_{1,2}(t, q) + (1 - 2tq)P_{1,2}(qt, q) + q^2tP_{1,2}(q^2t, q) = P_{1,1}(qt, q). \quad (107)$$

Since  $P_{1,1} \in \mathbb{Q}(t, q)$  satisfies (106), it is independent of  $t$ , i.e.,  $P_{1,1}(t, q) = P_{1,1}(q)$ . Therefore,  $P_{1,2}$  would be a  $P_{1,1}(q)$  multiple of a  $\mathbb{Q}(t, q)$ -valued solution to Equation (35). The only such solution is zero, thus  $P_{1,1} = P_{1,2} = 0$  which, together with (106) gives also  $P_{1,3} = 0$ , which violates the fact that  $P$  is invertible.  $\square$

**2.6. Categorical aspects.** In this section we briefly recall some categorical aspects of modules over the  $q$ -Weyl algebra and give a proof of Lemma 1.3.

To begin with, a gauge transformation  $X = P^{-1}Y$  changes (3) to  $\sigma Y = BY$  where  $B = \sigma P A P^{-1}$ , changes a fundamental solution  $U$  of (3) to  $P^{-1}U$ , and consequently changes  $\Omega_\gamma$  to  $(P|_\kappa \gamma)^{-1} \Omega P$ . Hence, if  $P \in \mathrm{GL}_r(\mathbb{Q}(t, q))$ , then modularity is a property of the gauge equivalence class of a linear  $q$ -difference equation, i.e., a property of a  $q$ -holonomic module, independent of a choice of a cyclic vector. This concludes part (a) of Lemma 1.3.

For part (b), we use the covariant function  $M \mapsto \mathrm{Sol}(M) := \mathrm{Ker}(\sigma, \mathcal{F} \otimes_{\mathbb{Q}(t, q)} M)$  where  $\mathcal{F}$  denotes a universal  $q$ -difference field; see for example [67, Sec.2.2] for the case of modules over the Weyl algebra and [42] for its extension for the  $q$ -Weyl algebra  $\mathcal{W}$ . This functor by definition satisfies [67, Lem.2.16]

$$\mathrm{Sol}(M) \cong \{y \in \mathcal{F}^r \mid \sigma y = Ay\} \quad (108)$$

where  $A$  is the matrix obtained by a choice of a cyclic vector of  $M$ . Moreover, if

$$0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$$

is a short exact sequence of finitely generated  $\mathcal{W}$ -modules, then

$$0 \rightarrow \mathrm{Sol}(M') \rightarrow \mathrm{Sol}(M) \rightarrow \mathrm{Sol}(M'') \rightarrow 0 \quad (109)$$

is a short exact sequence of vector spaces over  $\mathbb{Q}(t, q)$ ; see [67, Sec.2.2]. In addition,  $M$  has a canonical filtration at  $t = 0$  (and also at  $t = \infty$ ) independent of the choice of cyclic

vector [66] compatible with submodules and quotient modules. Converting the above into matrices, it follows that the filtration preserving fundamental matrices  $U$ ,  $U'$  and  $U''$  of  $M$ ,  $M'$  and  $M''$  (and likewise,  $V$ ,  $V'$  and  $V''$ ) and their corresponding cocycles are related by

$$U = \begin{pmatrix} U' & * \\ 0 & U'' \end{pmatrix}, \quad \Omega_U = \begin{pmatrix} \Omega_{U'} & * \\ 0 & \Omega_{U''} \end{pmatrix}. \quad (110)$$

It follows that if  $\Omega_U$  extends to the cut plane, so does  $\Omega_{U'}$  and  $\Omega_{U''}$ , concluding part (b). It is unlikely that the converse to part (b) holds, namely if extensions of modular  $q$ -holonomic modules are  $q$ -holonomic, but not necessarily modular.

For part (c), observe that if  $U$  is a fundamental matrix for  $M$ , then  $U^\wedge(t, q) := U(t, q^{-1})$  and  $U^\vee = (U^{-1})^t$  are fundamental matrices for  $M^\wedge$  and  $M^\vee$ . It follows that if  $\Omega$  is a cocycle of  $M$  then  $\Omega^\wedge(z, \tau) := \Omega(z, -\tau)$  and  $\Omega^\vee = (\Omega^{-1})^t$  are cocycles for  $M^\wedge$  and  $M^\vee$ . Part (c) follows. This concludes the proof of Lemma 1.3.  $\square$

### 3. HEINE'S $q$ -HYPERGEOMETRIC FUNCTIONS

This section is devoted to the proof of Theorem 1.4.

**3.1. Solutions.** In this section we describe the solutions of the generalised  $q$ -hypergeometric equation (11). Since that equation depends on parameters, it will be convenient to consider the following system of equations

$$\begin{aligned} \left( \prod_{j=0}^{r-1} (1 - q^{-1} b_j \sigma_t) - t \prod_{j=1}^r (1 - a_j \sigma_t) \right) f &= 0 \\ (\sigma_{a_i}^{-1} - q^{-1} a_i \sigma_t \sigma_{a_i}^{-1} - (1 - q^{-1} a_i)) f &= 0 \\ (1 - q^{-1} b_i \sigma_t - (1 - q^{-1} b_i) \sigma_{b_i}^{-1}) f &= 0. \end{aligned} \quad (111)$$

The first equation describes the  $q$ -difference equation in  $t$ , namely Equation (11) whose Newton polygon shown in Figure 1.

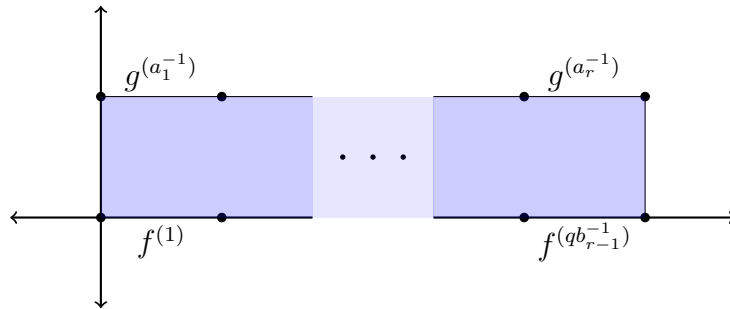


FIGURE 1. The Newton polygon of the first Equation (111).

We see that there are no slopes of the Newton polygon and therefore all solutions are determined by the top and bottom edges and their indicial polynomials. We will normalise



the solutions coming from the Frobenius algorithm so that they satisfy the full system of Equations (111). The bottom edge of the Newton polygon in Figure 1 has indicial polynomial

$$(1 - \rho)(1 - q^{-1}b_1\rho) \dots (1 - q^{-1}b_{r-1}\rho) = 0. \quad (112)$$

The solutions corresponding to the roots are given by  $f^{(q^{-1}b_j)}$  in Equation (14), with the convention that  $b_0 = q$ . The top edge of the Newton polygon in Figure 1 has indicial polynomial

$$(1 - a_1\rho) \dots (1 - a_r\rho) = 0. \quad (113)$$

The solutions corresponding to the roots are given by  $g^{(a_j^{-1})}$  in Equation (16). The companion matrix of Equation (11) is given by

$$A(t, a, b, q) = \begin{pmatrix} 0 & 1 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 1 \\ -(-1)^r \frac{1-t}{e_r(b/q) - te_r(a)} & (-1)^r \frac{e_1(b/q) - te_1(a)}{e_r(b/q) - te_r(a)} & \dots & \frac{e_{r-1}(b/q) - te_{r-1}(a)}{e_r(b/q) - te_r(a)} \end{pmatrix} \quad (114)$$

where

$$e_k(x) = \sum_{1 \leq j_1 < \dots < j_k \leq r} x_{j_1} \dots x_{j_k} \quad (115)$$

are the elementary symmetric polynomials and so with  $U$  and  $V$  in Equation (13),

$$U(qt, a, b, q) = A(t, a, b, q)U(t, a, b, q) \quad \text{and} \quad V(qt, a, b, q) = A(t, a, b, q)V(t, a, b, q). \quad (116)$$

**3.2. Self duality.** We will introduce a state integral in Section 3.3 which factorises as a finite sum of products of solutions of the module  $M$  associated to Equations (111) and its dual  $M^\wedge$  (see Section 2.5 for the definitions of the two duals). To prove modularity we must factorise the state integral as a finite sum of products of solutions of  $M$  and  $M^\vee$ . To do so, we need to give an explicit isomorphism between  $M^\vee$  and  $M^\wedge$ . This is the content of the following proposition, which after some change of variables, is equivalent to Beukers–Jouhet [8, Thm.1.3]. For completeness, we will give an independent proof using the methods of our paper.

**Proposition 3.1.** [8, Thm.1.3] If  $M$  is the module associated to Equations (111) then

$$M^\vee \cong M^\wedge. \quad (117)$$

Explicitly, there exists  $Q(t, a, b, q) \in GL_r(\mathbb{Q}(t, a, b, q))$  such that

$$Q(qt, a, b, q)A(qt, a, b, q^{-1})^{-1} = A(t, a, b, q)^{-T}Q(t, a, b, q). \quad (118)$$

In addition,  $Q$  satisfies

$$U(t, a, b, q)^{-T} = Q(t, a, b, q)U(t, a, b, q^{-1})\text{diag}(1, -1, -1, \dots, -1). \quad (119)$$

*Proof.* To prove Equation (118) let

$$(1 - q^{-1}b_1) \dots (1 - q^{-1}b_{r-1})P(t, a, b, q) = \begin{pmatrix} 1-t & 0 & 0 & \dots & 0 \\ 0 & -(e_2(b/q) - q^{-1}te_2(a)) & e_3(b/q) - q^{-2}te_3(a) & \dots & (-1)^{r-1}(e_r(b/q) - q^{-r+1}te_r(a)) \\ 0 & (e_3(b/q) - q^{-1}te_3(a)) & -(e_4(b/q) - q^{-2}te_4(a)) & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & (-1)^{r-1}(e_r(b/q) - q^{-1}te_r(a)) & 0 & \dots & 0 \end{pmatrix}. \quad (120)$$

Using the fact that  $e_k(\lambda x) = \lambda^k e_k(x)$ , one can then see that

$$\begin{aligned} & (1 - q^{-1}b_1) \dots (1 - q^{-1}b_{r-1}) P(qt, a, b, q) A(qt, q^{-1}a, q^{-2}b, q^{-1})^{-1} \\ &= (1 - q^{-1}b_1) \dots (1 - q^{-1}b_{r-1}) A(t, a, b, q)^{-T} P(t, a, b, q) \\ &= \begin{pmatrix} e_1(b/q) - te_1(a) & -(e_2(b/q) - q^{-1}te_2(a)) & \dots & (-1)^{r-1}(e_r(b/q) - q^{-r+1}te_r(a)) \\ -(e_2(b/q) - te_2(a)) & e_3(b/q) - q^{-1}te_3(a) & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ (-1)^{r-1}(e_r(b/q) - te_r(a)) & 0 & \dots & 0 \end{pmatrix}. \end{aligned} \quad (121)$$

Now notice that the second and third Equations (111) give gauge equivalences between the modules in  $t$  thinking of  $a, b$  as constants when we shift  $a_j \mapsto qa_j$  and  $b_i \mapsto qb_i$ . Therefore, multiplying  $P$  by these gauge equivalences gives the desired  $Q$ . Now to prove Equation (119) we note that, from Equation (118)

$$E(t, a, b, q) := U(t, q^{-1}a, q^{-2}b, q^{-1})^{-1} P(t, a, b, q)^{-1} U(t, a, b, q)^{-T} \quad (122)$$

is an elliptic function. However,

$$H(t, a, b, q) := \text{diag} \left( \frac{\theta(q^{-2}b_i t; q)}{\theta(q^{-1}t; q)} \right)^{-1} E(t, a, b, q) \text{diag} \left( \frac{\theta(q^{-1}b_i t; q)}{\theta(t; q)} \right) \quad (123)$$

is holomorphic at  $t = 0$ . Therefore, we see that

$$H_{i,j}(t, a, b, q) = -t \frac{\theta(q^{-1}b_{j-1}t; q)}{\theta(q^{-2}tb_{i-1}; q)} E_{i,j}(t, a, b, q) \quad (124)$$

is holomorphic. This implies that if  $i \neq j$  that  $E_{i,j}(t, a, b, q)$  has at most has simple poles at  $q \in b_{j-1}^{-1}q^{\mathbb{Z}}$  and must have zeros at  $t \in b_{i-1}^{-1}q^{\mathbb{Z}}$  and there is no such non-zero elliptic function and therefore  $E_{i,j}(t, a, b, q) = 0$ . Then notice that

$$H_{ii}(t, a, b, q) = -t \frac{\theta(q^{-1}b_{i-1}t; q)}{\theta(q^{-2}tb_{i-1}; q)} E_{i,i}(t, a, b, q) = qb_{i-1}^{-1} E_{i,i}(t, a, b, q) \quad (125)$$

is holomorphic at  $t = 0$  and so as  $E_{i,i}(t, a, b, q)$  is also elliptic in  $t$  it is constant in  $t$ . Now notice that

$$\begin{aligned} & \lim_{t \rightarrow 0} U(t, q^{-1}a, q^{-2}b, q^{-1}) \text{diag} \left( \frac{\theta(q^{-2}b_i t; q)}{\theta(q^{-1}t; q)} \right) \\ &= \begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & qb_1^{-1} & \dots & qb_{r-1}^{-1} \\ \vdots & \vdots & \dots & \vdots \\ 1 & (qb_1^{-1})^{r-1} & \dots & (qb_{r-1}^{-1})^{r-1} \end{pmatrix} \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & B_1(q^{-1}a, q^{-2}b, q^{-1}) & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & B_{r-1}(q^{-1}a, q^{-2}b, q^{-1}) \end{pmatrix} \end{aligned} \quad (126)$$

and

$$\begin{aligned} & \lim_{t \rightarrow 0} \text{diag} \left( \frac{\theta(q^{-1}b_i t; q)}{\theta(t; q)} \right)^{-1} U(t, a, b, q)^T \\ &= \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & B_1(a, b, q) & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & B_{r-1}(a, b, q) \end{pmatrix} \begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & qb_1^{-1} & \dots & (qb_1^{-1})^{r-1} \\ \vdots & \vdots & \dots & \vdots \\ 1 & qb_{r-1}^{-1} & \dots & (qb_{r-1}^{-1})^{r-1} \end{pmatrix}. \end{aligned} \quad (127)$$

Then using the fact that

$$\sum_{j=0}^r (-1)^j e_j(b/q) \rho^j = (1 - \rho)(1 - q^{-1}b_1\rho) \dots (1 - q^{-1}b_{r-1}\rho) \quad (128)$$

vanishes at  $\rho \in \{qb_i^{-1}\}$  we can show the matrix

$$\prod_{k=1}^r (1 - q^{-1}b_k) \begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & qb_1^{-1} & \dots & (qb_1^{-1})^{r-1} \\ \vdots & \vdots & \dots & \vdots \\ 1 & qb_{r-1}^{-1} & \dots & (qb_{r-1}^{-1})^{r-1} \end{pmatrix} P(t, a, b, q) \begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & qb_1^{-1} & \dots & qb_{r-1}^{-1} \\ \vdots & \vdots & \dots & \vdots \\ 1 & (qb_1^{-1})^{r-1} & \dots & (qb_{r-1}^{-1})^{r-1} \end{pmatrix} \quad (129)$$

has  $(i+1, j+1)$ -th entry

$$\begin{aligned} & \sum_{k=0}^{r-1} (-1)^j e_j(b/q) (qb_j^{-1})^k \sum_{\ell=0}^{r-1-k} (b_i b_j^{-1})^\ell \\ & = \delta_{i,j} (1 - b_0/b_j) \dots (1 - b_{j-1}/b_j) (1 - b_{j+1}/b_j) \dots (1 - b_{r-1}/b_j) \end{aligned} \quad (130)$$

where we have used the fact that

$$\begin{aligned} & \sum_{k=0}^{r-1} (-1)^j e_j(b/q) x^k \sum_{\ell=0}^{r-1-k} (xy^{-1})^\ell = \sum_{k=0}^{r-1} (-1)^j e_j(b/q) x^k \frac{1 - (xy^{-1})^{r-k}}{1 - xy^{-1}} \\ & = \frac{1}{1 - xy^{-1}} ((1-x)(1 - q^{-1}b_1x) \dots (1 - q^{-1}b_{r-1}x) - x^r y^{-r} (1-y)(1 - q^{-1}b_1y) \dots (1 - q^{-1}b_{r-1}y)) . \end{aligned} \quad (131)$$

Therefore, with the convention that  $(x; q^{-1})_\infty = (qx; q)_\infty^{-1}$  and  $(q^{-1}; q^{-1})_\infty = (q; q)_\infty^{-1}$  when  $|q| < 1$ , for  $j > 0$

$$\begin{aligned} & B_j(a, b, q) B_j(q^{-1}a, q^{-2}b, q^{-1}) \\ & = -q^{-1}b_j \frac{(1 - q^{-1}b_1) \dots (1 - q^{-1}b_{r-1})}{(1 - q/b_j)(1 - b_1/b_j) \dots (1 - b_{j-1}/b_j)(1 - b_{j+1}/b_j) \dots (1 - b_{r-1}/b_j)} \end{aligned} \quad (132)$$

and we see that

$$\begin{aligned} & \begin{pmatrix} E_{1,1}(t, a, b, q) & 0 & 0 & \dots & 0 \\ 0 & qb_1^{-1} E_{2,2}(t, a, b, q) & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & qb_{r-1}^{-1} E_{r,r}(t, a, b, q) \end{pmatrix} \\ & = \lim_{t \rightarrow 0} H(t, a, b, q) = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & -qb_1^{-1} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & -qb_{r-1}^{-1} \end{pmatrix} \end{aligned} \quad (133)$$

Therefore, again using the gauge equivalence in the second and third Equations (111) complete the proof.  $\square$

**Remark 3.2.** When  $r = 2$ , Proposition 3.1 is equivalent to the identity [38, Eqn.(1.4.3)]

$${}_2\phi_1(a, b; c; q, t) = \frac{(abc^{-1}t; q)_\infty}{(t; q)_\infty} {}_2\phi_1(ca^{-1}, cb^{-1}; c^{-1}; q, abc^{-1}t) \quad (134)$$

where we note that the ratio of Pochhammers is related to the determinant of  $U$ . This is the  $q$ -analogue of Euler's transformation formula for  ${}_2F_1$  [38, Eqn.(1.4.2)]

$${}_2F_1(a, b; c; t) = (1-t)^{c-a-b} {}_2F_1(c-a, c-b; c; t) \quad (135)$$

where

$${}_rF_{r-1}(a; b; t) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_r)_k}{(b_1)_k \cdots (b_{r-1})_k} \frac{t^k}{k!}. \quad (136)$$

**Remark 3.3.** The  $q \rightarrow 1$  limit of Proposition 3.1 is discussed in full generality in [8, Thm.1.1], where it is shown that the dual of the module associated to  ${}_rF_{r-1}(a; b; t)$  is the module associated to  ${}_rF_{r-1}(1-a; 2-b; t)$ .

**3.3. State integral.** Consider the following state integral

$$I(z, \alpha, \beta, \tau) = \int_{\mathcal{C}} \frac{\Phi_{\mathbf{b}}(x) \prod_{j=1}^{r-1} \Phi_{\mathbf{b}}(x + i\mathbf{b}^{-1}\beta_j + i\mathbf{b}^{-1} - i\mathbf{b})}{\prod_{j=1}^r \Phi_{\mathbf{b}}(x + i\mathbf{b}^{-1}\alpha_j + i\mathbf{b}^{-1} - i\mathbf{b})} \exp\left(-2\pi \frac{zx}{\mathbf{b}}\right) dx. \quad (137)$$

where  $\mathcal{C}$  is a contour in the complex plane asymptotic to  $\mathbb{R} + i\varepsilon$  that separates the poles of the numerator from the zeros of the denominator of the integrand,  $a_i = \mathbf{e}(\alpha_i)$ ,  $\tilde{a}_i = \mathbf{e}(\alpha_i/\tau)$ ,  $b_i = \mathbf{e}(\beta_i)$  and  $\tilde{b}_i = \mathbf{e}(\beta_i/\tau)$ , and  $\Phi_{\mathbf{b}}$  is the Faddeev quantum dilogarithm function [23, 24] and  $\mathbf{b} = \sqrt{\tau}$ .

We first discuss convergence of the above integral for  $\tau = \mathbf{b}^2$  in the upper half-plane. Using the asymptotic behavior  $\Phi_{\mathbf{b}}(x) \sim e^{2\pi i x^2}$  (resp., 1) when  $\operatorname{Re}(x) \gg 0$  (resp.,  $\operatorname{Re}(x) \ll 0$ ) (see for example, [3, Eqn.(46)]), it follows that when  $\operatorname{Re}(x) \gg 0$ , the integrand of (137) is given by a constant times  $e^{-2\pi x \mathbf{b}^{-1}(\gamma+z)}$  where  $\gamma = \sum_{j=0}^{r-1} \beta_j - \sum_{j=1}^r \alpha_j$ , and setting  $x = x_0 + it$  with  $t \gg 0$ , it follows that the absolute value of the integrand is a constant times  $e^{-2\pi t \operatorname{Im}(\mathbf{b}^{-1}(\gamma+z))}$ , which is exponentially decaying when  $\operatorname{Im}(\mathbf{b}^{-1}(\gamma+z)) < 0$ . Likewise, when  $\operatorname{Re}(x) \ll 0$ , the integrand is exponentially decaying when  $\operatorname{Im}(\mathbf{b}^{-1}z) > 0$ .

Finally, the state integral satisfies difference equations when we shift  $\alpha, \beta, z$  by either 1 or  $\tau$ . This can be used to analytically extend to a meromorphic function for  $\tau \in \mathbb{C}'$  and  $\alpha \in \mathbb{C}^r$ ,  $\beta \in \mathbb{C}^{r-1}$ , and  $z \in \mathbb{C}$ .

From its very definition, the state integral is a well-defined holomorphic function of  $\tau \in \mathbb{C}'$ . Moreover, after moving the contour of integration upwards and using the residue theorem (see eg. [30]), the state integral in Equation (137) can be written in the factorised form

$$I(z, \alpha, \beta, \tau) = -t^{1/2} \tilde{t}^{1/2} \frac{\tau}{2\pi i} \mathcal{I}(z, \alpha, \beta, \tau) \quad (138)$$

where

$$\mathcal{I}(z, \alpha, \beta, \tau) = f^{(1)}(t, a, b, q) f^{(1)}(\tilde{t}, \tilde{a}, \tilde{b}, \tilde{q}^{-1}) - \tau \sum_{j=1}^r f^{(qb_j^{-1})}(t, a, b, q) f^{(qb_j^{-1})}(\tilde{t}, \tilde{a}, \tilde{b}, \tilde{q}^{-1}). \quad (139)$$

Now  $\mathcal{I}(z+k+j\tau, \alpha, \beta, \tau)$  is nothing but the  $(j+1, k+1)$  entry of the matrix

$$U(t, a, b, q) \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & -\tau & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & -\tau \end{pmatrix} U(\tilde{t}, \tilde{a}, \tilde{b}, \tilde{q}^{-1})^T, \quad (140)$$

and therefore, by Equation (119), the  $(j + 1, k + 1)$  entry of

$$U(t, a, b, q) \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & \tau & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & \tau \end{pmatrix} U(\tilde{t}, \tilde{a}, \tilde{b}, \tilde{q})^{-1} Q(\tilde{t}, \tilde{a}, \tilde{b}, \tilde{q})^{-T}. \quad (141)$$

Therefore, inverting  $Q$ , we see that the cocycle

$$\Omega(z, \alpha, \beta, \tau) = U(\tilde{t}, \tilde{a}, \tilde{b}, \tilde{q}) \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & \tau & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & \tau \end{pmatrix}^{-1} U(t, a, b, q)^{-1} \quad (142)$$

extends to a meromorphic function for  $\tau \in \mathbb{C}'$ . This complete the proof of Theorem 1.4.

**Remark 3.4.** Note that the state integral (137) is absolutely convergent and its contour of integration can be pushed either upwards or downwards. Doing so, the integral factorises in two different ways, one giving the  $U$ -cocycle and another giving the  $V$ -cocycle. This explains the equality of the two cocycles from first principles.

**3.4. Resonance.** In this section we discuss in detail the resonant generalised  $q$ -hypergeometric equation (11), i.e., the case where at least one of the ratios  $a_i/a_j$  (for  $i \neq j$ ),  $b_i/b_j$  (for  $i \neq j$ ) or  $a_i/b_j$  (for some  $i$  and  $j$ ) is an integer power of  $q$ . For simplicity, we will consider only the case of  $r = 2$ , although our arguments remain valid for all  $r$ . When  $r = 2$ , the system of equations (111) is given by

$$\begin{aligned} (1-t)f(t, a, b, c, q) - (q^{-1}c + 1 - t(a+b))f(qt, a, b, c, q) + (q^{-1}c - tab)f(q^2t, a, b, c, q) &= 0 \\ f(t, q^{-1}a, b, c, q) - q^{-1}af(qt, q^{-1}a, b, c, q) - (1 - q^{-1}a)f(t, a, b, c, q) &= 0 \\ f(t, a, q^{-1}b, c, q) - q^{-1}bf(qt, a, q^{-1}b, c, q) - (1 - q^{-1}b)f(t, a, b, c, q) &= 0 \\ f(t, a, b, c, q) - q^{-1}cf(qt, a, b, c, q) - (1 - q^{-1}c)f(t, a, b, q^{-1}c, q) &= 0, \end{aligned} \quad (143)$$

with Equation (1) being one such solution. We can specialise  $a, b, c$  so that some of  $a, b, c, a/b, a/c$  or  $b/c$  lies in  $q^{\mathbb{Z}}$ . All of these conditions can be seen to be special points of the monodromy matrix (18) and various special properties of the equations appear like, for example, submodules.

We now present two examples of these special points in the simplest case or  $r = 2$ . These can all be deduced from Theorem 1.4.

- $b = c$ . The first Equation (143) now takes the form

$$\begin{aligned} (1 - q^{-1}b\sigma_t) ((1 - \sigma_t) - t(1 - a\sigma_t)) \\ = ((1 - \sigma_t)(1 - q^{-1}b\sigma_t) - t(1 - a\sigma_t)(1 - b\sigma_t)) f \\ = 0. \end{aligned} \quad (144)$$

This has normalised solutions

$$\begin{aligned} f^{(1)}(t, a, b, q) &= \sum_{k=0}^{\infty} \frac{(a; q)_k}{(q; q)_k} t^k \\ f^{(qb^{-1})}(t, a, b, q) &= \frac{(a; q)_{\infty} (q^2 b^{-1}; q)_{\infty} \theta(q^{-1} b t; q) (q; q)_{\infty}^2}{(q a b^{-1}; q)_{\infty} \theta(q^{-1} b; q) \theta(t; q)} \sum_{k=0}^{\infty} \frac{(q a b^{-1}; q)_k}{(q^2 b^{-1}; q)_k} t^k. \end{aligned} \quad (145)$$

Note that the first generates a submodule and indeed satisfies

$$(((1 - \sigma_t) - t(1 - a\sigma_t)) f^{(1)})(t, q) = (1 - t)f^{(1)}(t, a, b, q) - (1 - at)f^{(1)}(qt, a, b, q) = 0. \quad (146)$$

We note that for the value at  $t = 0$  and the  $q$ -difference equation we see that

$$f^{(1)}(t, a, b, q) = \frac{(at; q)_{\infty}}{(t; q)_{\infty}} \quad (147)$$

a classical result known as the  $q$ -binomial theorem. From Theorem 1.5 or even Theorem 1.4 with  $r = 1$ , we then see that this is a modular  $q$ -holonomic submodule. Now the second solution satisfies the inhomogeneous equation

$$(1 - t)f^{(qb^{-1})}(t, a, b, q) - (1 - at)f^{(qb^{-1})}(qt, a, b, q) = \frac{(a; q)_{\infty} (q^2 b^{-1}; q)_{\infty} \theta(q^{-1} b t; q) (q; q)_{\infty}^2}{(q a b^{-1}; q)_{\infty} \theta(q^{-1} b; q) \theta(t; q)} \quad (148)$$

where we note that the RHS is of course annihilated by  $(1 - q^{-1} b \sigma_t)$ . The full module can then be shown to be modular using elementary functions holomorphic for  $\tau \in \mathbb{C}'$  times the state integral

$$\int_{\mathcal{C}} \frac{\Phi_{\mathbf{b}}(x)}{\Phi_{\mathbf{b}}(x + i\mathbf{b}^{-1}\alpha)} \frac{\exp\left(-2\pi \frac{zx}{\mathbf{b}}\right)}{1 + \tilde{q}^{1/2} \exp\left(-2\pi \frac{x}{\mathbf{b}}\right) \mathbf{e}(\beta/\tau)} dx. \quad (149)$$

Note that this state integral is of course the same as the one in Equation (137) where  $\beta_1 = \gamma = \beta + 1 = \alpha_2 + 1$ .

•  $c = q$ . The first Equation (143) now takes the form

$$(1 - t)f(t, a, b, q) - (2 - t(a + b))f(qt, a, b, q) + (1 - tab)f(q^2 t, a, b, q) = 0. \quad (150)$$

Notice that this now has indicial polynomial  $(1 - \rho)^2$ . Therefore, we expand using the Frobenius method to find solutions which are the coefficients of  $\varepsilon$  in the expansion to order  $O(\varepsilon^2)$  of

$$\begin{aligned} f^{(1,0)}(t, a, b, q) &= \frac{(qe^{\varepsilon}; q)_{\infty}^2}{(ae^{\varepsilon}; q)_{\infty} (be^{\varepsilon}; q)_{\infty}} \sum_{k=0}^{\infty} \frac{(ae^{\varepsilon}; q)_k (be^{\varepsilon}; q)_k}{(qe^{\varepsilon}; q)_k^2} t^k \frac{\theta(e^{-\varepsilon} t; q)}{\theta(t; q)} \\ &= f^{(1,0)}(t, a, b, q) + f^{(1,1)}(t, a, b, q)\varepsilon + O(\varepsilon^2). \end{aligned} \quad (151)$$

Then considering the state integral

$$\int_{\mathcal{C}} \frac{\Phi_{\mathbf{b}}(x)^2}{\Phi_{\mathbf{b}}(x + i\mathbf{b}^{-1}\alpha)\Phi_{\mathbf{b}}(x + i\mathbf{b}^{-1}\beta)} \exp\left(-2\pi \frac{zx}{\mathbf{b}}\right) dx. \quad (152)$$

along with Equation (52) we can show that the module is modular with this special value.

4. PROOF OF THEOREMS 1.5 AND 1.6

4.1. **The  $q$ -Pochhammer symbol.** This section is devoted to the proof of Theorem 1.5. The  $q$ -difference equation (19) can be written in operator form as  $((1 - qt)\sigma - 1)f = 0$ , with the Newton polygon shown in Figure 2.

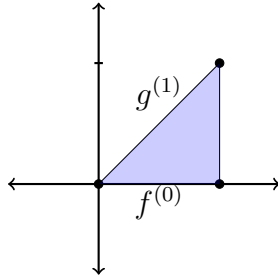


FIGURE 2. The Newton polygon of Equation (19).

The lower Newton polygon has one edge of slope zero. Applying the Frobenius method, we seek a formal power series solution of the form

$$f^{(0)}(t, q) = \sum_{k=0}^{\infty} \alpha_k(q) t^k \frac{\theta(\rho^{-1}t; q)}{\theta(t; q)} \quad \text{where} \quad q^k \rho \alpha_k(q) - q^k \rho \alpha_{k-1}(q) - \alpha_k(q) = 0. \quad (153)$$

Since  $\alpha_k(q) = 0$  for  $k < 0$ , setting  $k = 0$  in the above equation implies the vanishing of the indicial polynomial

$$(\rho - 1)\alpha_0(q) = 0, \quad (154)$$

giving  $\rho = 1$ . Then we have

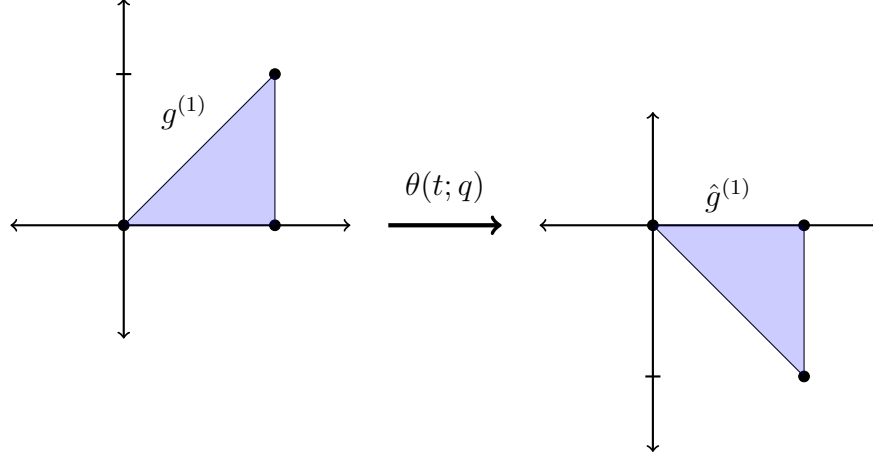
$$\frac{\alpha_k(q)}{\alpha_{k-1}(q)} = -\frac{q^k}{1 - q^k}. \quad (155)$$

Therefore, normalising so that  $\alpha_0(q) = 1$ ,

$$f^{(0)}(t, q) = \sum_{k=0}^{\infty} (-1)^k \frac{q^{k(k+1)/2}}{(q; q)_k} t^k = (qt; q)_{\infty}. \quad (156)$$

This solution is convergent at  $t = 0$ , in fact it is an entire function of  $t$ .

The upper Newton polygon has one edge of slope of one. Therefore, we must multiply by a  $\theta$ -function to get a slope zero, *i.e.* a power series solution. The new Newton polygon is as follows.



Therefore, the top edge has solution of the form

$$g^{(1)}(t, q) = \theta(t; q)\hat{g}^{(1)}(t, q), \quad \hat{g}^{(1)}(t, q) = \theta(t; q) \sum_{k=0}^{\infty} \beta_k(q)t^{-k} \frac{\theta(\rho^{-1}t; q)}{\theta(t; q)} \quad (157)$$

where

$$-q^{-k}\rho\beta_{k-1}(q) + q^{-k}\rho\beta_k(q) - \beta_k(q) = 0. \quad (158)$$

As  $\beta_k(q) = 0$  for  $k < 0$ , we get indicial polynomial

$$(\rho - 1)\beta_0(q) = 0 \quad (159)$$

and so  $\rho = 1$ . Then we have

$$\frac{\beta_k}{\beta_{k-1}} = \frac{1}{1 - q^k}. \quad (160)$$

Therefore, normalising so that  $\beta_0(q) = \frac{1}{(q; q)_{\infty}}$ , we obtain that

$$g^{(1)}(t, q) = \frac{\theta(t; q)}{(q; q)_{\infty}} \sum_{k=0}^{\infty} \frac{t^{-k}}{(q; q)_k} = \frac{\theta(t; q)}{(t^{-1}; q)_{\infty}(q; q)_{\infty}}. \quad (161)$$

It follows that the monodromy is given by

$$\vec{M}(t, q) = V(t, q)^{-1}U(t, q) = \frac{(qt; q)_{\infty}(t^{-1}; q)_{\infty}(q; q)_{\infty}}{\theta(t; q)} = 1 \quad (162)$$

and the cocycles are equal and given by

$$\Omega_U(z, \tau) = (U|_0S)(z, \tau)U(z, \tau)^{-1} = \Omega_U(z, \tau) = (V|_0S)(z, \tau)V(z, \tau)^{-1} = \frac{(\tilde{q}t; \tilde{q})_{\infty}}{(qt; q)_{\infty}}. \quad (163)$$

Notice that

$$\frac{(\tilde{q}t; \tilde{q})_{\infty}}{(qt; q)_{\infty}} = \Phi_{\mathbf{b}} \left( \frac{iz}{\mathbf{b}} + \frac{i\mathbf{b}}{2} + \frac{1}{2i\mathbf{b}} \right)^{-1} \quad (164)$$

where  $\Phi$  is the Faddeev quantum dilogarithm function [23, 24]. This function extends to a meromorphic function of  $(z, \tau) \in \mathbb{C} \times \mathbb{C}'$ , with poles at  $z \in \mathbb{Z}_{\geq 0} + \mathbb{Z}_{\geq 0}\tau$ . Noting that  $\Omega_T = 1$ , it then follows from Theorem 1.2 that  $\Omega_{U, \gamma}$  also extends. This is discussed in detail in upcoming work [31]. This proves that this is a modular  $q$ -holonomic module.



4.2. **The Appell-Lerch sums.** This section is devoted to the proof of Theorem 1.6. The  $q$ -difference equation (22) has Newton polygon shown in Figure 3.

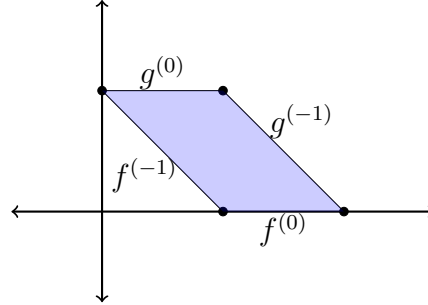


FIGURE 3. The Newton polygon of Equation (22).

To begin with, the boundary of the lower Newton polygon consists of one edge of slope zero and one edge of slope 1. The edge of slope zero has a solution of the form

$$f^{(0)}(t, q) = \sum_{k=0}^{\infty} \alpha_k(q) t^k \frac{\theta(\rho^{-1}t; q)}{\theta(t; q)} \text{ where } \alpha_k(q) q^{2k} \rho^2 + q^k \rho \alpha_{k-1}(q) - q^k \rho \alpha_k(q) - \alpha_{k-1}(q) = 0. \quad (165)$$

Therefore, as  $\alpha_k(q) = 0$  for  $k < 0$ , we get the indicial polynomial

$$\alpha_0(q)(\rho^2 - \rho) = 0 \quad (166)$$

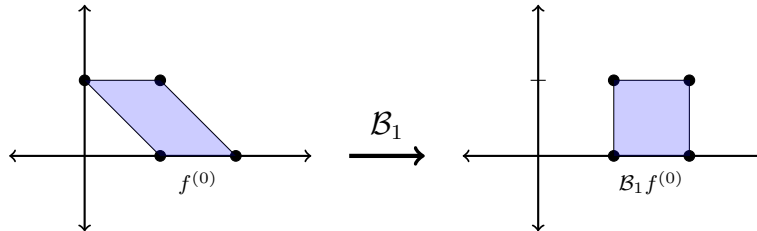
and so  $\rho = 1$ . Then we have

$$(q^{2k} - q^k) \alpha_k(q) - (1 - q^k) \alpha_{k-1}(q) = 0 \quad \text{so} \quad -q^k a_k = a_{k-1}. \quad (167)$$

Therefore, normalising so that  $\alpha_0(q) = 1$ ,

$$\hat{f}^{(0)}(t, q) = \sum_{k=0}^{\infty} (-1)^k q^{-k(k+1)/2} t^k. \quad (168)$$

This is of course divergent for  $|q| < 1$  so we must  $q$ -Borel resum. The affect on the Newton polygon is as follows



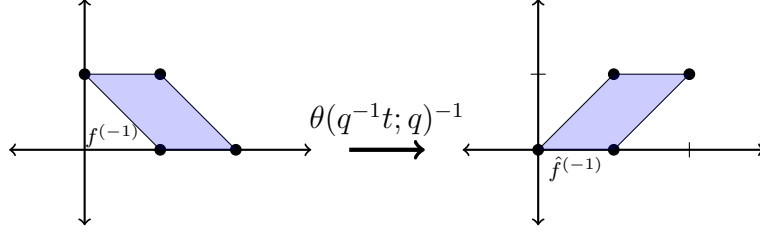
Now notice that

$$\mathcal{B}_1 \hat{f}^{(0)}(\xi, q) = \sum_{k=0}^{\infty} \xi^k = \frac{1}{1 - \xi} \quad (169)$$

and so we find

$$f^{(0)}(t, \lambda, q) = \mathcal{L}_1 \mathcal{B}_1(\hat{f}^{(0)})(t, \lambda, q) = \frac{1}{\theta(\lambda; q)} \sum_{\ell} (-1)^{\ell} \frac{q^{\ell(\ell+1)/2} \lambda^{\ell}}{1 - \lambda t q^{\ell}}. \quad (170)$$

This is the Appell-Lerch sum whose modular properties were studied in Zwegers thesis [74]. Now the bottom edge of slope  $-1$  must be divided by a  $\theta$ -function to get a zero slope. The effect on the Newton polygon is as follows.



Therefore, the bottom edge of slope minus one has a solution of the form

$$f^{(-1)}(t, q) = \theta(q^{-1}t; q)^{-1} \hat{f}^{(-1)}(t, q) = \theta(q^{-1}t; q)^{-1} \sum_{k=0}^{\infty} \alpha_k(q) t^k \frac{\theta(\rho^{-1}t; q)}{\theta(t; q)} \quad (171)$$

where

$$(1 - q^k \rho) \alpha_k(q) + q^k \rho (1 - q^{k-1} \rho) \alpha_{k-1}(q) = 0. \quad (172)$$

Therefore, the indicial polynomial is given by

$$(1 - \rho) \alpha_0(q) = 0 \quad (173)$$

and so  $\rho = 1$ . Then we see that

$$(1 - q) \alpha_1(q) = 0 \quad (174)$$

and so  $\alpha_k = 0$  for  $k \neq 0$ . Therefore,

$$f^{(-1)}(t, q) = \theta(q^{-1}t; q)^{-1}. \quad (175)$$

Now following similar calculations we find that

$$\mathcal{B}_1 \hat{g}^{(0)}(\xi, q) = - \sum_{k=-\infty}^{-1} \xi^k = - \frac{\xi^{-1}}{1 - \xi^{-1}} = \frac{1}{1 - \xi} \quad (176)$$

and so

$$g^{(0)}(t, \mu, q) = \frac{1}{\theta(\mu; q)} \sum_{\ell} (-1)^{\ell} \frac{q^{\ell(\ell+1)/2} \mu^{\ell}}{1 - \mu t q^{\ell}}. \quad (177)$$

Then finally, we have

$$g^{(-1)}(t, q) = \theta(q^{-1}t; q)^{-1}. \quad (178)$$

Consider the inhomogeneous gauge transformation (see Section 2.6)

$$U(t, \lambda, q) = \begin{pmatrix} 0 & 1 \\ 1 & -t \end{pmatrix} \begin{pmatrix} 1 & 0 \\ f^{(0)}(t, \lambda, q) & f^{(-1)}(t, q) \end{pmatrix} \quad (179)$$

and the similar one for  $V$ . With the definition of the fundamental matrices given in Equations (23) and associated monodromy, we then have that

$$\begin{aligned}
 \vec{M}(t, \lambda, q) &= \begin{pmatrix} 1 & 0 \\ g^{(0)}(t, \mu, q) & g^{(-1)}(t, q) \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ f^{(0)}(t, \lambda, q) & f^{(-1)}(t, q) \end{pmatrix} \\
 &= \theta(q^{-1}t, q) \begin{pmatrix} \theta(q^{-1}t, q)^{-1} & 0 \\ -g^{(0)}(t, \mu, q) & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ f^{(0)}(t, \lambda, q) & \theta(q^{-1}t, q)^{-1} \end{pmatrix} \\
 &= \begin{pmatrix} 1 & 0 \\ \vec{M}_{2,1}(t, \lambda, \mu, q) & 1 \end{pmatrix}
 \end{aligned} \tag{180}$$

where

$$\begin{aligned}
 \vec{M}_{2,1}(t, \lambda, \mu, q) &= \theta(q^{-1}t; q) f^{(0)}(t, \lambda, q) - \theta(q^{-1}t; q) g^{(0)}(t, \mu, q) \\
 &= \frac{\theta(q^{-1}t; q)}{\theta(\lambda; q)} \sum_{\ell} (-1)^{\ell} \frac{q^{\ell(\ell+1)/2} \lambda^{\ell}}{1 - \lambda t q^{\ell}} - \frac{\theta(q^{-1}t; q)}{\theta(\mu; q)} \sum_{\ell} (-1)^{\ell} \frac{q^{\ell(\ell+1)/2} \mu^{\ell}}{1 - \mu t q^{\ell}}.
 \end{aligned} \tag{181}$$

Using Equation (7) of Proposition 1.4 of [74], we can deduce that

$$\vec{M}_{2,1}(t, \lambda, \mu, q) = - \frac{(q; q)_{\infty}^3 \theta(q^{-1}t; q) \theta(\lambda^{-1}\mu; q) \theta(\lambda^{-1}\mu^{-1}t^{-1}; q)}{\theta(\lambda^{-1}; q) \theta(\mu; q) \theta(\lambda^{-1}t^{-1}; q) \theta(\mu^{-1}t^{-1}; q)}. \tag{182}$$

We now give a second proof of the above equation using elliptic functions and residues, which is more general and applicable to our third example. Let  $m_{2,1}$  denote the function defined by the right-hand side of (182). We need to prove that the function  $\mathcal{E}_1$ , defined by

$$\mathcal{E}_1(t, \lambda, \mu, q) := f^{(0)}(t, \lambda, q) - g^{(0)}(t, \mu, q) - \frac{m_{2,1}(t, \lambda, \mu, q)}{\theta(q^{-1}t; q)} \tag{183}$$

is identically zero. This will follow from the facts, that  $\mathcal{E}_1$  is holomorphic at  $\mathbb{C}^{\times}$  (see Lemma 4.2 below), and a solution of a first order equation (184) (see Lemma 4.1).

**Lemma 4.1.** If  $h(t, q)$  is a solution to the equation

$$h(qt, q) + th(t, q) = 0 \tag{184}$$

that is holomorphic for  $t \in \mathbb{C}^{\times}$  then  $h(t, q) = 0$ .

*Proof.* Every solution is of the form  $C(t, q)/\theta(q^{-1}t; q)$  for some elliptic function  $C(t, q)$ . Therefore,  $C(t, q) = h(t, q)\theta(q^{-1}t; q)$  is a holomorphic elliptic function and therefore constant. However, this implies that  $h$  has simple poles unless  $C = 0$ .  $\square$

**Lemma 4.2.** (a) The function  $f^{(0)}(t, \lambda, q)$  has simple poles at  $t \in \lambda^{-1}q^{\mathbb{Z}}$  with residue

$$\begin{aligned}
 \text{Res}_{t=q^{-m}\lambda^{-1}} f^{(0)}(t, \lambda, q) \frac{dt}{2\pi it} &= \frac{-1}{\theta(q^m \lambda; q)} \\
 \text{Res}_{t=q^{-m}\mu^{-1}} g^{(0)}(t, \mu, q) \frac{dt}{2\pi it} &= \frac{-1}{\theta(q^m \mu; q)}.
 \end{aligned} \tag{185}$$

(b) The function  $\theta(q^{-1}t; q)^{-1}m_{2,1}(t, \lambda, \mu, q)$  has simple poles at  $t \in q^{\mathbb{Z}}\lambda^{-1}, q^{\mathbb{Z}}\mu^{-1}$  with residues

$$\begin{aligned} \operatorname{Res}_{t=q^{-m}\lambda^{-1}}\theta(q^{-1}t; q)^{-1}m_{2,1}(t, \lambda, \mu, q)\frac{dt}{2\pi it} &= \frac{-1}{\theta(q^m\lambda; q)} \\ \operatorname{Res}_{t=q^{-m}\mu^{-1}}\theta(q^{-1}t; q)^{-1}m_{2,1}(t, \lambda, \mu, q)\frac{dt}{2\pi it} &= \frac{1}{\theta(q^m\mu; q)}. \end{aligned} \quad (186)$$

*Proof.* Part (a) follows from a calculation of the residue of the only term in the sum that contributes to the residue in Equations (170) (177) and Equation (48). Now for part (b) we note that

$$\operatorname{Res}_{t=1}\frac{1}{\theta(t; q)}\frac{dt}{2\pi it} = \frac{1}{(q; q)_{\infty}^3} \quad (187)$$

which follows easily from the Jacobi triple product (46) for example. Therefore, we see

$$\begin{aligned} \operatorname{Res}_{t=q^{-m}\lambda^{-1}}\frac{m_{2,1}(t, \lambda, \mu, q)}{\theta(q^{-1}t; q)}\frac{dt}{2\pi it} &= (-1)^m q^{m(m+1)/2} \operatorname{Res}_{t=1} - \frac{(q; q)_{\infty}^3 \theta(\lambda^{-1}\mu; q)\theta(\mu^{-1}t^{-1}; q)}{\theta(\lambda^{-1}; q)\theta(\mu; q)\theta(t^{-1}; q)\theta(\lambda\mu^{-1}t^{-1}; q)}\frac{dt}{2\pi it} \\ &= (-1)^m q^{m(m+1)/2} \frac{\theta(\lambda^{-1}\mu; q)\theta(\mu^{-1}; q)}{\theta(\lambda^{-1}; q)\theta(\mu; q)\theta(\lambda\mu^{-1}; q)} \\ &= \frac{-1}{\theta(q^m\lambda; q)} \end{aligned} \quad (188)$$

where we have repeatedly used Equations (47) and (48). A similar computation calculates the other residues or using Equation (47) we can show that

$$m_{2,1}(t, \lambda, \mu, q) = -m_{2,1}(t, \mu, \lambda, q). \quad (189)$$

□

Noting that  $\mathcal{E}_1(t, \lambda, \mu, q)$  satisfies Equation (184) these lemmata show that the potential simple poles of  $\mathcal{E}_1(t, \lambda, \mu, q)$  cancel and therefore it is holomorphic on  $t \in \mathbb{C}^{\times}$  and therefore vanishes.

For completeness, we give a third proof of (182) using Lemma 2.1.

$$\begin{aligned} \vec{M}_{2,1}(t, \lambda, \mu, q) &= \theta(q^{-1}t; q)\mathcal{L}_1\left(\frac{1}{1-\xi}\right)(t, \lambda; q) - \theta(q^{-1}t; q)\mathcal{L}_1\left(\frac{1}{1-\xi}\right)(t, \mu; q) \\ &= \frac{\theta(q^{-1}t; q)\theta(\lambda^{-1}\mu; q)(q; q)_{\infty}^3}{\theta(\lambda^{-1}; q)\theta(\mu; q)} \operatorname{Res}_{\xi=1} \frac{\theta(\lambda^{-1}\mu^{-1}t^{-1}\xi; q)}{\theta(\xi\lambda^{-1}t^{-1}; q)\theta(\xi\mu^{-1}t^{-1}; q)(1-\xi)} \\ &= -\frac{(q; q)_{\infty}^3 \theta(q^{-1}t; q)\theta(\lambda^{-1}\mu; q)\theta(\lambda^{-1}\mu^{-1}t^{-1}; q)}{\theta(\lambda^{-1}; q)\theta(\mu; q)\theta(\lambda^{-1}t^{-1}; q)\theta(\mu^{-1}t^{-1}; q)} \end{aligned} \quad (190)$$

concluding the proof of Equation (24). From the explicit expression for  $\vec{M}_{2,1}$  from Equation (25) and the modularity of the Dedekind  $\eta$ -function and the Jacobi  $\theta$ -function it follows that  $M$  satisfies Equation (6) with weight  $\kappa = (0, 1)$ . It follows that the two cocycles of Equation (23) agree. This implies that neither cocycle depends on  $\lambda$  or  $\mu$ . This is equivalent to observations in [74] that the slash operator acting on the Appell-Lerch sums depends only

on the difference of two Jacobi variables. Then using [74, Proposition 1.5] we can give an explicit formula for the cocycle in terms of the Mordell integral (27) and elementary functions. In particular, we have

$$\begin{aligned}
 f^{(0)}(qt, \lambda, q) + tf^{(0)}(t, \lambda, q) &= 1 \\
 f^{(-1)}(qt, q) + tf^{(-1)}(t, q) &= 0 \\
 \tilde{f}^{(0)}(t, \lambda, q) \mathbf{e} \left( \frac{(z + 1/2 - \tau/2)^2}{2\tau} - \frac{1}{8} \right) \sqrt{\tau} &= \tau f^{(0)}(t, \lambda, q) - \tau t^{-\frac{1}{2}} q^{\frac{1}{8}} h(t, q) \\
 f^{(-1)}(\tilde{t}, \tilde{q}) \mathbf{e} \left( \frac{(z + 1/2 - \tau/2)^2}{2\tau} - \frac{1}{8} \right) \sqrt{\tau} &= f^{(-1)}(t, q).
 \end{aligned} \tag{191}$$

This means that

$$\begin{aligned}
 \det U(t, \lambda, q) &= f^{(-1)}(qt, q) f^{(0)}(t, q) - f^{(-1)}(t, q) f^{(0)}(qt, q) \\
 &= f^{(-1)}(t, q) (-tf^{(0)}(t, q) - f^{(0)}(qt, q)) \\
 &= -f^{(-1)}(t, q).
 \end{aligned} \tag{192}$$

Therefore, we see that

$$\Omega_S(z, \tau) = \begin{pmatrix} 0 & 1 \\ 1 & -\tilde{t} \end{pmatrix} \begin{pmatrix} \tau^{-1} & 0 \\ \frac{\mathbf{e}(\frac{1}{8}) \tilde{t}^{-\frac{1}{2}} \tilde{q}^{\frac{1}{8}} \mathbf{e}(\frac{z^2}{2\tau})}{\sqrt{\tau}} h(z, \tau) & \frac{\mathbf{e}(\frac{1}{8})}{\sqrt{\tau}} \mathbf{e}(-\frac{(z+1/2-\tau/2)^2}{2\tau}) \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & -t \end{pmatrix}^{-1}. \tag{193}$$

We can vary the contour in Equation (27) to get the integral

$$\int_{\zeta\mathbb{R}} \frac{e^{\pi i(\tau x^2 + 2izx)}}{2 \cosh(\pi x)} dx \tag{194}$$

for  $|\zeta| = 1$  and  $\zeta \neq \pm i$ . This is a convergent integral to a holomorphic function in  $(z, \tau)$  when  $\text{Im}(\tau\zeta^2) > 0$ . Therefore, we see that from the uniqueness properties of the solutions to the functional equations of  $h$  the Mordell integral [74, Proposition 1.2] these functions give an analytic extension of  $h$  the Mordell integral to the cut plane  $\mathbb{C}'$ . Therefore, the cocycle  $\Omega_{U,S}(t, q)$  extends to a holomorphic function for  $z \in \mathbb{C}$  and  $\tau \in \mathbb{C}'$ . Since  $\Omega_{U,T} = I$ , part (c) of Theorem 1.2 concludes that the cocycle  $\Omega_U$  is modular.

## 5. A $q$ -DIFFERENCE EQUATION OF THE $4_1$ KNOT

This section is devoted to the proof of Theorem 1.7, Theorem 1.8 and Theorem 1.9.

**5.1. Solutions.** The  $q$ -difference equation (28) has Newton polygon shown in Figure 4. The boundary of the lower Newton polygon has edges of slope  $-1$  and  $1$ . The edge with slope  $-1$  must be divided by a  $\theta$ -function to get a power series solution. The effect on the Newton polygon is as follows.

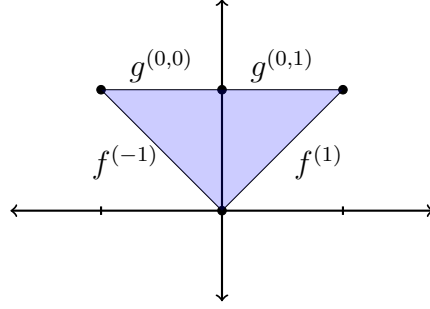
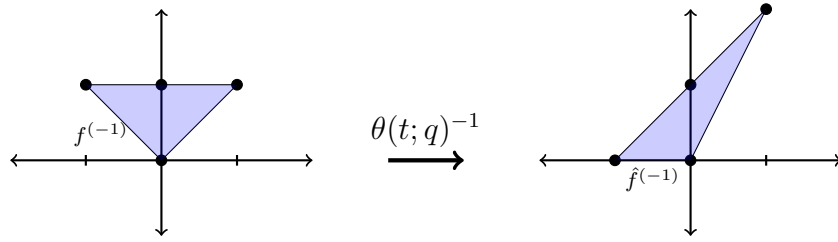


FIGURE 4. The Newton polygon of Equation (28).



Therefore, the bottom edge of the Newton polygon with slope  $-1$  has a solution of the form

$$f^{(-1)}(t, q) = \theta(t; q)^{-1} \hat{f}^{(-1)}(t, q) = \theta(t; q)^{-1} \sum_{k=0}^{\infty} \alpha_k^{(-1)}(q) t^k \frac{\theta(\rho^{-1}t; q)}{\theta(t; q)} \quad (195)$$

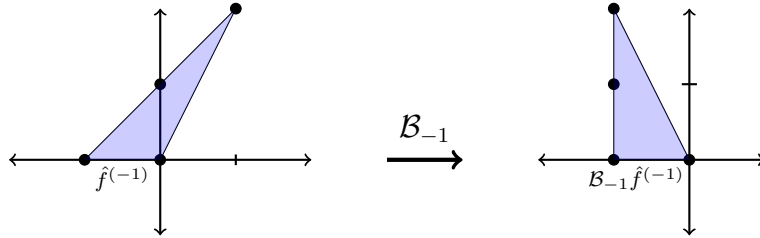
where

$$(1 - q^{-k-1} \rho^{-1}) \alpha_k^{(-1)}(q) - 2\alpha_{k-1}^{(-1)}(q) - q^k \rho \alpha_{k-2}^{(-1)}(q) = 0. \quad (196)$$

Therefore, this edge has indicial polynomial

$$(1 - q^{-1} \rho^{-1}) \alpha_0^{(-1)}(q) \quad (197)$$

and so  $\rho = q^{-1}$ . Now notice that if we take the  $(-1)$ - $q$ -Borel transform we see the effect on the Newton polygon is as follows.



This means that this Borel transform will satisfy the  $q$ -difference equation

$$(1 - q^{-1}t)^2 \mathcal{B}_{-1} \hat{f}^{(-1)}(q^{-1}t; q) = \mathcal{B}_{-1} \hat{f}^{(-1)}(t, q). \quad (198)$$

Therefore, normalising so that  $\alpha_0(q) = -(q; q)_{\infty}^2$  we have

$$\mathcal{B}_{-1} \hat{f}^{(-1)}(t, q) = \sum_{k=0}^{\infty} (-1)^k q^{-k(k+1)/2} \alpha_k^{(-1)}(q) t^k = \frac{-(q; q)_{\infty}^2}{(t; q)_{\infty}^2}. \quad (199)$$

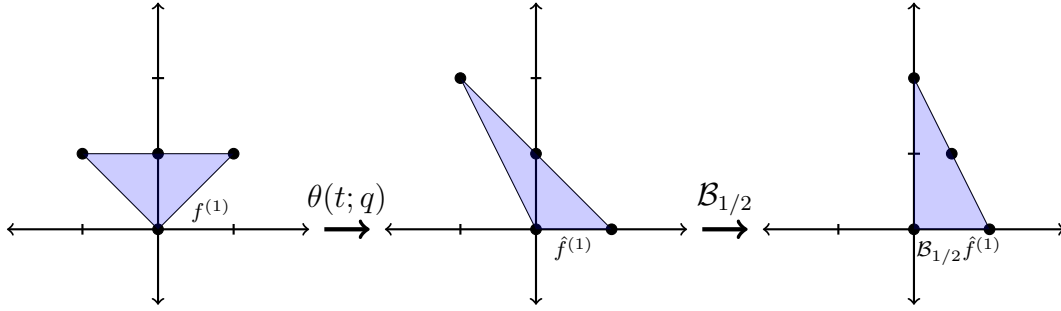
Therefore, we see that

$$\alpha_k^{(-1)}(q) = (q; q)_\infty^2 (-1)^{k+1} q^{k(k+1)/2} \sum_{\ell=0}^k \frac{1}{(q; q)_\ell (q; q)_{k-\ell}}. \quad (200)$$

In particular, we have

$$f^{(-1)}(t, q) = (q; q)_\infty^2 \theta(q^{-1}t; q)^{-1} \sum_{k=0}^{\infty} \sum_{\ell=0}^k (-1)^k \frac{q^{k(k+1)/2}}{(q; q)_\ell (q; q)_{k-\ell}} t^k. \quad (201)$$

Now the bottom edge of slope 1 must be multiplied by a  $\theta$ -function to get a power series solution. However, this will then be divergent so we must take  $(1/2)$ - $q$ -Borel resummation. The effect on the Newton polygon is as follows.



By symmetry of the  $q$ -difference equation, one can easily use the previous solution to check that the formal solution to this edge is given by  $q \mapsto q^{-1}$  which gives

$$f^{(1)}(t, q) = \frac{\theta(t; q)}{(q; q)_\infty^2} \hat{f}^{(1)}(t, q) = \frac{\theta(t; q)}{(q; q)_\infty^2} \sum_{k=0}^{\infty} \sum_{\ell=0}^k \frac{q^{\ell^2 - \ell k}}{(q; q)_\ell (q; q)_{k-\ell}} t^k. \quad (202)$$

One then sees that

$$\mathcal{B}_{1/2} \hat{f}^{(1)}(\xi, q) = \sum_{k=0}^{\infty} \sum_{\ell=0}^k (-1)^k \frac{q^{k(k+1)/4 + \ell^2 - \ell k}}{(q; q)_\ell (q; q)_{k-\ell}} \xi^k = \sum_{k, \ell=0}^{\infty} (-1)^{k+\ell} \frac{q^{k(k+1)/4 - k\ell/2 + \ell(\ell+1)/4}}{(q; q)_k (q; q)_\ell} \xi^{k+\ell} \quad (203)$$

is holomorphic for  $|\xi| < |q^{-1/4}|$ . Then, using the functional equation

$$(1 - q^{1/2} \xi^2) \mathcal{B}_{1/2} \hat{f}^{(1)}(\xi, q) + 2\xi \mathcal{B}_{1/2} \hat{f}^{(1)}(q^{1/2} \xi, q) - \mathcal{B}_{1/2} \hat{f}^{(1)}(q\xi, q) = 0, \quad (204)$$

we can analytically extend away from  $\xi \in \pm q^{-1/4 + \frac{1}{2}\mathbb{Z} \leq 0}$  and we see that there are poles at  $\xi \in \pm q^{-1/4 + \frac{1}{2}\mathbb{Z} \leq 0}$ . Therefore, we finally define

$$\begin{aligned} f^{(1)}(t, \lambda, q) &= \frac{\theta(t; q)}{(q; q)_\infty^2} \mathcal{L}_{1/2} \mathcal{B}_{1/2} \hat{f}^{(1)}(t, \lambda, q) \\ &= \frac{\theta(t; q)}{(q; q)_\infty^2} \sum_{n \in \mathbb{Z}} \frac{\mathcal{B}_{1/2} \hat{f}^{(1)}(q^{\frac{n}{2}} \lambda t, q)}{\theta(q^{\frac{n}{2}} \lambda; q^{\frac{1}{2}})} \\ &= \frac{\theta(t; q)}{(q; q)_\infty^2 \theta(\lambda; q^{1/2})} \sum_{n \in \mathbb{Z}} (-1)^n q^{n(n+1)/4} \lambda^n \mathcal{B}_{1/2} \hat{f}^{(1)}(q^{\frac{n}{2}} \lambda t, q). \end{aligned} \quad (205)$$

Now the top of the Newton polygon has one edge of slope 0. We find the solution satisfies

$$g^{(0)}(t, q) = \sum_{k=0}^{\infty} \beta_k(q) t^{-k} \frac{\theta(\rho^{-1}t; q)}{\theta(t; q)} \quad \text{where} \quad \beta_k(q) - q^{-k} \rho^{-1} (1 - q^k \rho^{-1})^2 \beta_{k+1}(q) = 0. \quad (206)$$

Therefore, as the indicial polynomial is

$$(1 - q^{-1} \rho^{-1})^2 \beta_0(q) = 0 \quad (207)$$

we take  $\rho = q^{-1} e^\varepsilon$  and expand to order  $\varepsilon^2$  to find solutions

$$\begin{aligned} g^{(0)}(t, \varepsilon, q) &= \sum_{k=0}^{\infty} (-1)^k \frac{q^{k(k+1)/2}}{(q; q)_k^2} t^{-1-k} \\ &+ \left( \sum_{k=0}^{\infty} \left( \frac{1}{2} E_1(q) - \frac{1}{2} - \frac{\theta'(t^{-1}; q)}{\theta(t^{-1}; q)} + \sum_{j=1}^k \frac{1 + q^j}{1 - q^j} \right) (-1)^k \frac{q^{k(k+1)/2}}{(q; q)_k^2} t^{-1-k} \right) \varepsilon + O(\varepsilon^2), \end{aligned} \quad (208)$$

where  $E_1(q) = 1 - 4 \sum_{j=1}^{\infty} \frac{q^j}{1 - q^j}$ . Therefore, we have solutions

$$\begin{aligned} g^{(0,0)}(t, q) &= \sum_{k=0}^{\infty} (-1)^k \frac{q^{k(k+1)/2}}{(q; q)_k^2} t^{-1-k} \\ g^{(0,1)}(t, q) &= \sum_{k=0}^{\infty} \left( \frac{1}{2} E_1(q) - \frac{1}{2} - \frac{\theta'(t^{-1}; q)}{\theta(t^{-1}; q)} + \sum_{j=1}^k \frac{1 + q^j}{1 - q^j} \right) (-1)^k \frac{q^{k(k+1)/2}}{(q; q)_k^2} t^{-1-k}. \end{aligned} \quad (209)$$

**5.2. Monodromy.** Consider the fundamental matrices  $U$  and  $V$  given by Equations (29) and the associated monodromy. Using the modular transformation properties of the Jacobi  $\theta$ -function Equation (49) and the Dedekind  $\eta$ -function, it is easy to see that Equation (31) implies (32).

Note that each of the functions  $\vec{M}_{2,1}, \vec{M}_{2,1}|_{-1}T, \vec{M}_{2,1}|_{-1}S$  has  $\text{SL}_2(\mathbb{Z})$ -stabiliser  $\langle T^2, TST \rangle, \langle T^2, S \rangle$  and  $\langle ST^2S, T \rangle$ , respectively, and that the second group is the  $\theta$ -subgroup and the last group is  $\Gamma_0(2)$ . The appearance of  $\Gamma_0(2)$  is a consequence of the  $(1/2)$ - $q$ -Borel transform below.

The rest of this section is devoted to the proof that the monodromy is given by Equation (31). To achieve this, we need to write  $f^{(-1)}$  and  $f^{(1)}$  as linear combinations of  $g^{(0,0)}$  and  $g^{(0,1)}$ , the coefficients in these expressions give the entries of the monodromy matrix.

Firstly we determine the second column of the monodromy, using an adaption of an argument in [54]. Using the  $(-1)$ - $q$ -Laplace transform or the Meinardus trick (see for example [51])



and [71, p.54]) and shifting the contour of  $\oint := \int_{|t|=\varepsilon}$  we obtain that

$$\begin{aligned}
 \hat{f}^{(-1)}(t, q) &= (q; q)_\infty^2 \oint_0 \frac{\theta(t/\xi; q)}{(\xi; q)_\infty^2} \frac{d\xi}{2\pi i \xi} = -(q; q)_\infty^2 \sum_{k=0}^{\infty} \left( \text{Res}_{\xi=q^{-k}} \right) \frac{\theta(t/\xi; q)}{(\xi; q)_\infty^2} \frac{d\xi}{2\pi i \xi} \\
 &= -(q; q)_\infty^2 \sum_{k=0}^{\infty} \left( \text{Res}_{\varepsilon=0} \right) \frac{\theta(tq^k e^\varepsilon; q)}{(q^{-1}e^{-\varepsilon}; q^{-1})_k^2 (e^{-\varepsilon}; q)_\infty^2} \frac{-d\varepsilon}{2\pi i} \\
 &= -(q; q)_\infty^4 \sum_{k=0}^{\infty} \left( \text{Res}_{\varepsilon=0} \right) \frac{(-1)^k q^{-k(k+1)/2} t^{-k} e^{-k\varepsilon} \theta(te^\varepsilon; q)}{\theta(e^\varepsilon; q)^2} \frac{(qe^\varepsilon; q)_\infty^2}{q^{-k(k+1)} e^{-2k\varepsilon} (qe^\varepsilon; q)_k^2} \frac{-d\varepsilon}{2\pi i} \\
 &= (q; q)_\infty^6 \text{Res}_{\varepsilon=0} \frac{\theta(te^\varepsilon; q)}{\theta(e^\varepsilon; q)^2} \frac{(qe^\varepsilon; q)_\infty^2}{(q; q)_\infty^2} \sum_{k=0}^{\infty} \frac{(-1)^k q^{k(k+1)/2} t^{-k} e^{k\varepsilon}}{(qe^\varepsilon; q)_k^2} \frac{d\varepsilon}{2\pi i} \\
 &= -\theta(t^{-1}; q) \sum_{k=0}^{\infty} \left( \frac{1}{2} E_1(q) - \frac{1}{2} + \sum_{j=1}^k \frac{1+q^j}{1-q^j} \right) \frac{(-1)^k q^{k(k+1)/2} t^{-k-1}}{(q; q)_k^2} + \theta'(t^{-1}; q) \sum_{k=0}^{\infty} \frac{(-1)^k q^{k(k+1)/2} t^{-k-1}}{(q; q)_k^2}.
 \end{aligned} \tag{210}$$

Therefore, we see that

$$f^{(-1)}(t, q) = g^{(0,1)}(t, q), \tag{211}$$

which implies that  $\vec{M}_{1,2} = 0$  and  $\vec{M}_{2,2} = 1$ . Lemma 5.4 below implies that  $\det(M) = -1$ , therefore

$$\vec{M}(t, \lambda, q) = \begin{pmatrix} -1 & 0 \\ * & 1 \end{pmatrix}. \tag{212}$$

To finish the proof we will show that the function

$$\mathcal{E}_2 := f^{(1)} + g^{(0,0)} - m_{2,1} g^{(0,1)} \tag{213}$$

vanishes identically, where  $m_{2,1}$  denotes the function given in Equation (31). The function  $\mathcal{E}_2$  is meromorphic of  $t \in \mathbb{C}^\times$  and has potential simple poles located at  $t \in \pm q^{-1/4 - \frac{1}{2}\mathbb{Z}} \lambda^{-1}$  (coming from  $f^{(1)}$  and  $\vec{M}_{2,1}$ ) and potential simple poles at  $q^{\mathbb{Z}}$  (coming from  $g^{(0,1)}$ ). Lemma 5.3 below gives

$$\begin{aligned}
 \text{Res}_{t=\pm q^{-1/4-n}\lambda^{-1}} f^{(1)}(t, \lambda, q) &\frac{dt}{2\pi i t} \\
 &= \frac{\theta(\pm q^{3/4}\lambda^{-1}; q)\theta(\pm q^{-1/4}; q)\theta(\pm q^{-3/4}; q)\theta(\pm q^{-3/4}\lambda; q)}{2(q; q)_\infty^6 \theta(q^{-1}\lambda; q)\theta(q^{-3/2}\lambda; q)} f^{(-1)}(\pm q^{-1/4-n}\lambda^{-1}, q).
 \end{aligned} \tag{214}$$

On the other hand,

$$\begin{aligned}
 \text{Res}_{t=\pm q^{-1/4-n}\lambda^{-1}} m_{2,1}(t, \lambda, q) &\frac{dt}{2\pi i t} \\
 &= \frac{\text{Res}_{t=\pm q^{-1/4-n}\lambda^{-1}} \theta(qt; q)\theta(t\lambda; q)\theta(t\lambda q^{-1/2}; q)\theta(t\lambda^2 q^{-1/2}; q)}{\theta(t\lambda q^{1/4}; q)\theta(-t\lambda q^{1/4}; q)\theta(t\lambda q^{-1/4}; q)\theta(-t\lambda q^{-1/4}; q)\theta(q^{-1}\lambda; q)\theta(q^{-3/2}\lambda; q)} \frac{dt}{2\pi i t} \\
 &= \frac{(-1)^n q^{n(n-1)/2} \theta(\pm q^{3/4-n}\lambda^{-1}; q)\theta(\pm q^{-1/4-n}; q)\theta(\pm q^{-3/4-n}; q)\theta(\pm q^{-3/4-n}\lambda; q)}{(q; q)_\infty^3 \theta(-q^{-n}; q)\theta(q^{-1/2-n}; q)\theta(-q^{-1/2-n}; q)\theta(q^{-1}\lambda; q)\theta(q^{-3/2}\lambda; q)} \\
 &= \frac{\theta(\pm q^{3/4}\lambda^{-1}; q)\theta(\pm q^{-1/4}; q)\theta(\pm q^{-3/4}; q)\theta(\pm q^{-3/4}\lambda; q)}{(q; q)_\infty^3 \theta(-1; q)\theta(q^{-1/2}; q)\theta(-q^{-1/2}; q)\theta(q^{-1}\lambda; q)\theta(q^{-3/2}\lambda; q)} \\
 &= \frac{\theta(\pm q^{3/4}\lambda^{-1}; q)\theta(\pm q^{-1/4}; q)\theta(\pm q^{-3/4}; q)\theta(\pm q^{-3/4}\lambda; q)}{2(q; q)_\infty^6 \theta(q^{-1}\lambda; q)\theta(q^{-3/2}\lambda; q)}.
 \end{aligned} \tag{215}$$

Therefore,

$$\operatorname{Res}_{t=\pm q^{-1/4-n}\lambda^{-1}} \mathcal{E}_2(t, \lambda, q) \frac{dt}{2\pi i t} = 0. \quad (216)$$

A similar computation or the fact everything is elliptic in  $\lambda \mapsto q^{1/2}\lambda$  also shows that

$$\operatorname{Res}_{t=\pm q^{-3/4-n}\lambda^{-1}} \mathcal{E}_2(t, \lambda, q) \frac{dt}{2\pi i t} = 0. \quad (217)$$

Now notice that  $m_{2,1}(q^{\mathbb{Z}}, \lambda, q) = 0$  which implies  $m_{2,1}g^{(0,1)}$  is holomorphic at  $t \in q^{\mathbb{Z}}$ . We then see that  $\mathcal{E}_2(t, \lambda, q)$  extends to a holomorphic function of  $t \in \mathbb{C}^\times$ . Therefore, from Corollary 5.5 we see that  $\mathcal{E}_2(t, \lambda, q) = C(q)g^{(0,0)}(t, q)$ . Finally, we note that

$$\frac{-C(q)}{(q; q)_\infty^2} = \lim_{r \rightarrow \infty} \frac{\mathcal{E}_2(q^r t, \lambda, q)}{\theta(q^r t; q)} = 0 \quad (218)$$

which follows from Lemma 5.1. Thus, Equation (31) follows from the residue equation (214) and from a computation of the determinants of  $U$  and  $V$ . We discuss these in the next sections.

**Lemma 5.1.** For  $t$  in some compact set where the functions are holomorphic, we have

$$\begin{aligned} \lim_{r \rightarrow \infty} \frac{g^{(0,0)}(q^r t, q)}{\theta(q^r t; q)} &= \frac{-1}{(q; q)_\infty^2} & \lim_{r \rightarrow \infty} \frac{f^{(-1)}(q^r t, q)}{\theta(q^r t; q)} &= 0 \\ \lim_{r \rightarrow \infty} \frac{g^{(0,1)}(q^r t, q)}{\theta(q^r t; q)} &= 0 & \lim_{r \rightarrow \infty} \frac{f^{(1)}(q^r t, q)}{\theta(q^r t; q)} &= \frac{1}{(q; q)_\infty^2}. \end{aligned} \quad (219)$$

*Proof.* We have

$$\begin{aligned} g^{(0,0)}(q^r t, q) &= \sum_{k=0}^{\infty} (-1)^k \frac{q^{k(k+1)/2 - rk - r}}{(q; q)_k^2} t^{-k-1} \\ &= q^{-r(r+1)/2} \sum_{k=0}^{\infty} (-1)^k \frac{q^{(k-r)(k-r+1)/2}}{(q; q)_k^2} t^{-k-1} \\ &= (-1)^r q^{-r(r+1)/2} t^{-r-1} \sum_{k=-r}^{\infty} (-1)^k \frac{q^{k(k+1)/2}}{(q; q)_{k+r}^2} t^{-k}. \end{aligned} \quad (220)$$

The first equality then follows from

$$\begin{aligned} \lim_{r \rightarrow \infty} \frac{g^{(0,0)}(q^r t, q)}{\theta(q^r t; q)} &= \lim_{r \rightarrow \infty} \frac{-1}{\theta(t^{-1}; q)} \sum_{k=-r}^{\infty} (-1)^k \frac{q^{k(k+1)/2}}{(q; q)_{k+r}^2} t^{-k} \\ &= \frac{-1}{\theta(t^{-1}; q)} \sum_{k \in \mathbb{Z}} (-1)^k \frac{q^{k(k+1)/2}}{(q; q)_\infty^2} t^{-k} = \frac{-1}{(q; q)_\infty^2}. \end{aligned}$$

We can show similarly that

$$g^{(0,1)}(q^r t, q) = (-1)^r q^{-r(r+1)/2} t^{-r-1} \sum_{k=-r}^{\infty} \left( \frac{1}{2} E_1(q) - r - \frac{1}{2} - \frac{\theta'(t^{-1}; q)}{\theta(t^{-1}; q)} + \sum_{j=1}^{k+r} \frac{1+q^j}{1-q^j} \right) (-1)^k \frac{q^{k(k+1)/2}}{(q; q)_{k+r}^2} t^{-k}. \quad (221)$$

Noting that  $E_1(q)/2 - 1/2 + \sum_{j=1}^k \frac{1+q^j}{1-q^j} = k + O(q^k)$  completes the proof of the limits of  $g^{(0,0)}, g^{(0,1)}$ . The limits of  $f^{(\pm 1)}$  follow from Watson's lemma for  $q$ -Borel resummation and keeping track of the  $\theta$  prefactors.  $\square$

**5.3. Residues.** Note that  $g^{(0,0)}$  is holomorphic on  $\mathbb{C}^\times \cup \{\infty\}$ . In this subsection we compute the residues of the meromorphic functions  $f^{(-1)} = g^{(0,1)}$  and  $f^{(1)}$ .

**Lemma 5.2.** The functions  $f^{(-1)}(t, q) = g^{(0,1)}(t, q)$  have simple poles at  $t = q^{\mathbb{Z}}$  with residues

$$\operatorname{Res}_{t=q^n} f^{(-1)}(t, q) \frac{dt}{2\pi i t} = \operatorname{Res}_{t=q^n} g^{(0,1)}(t, q) \frac{dt}{2\pi i t} = g^{(0,0)}(q^n, q). \quad (222)$$

*Proof.* We have

$$\operatorname{Res}_{t=q^k} \frac{\theta'(t^{-1}; q)}{\theta(t^{-1}; q)} \frac{dt}{2\pi i t} = -1. \quad (223)$$

To finish we note that the proof of the equality  $f^{(-1)} = g^{(0,1)}$  is independent of this Lemma and follows from Equation (210).  $\square$

**Lemma 5.3.** (a) The function  $\mathcal{B}_{1/2} \hat{f}^{(1)}(\xi, q)$  has simple poles at  $\xi = \pm q^{-1/4 - \mathbb{Z}/2}$  with residue

$$\operatorname{Res}_{\xi = \pm q^{-1/4 - n/2}} \mathcal{B}_{1/2} \hat{f}^{(1)}(\xi, q) \frac{d\xi}{2\pi i \xi} = \frac{-\theta(\pm q^{-1/4}; q^{1/2})}{2(q; q)_\infty^2} \sum_{\ell=0}^k (\pm 1)^k \frac{q^{\frac{k(k+2)}{4}}}{(q; q)_\ell (q; q)_{k-\ell}}. \quad (224)$$

(b) The function  $f^{(1)}(t, \lambda, q)$  has simple poles at  $t = \pm q^{-1/4 + \mathbb{Z}} \lambda^{-1}$  with residue

$$\begin{aligned} & \operatorname{Res}_{t = \pm q^{-1/4 - n} \lambda^{-1}} f^{(1)}(t, \lambda, q) \frac{dt}{2\pi i t} \\ &= \frac{\theta(\pm q^{3/4} \lambda^{-1}; q) \theta(\pm q^{-1/4}; q) \theta(\pm q^{-3/4}; q) \theta(\pm q^{-3/4} \lambda; q)}{2(q; q)_\infty^6 \theta(q^{-1} \lambda; q) \theta(q^{-3/2} \lambda; q)} f^{(-1)}(\pm q^{-1/4 - n} \lambda^{-1}, q). \end{aligned} \quad (225)$$

*Proof.* For part (a), consider the following auxiliary function,

$$H_r(\xi, q) = \sum_k \frac{\xi^{2k}}{(q; q)_k (q; q)_{k+r}}, \quad (226)$$

which we analytically continue to  $\xi \notin \pm q^{\frac{1}{2}\mathbb{Z} \leq 0}$ . From the power series at  $\xi = 0$ , we see that

$$\mathcal{B}_{1/2} \hat{f}^{(1)}(q^{-1/4} \xi, q) = \sum_{r \in \mathbb{Z}} (-1)^r q^{r^2/4} \xi^r H_r(\xi, q). \quad (227)$$

We have the relation

$$H_{r-1}(\xi, q) - \xi H_r(\xi, q) = -q^{1-r} (H_{r-2}(\xi, q) - H_{r-1}(\xi, q)) \quad (228)$$

and from this we can see that

$$\operatorname{Res}_{\xi=1} (H_{r-1}(\xi, q) - H_r(\xi, q)) \frac{d\xi}{2\pi i \xi} = (-1)^r q^{-r(r-1)/2} \operatorname{Res}_{\xi=1} (H_{-1}(\xi, q) - H_0(\xi, q)) \frac{d\xi}{2\pi i \xi}. \quad (229)$$

Since

$$\lim_{r \rightarrow \infty} H_r(\xi, q) = \frac{1}{(q; q)_\infty (\xi^2; q)_\infty}, \quad (230)$$

it follows that

$$\operatorname{Res}_{\xi=1} H_r(\xi, q) \frac{d\xi}{2\pi i \xi} = \operatorname{Res}_{\xi=1} H_0(\xi, q) \frac{d\xi}{2\pi i \xi} = \operatorname{Res}_{\xi=1} H_\infty(\xi, q) \frac{d\xi}{2\pi i \xi} = \frac{-1}{2(q; q)_\infty^2}. \quad (231)$$

Let

$$R_\pm(n, q) = \operatorname{Res}_{\xi=\pm q^{-1/4-n/2}} \mathcal{B}_{1/2} \hat{f}^{(1)}(\xi, q) \frac{d\xi}{2\pi i \xi}. \quad (232)$$

Then,

$$R(0, q) = \operatorname{Res}_{\xi=\pm 1} \sum_{r \in \mathbb{Z}} (-1)^r q^{r^2/4} \xi^r H_r(\xi, q) \frac{d\xi}{2\pi i \xi} = \frac{-\theta(\pm q^{-1/4}; q^{1/2})}{2(q; q)_\infty^2}. \quad (233)$$

Now from the functional equation of  $\mathcal{B}_{1/2} \hat{f}^{(1)}(\xi, q)$ , we deduce that

$$\begin{aligned} 0 &= \operatorname{Res}_{\xi=\pm q^{-1/4-n/2}} (1 - q^{1/2} \xi^2) \mathcal{B}_{1/2} \hat{f}^{(1)}(\xi, q) + 2\xi \mathcal{B}_{1/2} \hat{f}^{(1)}(q^{1/2} \xi, q) - \mathcal{B}_{1/2} \hat{f}^{(1)}(q\xi, q) \frac{d\xi}{2\pi i \xi} \\ &= (1 - q^{-n}) R_\pm(n, q) \pm 2q^{-1/4-(n-1)/2} R_\pm(n-1, q) - R_\pm(n-2, q). \end{aligned} \quad (234)$$

Note that  $R_\pm(n, q) = 0$  for  $n < 0$  and that when  $R_\pm(-1, q)$  is zero we have a unique solution determined by  $R_\pm(0, q)$ . One can then check that

$$R_\pm(n, q) = \frac{-\theta(\pm q^{-1/4}; q^{1/2})}{2(q; q)_\infty^2} \sum_{\ell=0}^n (\pm 1)^n \frac{q^{\frac{n(n+2)}{4}}}{(q; q)_\ell (q; q)_{n-\ell}}. \quad (235)$$

For part (b), we compute

$$\begin{aligned} &\operatorname{Res}_{t=\pm q^{-1/4-n} \lambda^{-1}} f^{(1)}(t, \lambda, q) \frac{dt}{2\pi i t} \\ &= \operatorname{Res}_{t=\pm q^{-1/4-n} \lambda^{-1}} \frac{\theta(t; q)}{(q; q)_\infty^2} \mathcal{L}_{1/2} \mathcal{B}_{1/2} \hat{f}^{(1)}(t, q) \frac{dt}{2\pi i t} \\ &= \frac{\theta(\pm q^{-1/4-n} \lambda^{-1}; q)}{(q; q)_\infty^2 \theta(\lambda; q^{1/2})} \sum_{k \in \mathbb{Z}} (-1)^k q^{(k-2n)(k-2n-1)/4} \lambda^{-k+2n} R_\pm(k, q) \\ &= \frac{\theta(\pm q^{-1/4-n} \lambda^{-1}; q)}{(q; q)_\infty^2 \theta(\lambda; q^{1/2})} \frac{-\theta(\pm q^{-1/4}; q^{1/2})}{2(q; q)_\infty^2} q^{n(2n+1)/2} \lambda^{2n} \sum_{k=0}^{\infty} \sum_{\ell=0}^k (-1)^k \frac{q^{\frac{k(k+1)}{2}}}{(q; q)_\ell (q; q)_{k-\ell}} (\pm q^{-1/4-n} \lambda^{-1})^k \\ &= -\frac{\theta(\pm q^{-1/4-n} \lambda^{-1}; q) \theta(\pm q^{-1/4}; q^{1/2})}{2(q; q)_\infty^4 \theta(\lambda; q^{1/2})} q^{n(2n+1)/2} \lambda^{2n} \hat{f}^{(-1)}(\pm q^{-1/4-n/2} \lambda^{-1}, q) \\ &= -\frac{\theta(\pm q^{-1/4-n} \lambda^{-1}; q) \theta(\pm q^{-1/4}; q) \theta(\pm q^{-3/4}; q) \theta(\pm q^{-5/4-n} \lambda^{-1}; q)}{2(q; q)_\infty^6 \theta(\lambda; q) \theta(q^{-1/2} \lambda; q) q^{-n(2n+1)/2} \lambda^{-2n}} f^{(-1)}(\pm q^{-1/4-n} \lambda^{-1}, q) \\ &= -\frac{\theta(\pm q^{-1/4} \lambda^{-1}; q) \theta(\pm q^{-1/4}; q) \theta(\pm q^{-3/4}; q) \theta(\pm q^{-5/4} \lambda^{-1}; q)}{2(q; q)_\infty^6 \theta(\lambda; q) \theta(q^{-1/2} \lambda; q)} f^{(-1)}(\pm q^{-1/4-n} \lambda^{-1}, q) \\ &= \frac{\theta(\pm q^{3/4} \lambda^{-1}; q) \theta(\pm q^{-1/4}; q) \theta(\pm q^{-3/4}; q) \theta(\pm q^{-3/4} \lambda; q)}{2(q; q)_\infty^6 \theta(q^{-1} \lambda; q) \theta(q^{-3/2} \lambda; q)} f^{(-1)}(\pm q^{-1/4-n} \lambda^{-1}, q). \end{aligned} \quad (236)$$

□

5.4. **Determinants.** The functional equation (28) for  $U, V$  defined in (29) can be written in the form

$$U(qt, \lambda, q) = A(t; q)U(t, \lambda, q) \quad \text{and} \quad V(qt, q) = A(t; q)V(t, q) \quad (237)$$

where

$$A(t, q) = \begin{pmatrix} 0 & 1 \\ -q^{-2} & 2q^{-1} - q^{-2}t^{-1} \end{pmatrix}. \quad (238)$$

Therefore, we see that

$$\det(U(qt, \lambda, q)) = \det(U(t, \lambda, q))q^{-2} \quad \text{and} \quad \det(V(qt, q)) = \det(V(t, q))q^{-2}. \quad (239)$$

**Lemma 5.4.** We have

$$-\det(U(t, \lambda, q)) = \det(V(t, q)) = q^{-1}t^{-2}. \quad (240)$$

*Proof.* Firstly notice that both  $\det(U(t, \lambda, q))t^2$  and  $\det(V(t, q))t^2$  are elliptic functions in  $t$ . Furthermore,

$$\det(U(t, \lambda, q)) = f^{(1)}(t, \lambda, q)f^{(-1)}(qt, q) - f^{(1)}(qt, \lambda, q)f^{(-1)}(t, q) \quad (241)$$

has potentially simple poles in  $t$  at  $\pm q^{-1/4-n/2}\lambda^{-1}$ . Lemma 5.3 implies that

$$\begin{aligned} \text{Res}_{t=\pm q^{-1/4-n}\lambda^{-1}} \det(U(t, \lambda, q)) &= \frac{\theta(\pm q^{3/4}\lambda^{-1}; q)\theta(\pm q^{-1/4}; q)\theta(\pm q^{-3/4}; q)\theta(\pm q^{-3/4}\lambda; q)}{2(q; q)_\infty^6 \theta(q^{-1}\lambda; q)\theta(q^{-3/2}\lambda; q)} \\ &\quad \times f^{(-1)}(\pm q^{-1/4-n}\lambda^{-1}, q)f^{(-1)}(\pm q^{3/4-n}\lambda^{-1}, q) \\ &\quad - \frac{\theta(\pm q^{3/4}\lambda^{-1}; q)\theta(\pm q^{-1/4}; q)\theta(\pm q^{-3/4}; q)\theta(\pm q^{-3/4}\lambda; q)}{2(q; q)_\infty^6 \theta(q^{-1}\lambda; q)\theta(q^{-3/2}\lambda; q)} \\ &\quad \times f^{(-1)}(\pm q^{3/4-n}\lambda^{-1}, q)f^{(-1)}(\pm q^{-1/4-n}\lambda^{-1}, q) \\ &= 0. \end{aligned}$$

A similar calculation shows that  $\text{Res}_{t=\pm q^{1/4-n}\lambda^{-1}} \det(U(t, \lambda, q)) = 0$ . Therefore,  $U(t, \lambda, q)t^2$  is elliptic and holomorphic in  $t \in \mathbb{C}^\times$ , hence it is constant in  $t$ . Now considering the limit as  $t \rightarrow 0$ , using the definition of  $f^{(\pm 1)}$  and their asymptotic expansions (given by their formal power series expansions, by a  $q$ -version of Watson's lemma), it follows that

$$\begin{aligned} \det(U(t, \lambda, q))t^2 &= \lim_{t \rightarrow 0} \det(U(t, \lambda, q))t^2 = \lim_{t \rightarrow 0} \frac{\theta(t; q)}{\theta(t; q)}t^2 - \lim_{t \rightarrow 0} \frac{\theta(qt; q)}{\theta(q^{-1}t; q)}t^2 \\ &= -\lim_{t \rightarrow 0} q^{-1} \frac{\theta(t; q)}{\theta(t; q)} = -q^{-1}. \end{aligned} \quad (242)$$

For  $V$  notice that

$$\det(V(t, q))t^2 = g^{(0,0)}(t, q)g^{(0,1)}(qt, q)t^2 - g^{(0,0)}(qt, q)g^{(0,1)}(t, q)t^2. \quad (243)$$

This has potentially simple poles at  $t \in q^\mathbb{Z}$ . There is no non-constant elliptic function that satisfies this [2]. Therefore,  $\det(V(t, q))t^2$  is constant. Then notice that

$$\det(V(t, q))t^2 = \lim_{t \rightarrow \infty} \det(V(t, q))t^2 = q^{-1}. \quad (244)$$

□

**Corollary 5.5.** If  $h(t, q)$  is holomorphic for  $t \in \mathbb{C}^\times$  and satisfies the  $q$ -difference equation (28) then

$$h(t, q) = C_0(q)g^{(0,0)}(t, q). \quad (245)$$

*Proof.* Every solution to equation (28) can be written in the form

$$h(t, q) = C_0(t, q)g^{(0,0)}(t, q) + C_1(t, q)g^{(0,1)}(t, q) \quad (246)$$

for some elliptic functions  $C_0$  and  $C_1$ . Now we see that

$$\det \begin{pmatrix} g^{(0,0)}(t, q) & h(t, q) \\ g^{(0,0)}(qt, q) & h(qt, q) \end{pmatrix} = C_1(t, q)q^{-1}t^{-2} \quad (247)$$

is holomorphic and therefore  $C_1(t, q) = C_1(q)$  is independent of  $t$ . Now notice that if  $g(t_0, q) = 0$  then  $g(qt_0, q) \neq 0$  as otherwise this would imply  $\det(V(t_0, q)) = 0$ . This means that if  $C_1(q) \neq 0$  then  $C_0(t, q)$  has simple poles at  $t \in q^{\mathbb{Z}}$  and no other poles for  $t \in \mathbb{C}^\times$  which contradicts the fact that  $C_0(t, q)$  is elliptic as this would give an isomorphism from the elliptic curve to  $\mathbb{CP}^1$ . Therefore  $C_1(q) = 0$ . Then again noting that if  $g(t_0, q) = 0$  then  $g(qt_0, q) \neq 0$  we see that  $C_0(t, q)$  must be constant in  $t$ .

We remark that an alternative proof can be given using the fact that all holomorphic solutions on  $\mathbb{C}^\times$  have a convergent Laurent/Fourier series expansion

$$\sum_{k \in \mathbb{Z}} \alpha_k(q)t^k. \quad (248)$$

This is a consequence of Cauchy's theorem and a detailed discussion of this can be found, for example, in [1]. In our case, the functional equations imply this expansion is divergent if  $\alpha_{-1} \neq 0$ . This forces  $\alpha_{-1} = 0$ , and the functional equation then implies that  $\alpha_k = 0$  for  $k < 0$  and that  $\alpha_k$  are uniquely determined by  $C_0(q) = \alpha_0(q)$  for  $k \geq 0$  which implies that our function is equal to  $C_0(q)g^{(0,0)}(t, q)$ .  $\square$

**5.5. State integral.** In this section we discuss the second part of Theorem 1.7 concerning the extension of the cocycle to the cut plane. The main idea is to use descendant state integrals, following [27, Eqn.(41)], defined by

$$\mathcal{I}_{A,B}(z, \tau) = \int_{\mathbb{R}+i\varepsilon} \Phi_{\mathbf{b}}(x)^B \exp\left(-A\pi i x^2 - 2\pi \frac{zx}{\mathbf{b}}\right) dx. \quad (249)$$

Using the holomorphic extension of the quantum dilogarithm  $\Phi$  from Theorem 1.5, this integral can be shown to extend to a holomorphic function in for  $(z, \tau) \in \mathbb{C} \times \mathbb{C}'$ . We are interested in the integrals  $\mathcal{I}_{1,2}(z, \tau)$  which, using the residue theorem, factorise as an elementary function holomorphic for  $\tau \in \mathbb{C}'$  times the following bilinear combination, see [27, Thm.3],

$$\begin{aligned} \mathcal{I}(z, \tau) &= \tau^{1/2} g^{(0,0)}(\tilde{t}, \tilde{q}) g^{(0,1)}(t, q) - \tau^{-1/2} g^{(0,1)}(\tilde{t}, \tilde{q}) g^{(0,0)}(t, q) \\ &\quad + \tau^{-1/2} \left( \tau \frac{\theta'(t^{-1}; q)}{\theta(t^{-1}; q)} - \frac{\theta'(\tilde{t}^{-1}; \tilde{q})}{\theta(\tilde{t}^{-1}; \tilde{q})} - \frac{1}{2} + \frac{\tau}{2} - z \right) g^{(0,0)}(\tilde{t}, \tilde{q}) g^{(0,0)}(t, q) \\ &= \tau^{1/2} g^{(0,0)}(\tilde{t}, \tilde{q}) g^{(0,1)}(t, q) - \tau^{-1/2} g^{(0,1)}(\tilde{t}, \tilde{q}) g^{(0,0)}(t, q). \end{aligned} \quad (250)$$

where the equality follows from Equation (52). Therefore,  $\mathcal{I}(z, \tau)$  extends to a holomorphic function for  $(z, \tau) \in \mathbb{C} \times \mathbb{C}'$ . Now, we see that

$$\begin{aligned} \Omega_{V,S}(z, \tau) &= (V|_{\kappa} S)(z, \tau) V(z, \tau)^{-1} \\ &= qt^2 \begin{pmatrix} g^{(0,0)}(\tilde{t}, \tilde{q}) & g^{(0,1)}(\tilde{t}, \tilde{q}) \\ g^{(0,0)}(\tilde{q}\tilde{t}, \tilde{q}) & g^{(0,1)}(\tilde{q}\tilde{t}, \tilde{q}) \end{pmatrix} \begin{pmatrix} \tau^{1/2} & 0 \\ 0 & \tau^{-1/2} \end{pmatrix} \begin{pmatrix} g^{(0,1)}(qt, q) & -g^{(0,1)}(t, q) \\ -g^{(0,0)}(qt, q) & g^{(0,0)}(t, q) \end{pmatrix}. \end{aligned} \quad (251)$$

The entries of  $\Omega_{V,S}$  are then elementary functions holomorphic for  $\tau \in \mathbb{C}'$  times  $\mathcal{I}(z + n + m\tau, \tau)$  for some  $n, m \in \{-1, 0, 1\}$ . This shows that the cocycle  $\Omega_{V,S}$  extends to a holomorphic function for  $(z, \tau) \in \mathbb{C} \times \mathbb{C}'$ ; see [26, Thm.14] following the proof of [30, Thm.1.1]. Now noting that

$$\text{Av}(\vec{M}) = \Delta_{\kappa, \gamma}(\text{Av}(\vec{M})|_{\kappa} \gamma) \quad (252)$$

and  $\Omega_{\text{Av}(U), T} = I$ , part (c) of Theorem 1.2 proves part (b) of Theorem 1.7.

**5.6. The inhomogeneous equation.** We now consider Equation (35). We can convert this into a homogeneous equation of one degree higher

$$tf(t; q) + (1 - 3qt)f(qt; q) + (3tq^2 - 1)f(q^2t; q) - tq^3f(q^3t; q) = 0 \quad (253)$$

with Newton polygon shown in Figure 5.

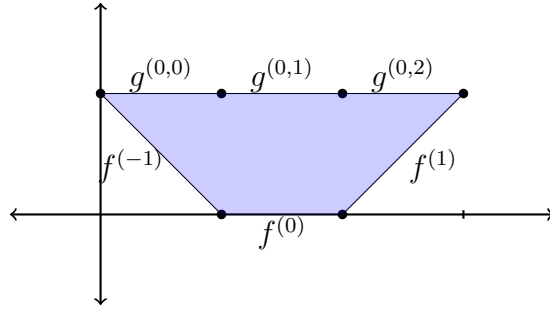


FIGURE 5. The Newton polygon of Equation (35).

Now from the general theory the solutions  $f^{(\pm 1)}$  and  $g^{(0,0)}$  and  $g^{(0,1)}$  are the same as the solutions in Section 5.1. We do find two additional solutions which are in fact solutions to the inhomogeneous equation, namely

$$\hat{f}^{(0)}(t, q) = \sum_{k=0}^{\infty} (-1)^k q^{-k(k+1)/2} (q; q)_k^2 t^k, \quad (254)$$

a divergent formal power series solution at  $t = 0$  and

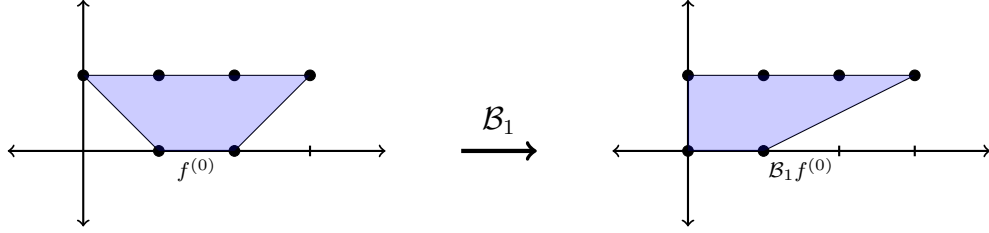
$$\begin{aligned} g^{(0,2)}(t; q) &= \sum_{k=0}^{\infty} \left( \frac{1}{2} \left( \frac{1}{2} E_1(q) - \frac{1}{2} + \sum_{j=1}^k \frac{1+q^j}{1-q^j} \right)^2 - \left( \frac{1}{2} E_1(q) - \frac{1}{2} + \sum_{j=1}^k \frac{1+q^j}{1-q^j} \right) \frac{\theta'(t^{-1}; q)}{\theta(t^{-1}; q)} \right. \\ &\quad \left. + \frac{1}{2} \frac{\theta''(t^{-1}; q)}{\theta(t^{-1}; q)} + \sum_{j=1}^k \frac{q^j}{(1-q^j)^2} - \frac{1}{24} - \frac{1}{24} E_2(q) \right) (-1)^k \frac{q^{k(k+1)/2}}{(q; q)_k^2} t^{-1-k}, \end{aligned} \quad (255)$$

where  $E_2(q) = 1 - 24 \sum_{k=0}^{\infty} q^k / (1 - q^k)^2$ , a convergent solution at  $t = \infty$ . So, we must resum  $\hat{f}^{(0)}(t, q)$ . For  $|\xi| < 1$ , we have

$$\mathcal{B}_1 \hat{f}^{(0)}(\xi, q) = \sum_{k=0}^{\infty} (q; q)_k^2 \xi^k \quad (256)$$

which can be analytically continued away from  $\xi \in q^{\mathbb{Z} \leq 0}$  using the relation

$$(1 - \xi) \mathcal{B}_1 \hat{f}^{(0)}(\xi, q) + 2q\xi \mathcal{B}_1 \hat{f}^{(0)}(q\xi, q) - q^2 \xi \mathcal{B}_1 \hat{f}^{(0)}(q^2 \xi, q) = 1. \quad (257)$$



Therefore, we define

$$f^{(0)}(t, \lambda_2, q) = (\mathcal{L}_1 \mathcal{B}_1 \hat{f}^{(0)})(t, \lambda_2, q) = \frac{1}{\theta(\lambda_2; q)} \sum_{k \in \mathbb{Z}} (-1)^k q^{k(k+1)} \lambda_2^k \mathcal{B}_1 \hat{f}^{(0)}(q^k \lambda_2 t, q). \quad (258)$$

Now we consider the matrices

$$\begin{aligned} U(t, \lambda_1, \lambda_2, q) &= W(f^{(1)}(t, \lambda_1, q), f^{(-1)}(t, q), f^{(0)}(t, \lambda_2, q)) \\ &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ q^{-2}t^{-1} & -q^{-2} & q^{-2}t^{-1} - 2q^{-1} \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ f^{(1)}(t, \lambda_1, q) & f^{(-1)}(t, q) & f^{(0)}(t, \lambda_2, q) \\ f^{(1)}(qt, \lambda_1, q) & f^{(-1)}(qt, \lambda_1, q) & f^{(0)}(qt, \lambda_2, q) \end{pmatrix} \end{aligned} \quad (259)$$

and

$$\begin{aligned} V(t, q) &= W(g^{(0,0)}(t, q), g^{(0,1)}(t, q), g^{(0,2)}(t, q)) \\ &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ q^{-2}t^{-1} & -q^{-2} & q^{-2}t^{-1} - 2q^{-1} \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ g^{(0,0)}(t, q) & g^{(0,1)}(t, q) & g^{(0,2)}(t, q) \\ g^{(0,0)}(qt, q) & g^{(0,1)}(qt, q) & g^{(0,2)}(qt, q) \end{pmatrix}. \end{aligned} \quad (260)$$

With these formulae we will now go on to prove Theorem 1.8.

*Proof of Theorem 1.8.* The identification of the first two columns of  $\vec{M}(t, \lambda_1, \lambda_2, q)$  with those in Equation (37) follows from Theorem 1.7. Equations (259) and (260), together with Lemma 5.4 imply that

$$-\det(U(t, \lambda_1, \lambda_2, q)) = \det(V(t, q)) = q^{-3}t^{-3}. \quad (261)$$

Hence,  $\det(\vec{M}(t, \lambda_1, \lambda_2, q)) = -1$  which in turn implies that  $\vec{M}_{3,3}(t, \lambda_2, q) = 1$ .

Now, Equation (37), written in the form  $U(t, \lambda_1, \lambda_2, q) = V(t, q) \vec{M}(t, \lambda_1, \lambda_2, q)$ , together with Equations (259) and (260) and the fact that  $\vec{M}_{3,3} = 1$  imply that

$$f^{(0)} = \vec{M}_{1,3} g^{(0,0)} + \vec{M}_{2,3} g^{(0,1)} + g^{(0,2)}. \quad (262)$$



Thus, to determine the two remaining entries of the monodromy matrix, we need to show that the function

$$\mathcal{E}_3 := f^{(0)} - (m_{1,3}g^{(0,0)} + m_{2,3}g^{(0,1)} + g^{(0,2)}) \quad (263)$$

vanishes identically, where

$$m_{1,3}(t, q) = \wp(t, q), \quad m_{2,3}(t, \lambda_2, q) = \frac{1}{2} \frac{\wp'(t, q) - \wp'(\lambda_2, q)}{\wp(t, q) - \wp(\lambda_2, q)} \quad (264)$$

denote the two entries of the matrix on the right hand-side of Equation (37). Note that  $\mathcal{E}_3(t, \lambda_2, q)$  is a meromorphic function of  $t$  with potential simple poles at  $\lambda_2^{-1}q^{\mathbb{Z}}$  (coming from  $f^{(0)}$  and  $m_{2,3}$ ) and potential double poles at  $q^{\mathbb{Z}}$  (coming from  $g^{(0,1)}$ ,  $g^{(0,2)}$ ,  $m_{1,3}$  and  $m_{2,3}$ ). Since  $\text{Res}_{t=q^{-m}\lambda_2^{-1}} m_{2,3}(t, \lambda_2, q) \frac{dt}{2\pi it} = 1$ , combined with equation (211) gives

$$\text{Res}_{t=q^{-m}\lambda_2^{-1}} m_{2,3}(t, \lambda_2, q) g^{(0,1)}(t, q) \frac{dt}{2\pi it} = f^{(-1)}(q^{-m}\lambda_2^{-1}, q), \quad (265)$$

we see from Lemma 5.6 that  $\mathcal{E}_3$  has no poles at  $\lambda_2^{-1}q^{\mathbb{Z}}$ . Now noting that

$$m_{2,3}(t, \lambda_2, q) = \frac{\theta'(t^{-1}; q)}{\theta(t^{-1}; q)} + \frac{\theta'(\lambda^{-1}; q)}{\theta(\lambda^{-1}; q)} - \frac{\theta'(\lambda^{-1}t^{-1}; q)}{\theta(\lambda^{-1}t^{-1}; q)} + \frac{1}{2}, \quad (266)$$

the only terms that contribute to the polar part of

$$g^{(0,2)}(t, q) + m_{2,3}(t, \lambda_2, q)g^{(0,1)}(t, q) + m_{1,3}(t, q)g^{(0,0)}(t, q) \quad (267)$$

at  $t = q^m \varepsilon$  for  $\varepsilon \sim 1$  are

$$\begin{aligned} & \sum_{k=0}^{\infty} \left( - \left( k - 2E_1^{(k)}(q) \right) \frac{\theta'(\varepsilon^{-1}; q)}{\theta(\varepsilon^{-1}; q)} + \frac{1}{2} \frac{\theta''(q^{-m}\varepsilon^{-1}; q)}{\theta(q^{-m}\varepsilon^{-1}; q)} \right. \\ & \quad + \left( k - m - \frac{\theta'(\varepsilon^{-1}; q)}{\theta(\varepsilon^{-1}; q)} - 2E_1^{(k)}(q) \right) \frac{\theta'(\varepsilon^{-1}; q)}{\theta(\varepsilon^{-1}; q)} - \frac{1}{2} \frac{\theta'(\varepsilon^{-1}; q)}{\theta(\varepsilon^{-1}; q)} \\ & \quad \left. + \frac{\theta'(\varepsilon^{-1}; q)^2}{\theta(\varepsilon^{-1}; q)^2} - \frac{\theta''(\varepsilon^{-1}; q)}{\theta(\varepsilon^{-1}; q)} \right) (-1)^k \frac{q^{k(k+1)/2 - m - mk}}{(q; q)_k^2} \varepsilon^{-1-k} \\ & = \sum_{k=0}^{\infty} \left( \frac{1}{2} \frac{\theta''(q^{-m}\varepsilon^{-1}; q)}{\theta(q^{-m}\varepsilon^{-1}; q)} - \left( m + \frac{1}{2} \right) \frac{\theta'(\varepsilon^{-1}; q)}{\theta(\varepsilon^{-1}; q)} - \frac{\theta''(\varepsilon^{-1}; q)}{\theta(\varepsilon^{-1}; q)} \right) (-1)^k \frac{q^{k(k+1)/2}}{(q; q)_k^2} \varepsilon^{-1-k}. \end{aligned} \quad (268)$$

This has a potentially simple pole at  $\varepsilon = 1$  however noting that

$$\begin{aligned} \text{Res}_{\varepsilon=1} \frac{\theta''(q^{-m}\varepsilon^{-1}; q)}{\theta(q^{-m}\varepsilon^{-1}; q)} \frac{d\varepsilon}{2\pi i \varepsilon} &= -2m + 1 \\ \text{Res}_{\varepsilon=1} \frac{\theta'(\varepsilon^{-1}; q)}{\theta(\varepsilon^{-1}; q)} \frac{d\varepsilon}{2\pi i \varepsilon} &= -1 \end{aligned} \quad (269)$$

we see that the residue vanishes and therefore (267) has a removable singularity at all points  $t \in q^{\mathbb{Z}}$ . Therefore, we see that  $\mathcal{E}_3$  is a holomorphic (for  $t \in \mathbb{C}^\times$ ) solution of Equation (28) and therefore from Corollary 5.5 we see that  $\mathcal{E}_3(t, \lambda_2, q) = C(q)g^{(0,0)}(t, q)$ . To finish the proof

we note that for  $t$  on some compact set where the functions are holomorphic, Lemma 5.1 and Lemma 5.7 implies that

$$\begin{aligned} -\frac{C(q)}{(q; q)_\infty^2} &= \lim_{r \rightarrow \infty} \frac{C(q)g^{(0,0)}(t^r, q)}{\theta(q^r t; q)} \\ &= \lim_{r \rightarrow \infty} \frac{f^{(0)}(t, \lambda_2, q) - (g^{(0,2)}(t, q) + m_{2,3}(t, \lambda_2, q)g^{(0,1)}(t, q) + m_{1,3}(t, q)g^{(0,0)}(t, q))}{\theta(q^r t; q)} \\ &= \frac{1}{(q; q)_\infty^2} (m_{1,3}(t^{-1}, q) - m_{1,3}(t, q)) = 0, \end{aligned}$$

where we note that  $m_{1,3}(t, q)$  is the Weierstrass  $\wp$ -function plus a constant which is even in  $z$  where  $t = \mathbf{e}(z)$ .

Our next task is to compute the residues of  $f^{(0)}$ .

**Lemma 5.6.** (a) The function  $\mathcal{B}_1 \hat{f}^{(0)}(\xi, q)$  has simple poles at  $\xi \in q^{\mathbb{Z}}$  with residue

$$R(m, q) := \text{Res}_{\xi=q^{-m}} \mathcal{B}_1 \hat{f}^{(0)}(\xi, q) \frac{d\xi}{2\pi i \xi} = -(q; q)_\infty^2 \sum_{\ell=0}^m \frac{q^m}{(q; q)_\ell (q; q)_{m-\ell}}. \quad (270)$$

(b) The function  $f^{(0)}(t, q)$  has simple poles at  $t \in \lambda_2^{-1} q^{\mathbb{Z}}$  with residue

$$\text{Res}_{t=q^{-m} \lambda_2^{-1}} f^{(0)}(t, \lambda_2, q) \frac{dt}{2\pi i t} = f^{(-1)}(q^{-m} \lambda_2^{-1}, q). \quad (271)$$

*Proof.* For part (a), notice that

$$\begin{aligned} \mathcal{B}_1 \hat{f}^{(0)}(\xi, q) &= \sum_{k=0}^{\infty} (q; q)_k^2 \xi^k = (q; q)_\infty^2 \sum_{k=0}^{\infty} \frac{1}{(q^{k+1}; q)_\infty} \xi^k \\ &= (q; q)_\infty^2 \sum_{k, \ell, n=0}^{\infty} \frac{q^{kn+k\ell+n+\ell}}{(q; q)_\ell (q; q)_n} \xi^k = (q; q)_\infty^2 \sum_{\ell, n=0}^{\infty} \frac{q^{n+\ell}}{(q; q)_\ell (q; q)_n (1 - q^{n+\ell} \xi)} \end{aligned} \quad (272)$$

is the analytic continuation of  $\mathcal{B}_1 \hat{f}^{(0)}$ . From this, it clearly follows that the only poles are simple, located at  $\xi = q^{\mathbb{Z}}$ . (Alternatively, one can also use the linear  $q$ -difference equation satisfied by this function, though this is not needed here). We compute the residue as follows:

$$\begin{aligned} R(m, q) &= \text{Res}_{\xi=q^{-m}} (q; q)_\infty^2 \sum_{\ell, n=0}^{\infty} \frac{q^{n+\ell}}{(q; q)_\ell (q; q)_n (1 - q^{n+\ell} \xi)} \frac{d\xi}{2\pi i \xi} \\ &= -(q; q)_\infty^2 \sum_{\ell=0}^m \frac{q^m}{(q; q)_\ell (q; q)_{m-\ell}}. \end{aligned} \quad (273)$$

For part (b), the pole structure is clear noting that Equation (258) is convergent if all the terms are and that  $\mathcal{B}_1 \hat{f}^{(0)}(q^k t \lambda_2, q)$  has poles at  $t \in \lambda_2^{-1} q^{\mathbb{Z} \leq -k}$ . We compute the residues as

follows:

$$\begin{aligned}
 \operatorname{Res}_{t=q^{-m}\lambda_2^{-1}} f^{(0)}(t, \lambda_2, q) \frac{dt}{2\pi it} &= \operatorname{Res}_{t=q^{-m}\lambda_2^{-1}} (\mathcal{L}_1 \mathcal{B}_1 \hat{f}^{(0)})(t, \lambda_2, q) \frac{dt}{2\pi it} \\
 &= \operatorname{Res}_{t=q^{-m}\lambda_2^{-1}} \frac{1}{\theta(\lambda_2, q)} \sum_{k \in \mathbb{Z}} (-1)^k q^{k(k+1)/2} \lambda_2^k \mathcal{B}_1 \hat{f}^{(0)}(q^k \lambda_2 t, q) \frac{dt}{2\pi it} \\
 &= \frac{1}{\theta(\lambda_2, q)} \sum_{k \in \mathbb{Z}} (-1)^k q^{k(k+1)/2} \lambda_2^k R(m-k, q) \\
 &= \frac{1}{\theta(\lambda_2, q)} (-1)^m q^{m(m+1)/2} \lambda_2^m \sum_{k=0}^{\infty} (-1)^k q^{k(k-1)/2} (q^{-m} \lambda_2^{-1})^k R(k, q) \\
 &= \frac{-(q; q)_{\infty}^2}{\theta(\lambda_2, q)} (-1)^m q^{m(m+1)/2} \lambda_2^m \sum_{k, \ell=0}^{\infty} (-1)^k \frac{q^{k(k+1)/2}}{(q; q)_{\ell} (q; q)_{k-\ell}} (q^{-m} \lambda_2^{-1})^k \\
 &= f^{(-1)}(q^{-m} \lambda_2^{-1}, q).
 \end{aligned} \tag{274}$$

□

**Lemma 5.7.** For  $t$  in some compact set where the functions are holomorphic, we have

$$\lim_{r \rightarrow \infty} \frac{g^{(0,2)}(q^r t, q)}{\theta(q^r t; q)} = \frac{1}{(q; q)_{\infty}^2} m_{1,3}(t^{-1}, q) \quad \lim_{r \rightarrow \infty} \frac{f^{(0)}(q^r t, q)}{\theta(q^r t; q)} = 0. \tag{275}$$

*Proof.* We have

$$\begin{aligned}
 &(-1)^r q^{r(r+1)/2} t^{r+1} g^{(0,2)}(q^r t, q) \\
 &= \sum_{k=-r}^{\infty} \left( \frac{1}{2} \left( \frac{1}{2} E_1(q) - r - \frac{1}{2} - \frac{\theta'(t^{-1}; q)}{\theta(t^{-1}; q)} + \sum_{j=1}^{k+r} \frac{1+q^j}{1-q^j} \right)^2 \right. \\
 &\quad \left. - \left( \frac{1}{2} E_1(q) - r - \frac{1}{2} - \frac{\theta'(t^{-1}; q)}{\theta(t^{-1}; q)} + \sum_{j=1}^{k+r} \frac{1+q^j}{1-q^j} \right) \frac{\theta'(t^{-1}; q)}{\theta(t^{-1}; q)} \right. \\
 &\quad \left. + \frac{1}{2} \frac{\theta''(t^{-1}; q)}{\theta(t^{-1}; q)} + \sum_{j=1}^{k+r} \frac{q^j}{(1-q^j)^2} - \frac{1}{24} - \frac{1}{24} E_2(q) \right) (-1)^k \frac{q^{k(k+1)/2}}{(q; q)_{k+r}^2} t^{-k}.
 \end{aligned} \tag{276}$$

Then noting that  $1/24 - E_2(q)/24 - \sum_{j=1}^k q^j/(1-q^j)^2 = O(q^{k+1})$  the proof then proceeds as in Lemma 5.1. □

To finish the proof of Theorem 1.8 we will use the state integrals introduced in [28, Eqn.(73)].

$$\int_{\mathbb{R}+i\epsilon} \Phi_{\mathbf{b}}(x)^2 \frac{\exp(-\pi i x^2 - 2\pi \frac{zx}{\mathbf{b}})}{1 + \tilde{q}^{1/2} \exp(-\frac{2\pi x}{\mathbf{b}})} dx. \tag{277}$$

The factorisation of this integral was done in [28, Thm.7]. The fact this module is not self dual (see Proposition 2.6) means that additional functions arise in the factorisation.

It was shown, using Equation (52), that Equation (277) factors as an elementary function holomorphic in  $\mathbb{C}'$  times

$$\mathcal{I}(z, \tau) = \tau^2 g^{(0,2)}(\tilde{t}, \tilde{q}) + \tau g^{(0,1)}(\tilde{t}, \tilde{q}) L_0(t, q) - g^{(0,0)}(\tilde{t}, \tilde{q}) L_1(t, q) \quad (278)$$

where

$$\begin{aligned} L_0(t, q) &= 1 - \frac{1}{2} E_1(q) + \frac{\theta'(t^{-1}; q)}{\theta(t^{-1}; q)} + \sum_{k=1}^{\infty} (-1)^k \frac{q^{k(k+1)/2}}{(q; q)_k^2 (1 - q^k)} t^{-k} \\ L_1(t, q) &= -\frac{5}{12} + \frac{1}{2} E_1 - E_1(q)^2 - \frac{1}{24} E_2(q) - \left( \frac{1}{8} - \frac{1}{8} E_1(q) \right) \frac{\theta'(t^{-1}; q)}{\theta(t^{-1}; q)} - \frac{\theta''(t^{-1}; q)}{2\theta(t^{-1}; q)} \\ &\quad + \sum_{k=1}^{\infty} (-1)^k \frac{q^{k(k+1)/2}}{(q; q)_k^2 (1 - q^k)} t^{-k} \left( \frac{1}{2} E_1(q) - \frac{1}{2} - \frac{\theta'(t^{-1}; q)}{\theta(t^{-1}; q)} + \sum_{j=1}^k \frac{1 + q^j}{1 - q^j} + \frac{q^k}{1 - q^k} \right). \end{aligned} \quad (279)$$

These functions  $L_i$  were then shown to satisfy

$$\begin{aligned} L_0(t, q) - L_0(qt, q) &= -qtg^{(0,0)}(qt, q) \\ L_1(t, q) - L_1(qt, q) &= -qtg^{(0,1)}(qt, q). \end{aligned} \quad (280)$$

Notice that functions satisfying these equations are unique up to the addition of an elliptic function. Therefore, with exactly the same argument in [28, Thm.4] with the addition of checking the principal parts of the LHS and RHS we can then show that

$$V(t, q)^{-1} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ q^{-2}t^{-1} & -q^{-2} & q^{-2}t^{-1} - 2q^{-1} \end{pmatrix} = \begin{pmatrix} -L_1(t, q) & -qt^2g^{(0,1)}(qt, q) & qt^2g^{(0,1)}(t, q) \\ L_0(t, q) & qt^2g^{(0,0)}(qt, q) & -qt^2g^{(0,0)}(qt, q) \\ 1 & 0 & 0 \end{pmatrix}. \quad (281)$$

Therefore, we see that the entries of  $\Omega_{V,S}$  are combinations of elementary functions times  $\mathcal{I}(z+n+m\tau, \tau)$  for  $m, n \in \{-2, -1, 0, 1, 2\}$ . Using the explicit expressions for the monodromy one can prove it satisfies Equation (32) with equal weights  $(2, 1, 0)$  and again using part (c) of Theorem 1.2 complete the proof.  $\square$

**5.7. The  $x$ -deformation.** We will first discuss the two variable holonomic system given by the homogeneous equations

$$tq^{-1}f(q^{-1}t, x, q) + (1 - (x^{-1} + x)t)f(t, x, q) + tqf(qt, x, q) = 0, \quad (282a)$$

$$\begin{aligned} (1 - qx)(1 - q^{-1}x^2)f(t, qx, q) \\ - (x - 1)^2(x + 1)(x^2t - x - (q^{-1} + q)t - x^{-1} + x^{-2}t)f(t, x, q) \\ + (1 - qx^2)(1 - q^{-1}x)f(t, q^{-1}x, q) = 0, \end{aligned} \quad (282b)$$

$$(1 - xq)(f(t, qx, q) - x^{-1}f(qt, qx, q)) = (1 - x^{-1})(f(t, x, q) - qxf(qt, x, q)). \quad (282c)$$

This is not a random system of equations, instead they are the defining equations of the  $t$ -deformation of the homogeneous  $\hat{A}$ -polynomial of the  $4_1$  knot. This system appeared in [29, Eqn.(10)] and [26, Eqn.(134)], and it is  $q$ -holonomic of rank 2 in the variables  $(t, x)$ . The system is symmetric under the involution  $x \leftrightarrow x^{-1}$  (which corresponds to the Weyl symmetry

in the color of the Jones polynomial of the knot) and our solutions will also be invariant under this involution. As a result the monodromy connecting  $x = 0$  to  $x = \infty$  is the identity.

To construct solutions, we will apply the Frobenius method to the Equation (282a). This has Newton polygon depicted in Figure 4. Notice that the indicial polynomial  $(\rho - 1)^2$  for the top edge of Equation (28) now becomes  $(\rho - x)(\rho - x^{-1})$  for Equation (282a). We can then normalise so that the solutions satisfy the full system of equations (282a), (282b), (282c). The solutions to this homogeneous system are then given by

$$\begin{aligned} f^{(-1)}(t, x, q) &= \frac{\theta(q^{-1}x; q)}{(1-x)\theta(q^{-1}t; q)(q; q)_\infty} \sum_{k=0}^{\infty} \sum_{\ell=0}^k (-1)^k \frac{q^{k(k+1)/2} x^{2\ell-k}}{(q; q)_\ell (q; q)_{k-\ell}} t^k \\ f^{(1)}(t, x, \lambda_1, q) &= \frac{\theta(t; q)(q; q)_\infty}{(1-x)\theta(x; q)} \mathcal{L}_{1/2} \mathcal{B}_{1/2} \hat{f}^{(1)}(t, x, \lambda_1, q) \\ g^{(0, x^{-1})}(t, x, q) &= \frac{\theta(q^{-1}x; q)\theta(tx; q)(qx^2; q)_\infty}{\theta(t; q)\theta(x^2; q)(1-x)(q; q)_\infty} \sum_{k=0}^{\infty} (-1)^k \frac{q^{k(k+1)/2} x^k}{(q; q)_k (qx^2; q)_k} t^{-k-1} \\ g^{(0, x)}(t, x, q) &= \frac{\theta(q^{-1}x^{-1}; q)\theta(tx^{-1}; q)(qx^{-2}; q)_\infty}{\theta(t; q)\theta(x^{-2}; q)(1-x^{-1})(q; q)_\infty} \sum_{k=0}^{\infty} (-1)^k \frac{q^{k(k+1)/2} x^{-k}}{(q; q)_k (qx^{-2}; q)_k} t^{-k-1} \end{aligned} \quad (283)$$

where

$$\hat{f}^{(1)}(t, x, q) = \sum_{k=0}^{\infty} \sum_{\ell=0}^k \frac{q^{\ell^2 - k\ell} x^{2\ell - k}}{(q; q)_\ell (q; q)_{k-\ell}} t^k. \quad (284)$$

Notice that we have

$$\mathcal{B}_{1/2} \hat{f}^{(1)}(t, x, q) = \sum_{k=0}^{\infty} \sum_{\ell=0}^k (-1)^k \frac{q^{k(k+1)/4 + \ell^2 - k\ell} x^{2\ell - k}}{(q; q)_\ell (q; q)_{k-\ell}} t^k \quad (285)$$

which satisfies

$$(1 - q^{1/2}\xi^2)\mathcal{B}_{1/2}\hat{f}^{(1)}(\xi, q) + (x + x^{-1})\xi\mathcal{B}_{1/2}\hat{f}^{(1)}(q^{1/2}\xi, q) - \mathcal{B}_{1/2}\hat{f}^{(1)}(q\xi, q) = 0. \quad (286)$$

We can complete this to include the inhomogeneous solutions to the system of Equations (39a), (39b), (39c). Equations (39a) can be made third order homogenous which has Newton polygon depicted in Figure 5. The indicial polynomial of the top edge now factors as  $(\rho - x)(\rho - x^{-1})(\rho - 1)$ . This gives two additional solutions as in Section 5.6. Firstly we have

$$\hat{f}^{(0)}(t, x, q) = \sum_{k=0}^{\infty} (-1)^k q^{-k(k+1)/2} (qx; q)_k (qx^{-1}; q)_k t^k, \quad (287)$$

a divergent formal power series solution at  $t = 0$ . Then

$$\mathcal{B}_1 f^{(0)}(\xi, x, q) = \sum_{k=0}^{\infty} (qx; q)_k (qx^{-1}; q)_k t^k, \quad (288)$$

which can be analytically continued away from  $\xi \in q^{\mathbb{Z} \leq 0}$  using the relation

$$(1 - \xi)\mathcal{B}_1 \hat{f}^{(0)}(\xi, q) + (x + x^{-1})q\xi\mathcal{B}_1 \hat{f}^{(0)}(q\xi, q) - q^2\xi\mathcal{B}_1 \hat{f}^{(0)}(q^2\xi, q) = 1. \quad (289)$$

So we define

$$f^{(0)}(t, x, \lambda_2, q) = (\mathcal{L}_1 \mathcal{B}_1 \hat{f}^{(0)})(t, x, \lambda_2, q) = \frac{1}{\theta(\lambda_2; q)} \sum_{k \in \mathbb{Z}} (-1)^k q^{k(k+1)} \lambda_2^k \mathcal{B}_1 \hat{f}^{(0)}(q^k \lambda_2 t, x, q). \quad (290)$$

The second solution is then given by

$$g^{(0)}(t, x, q) = \sum_{k=0}^{\infty} (-1)^k \frac{q^{k(k+1)/2}}{(x; q)_{k+1} (x^{-1}; q)_{k+1}} t^{-k-1}. \quad (291)$$

We then take

$$\begin{aligned} U(t, x, \lambda_1, \lambda_2, q) &= W(f^{(1)}(t, x, \lambda_1, q), f^{(-1)}(t, x, q), f^{(0)}(t, x, \lambda_2, q)), \\ V(t, q) &= W(g^{(0, x^{-1})}(t, x, q), g^{(0, x)}(t, x, q), g^{(0, 1)}(t, x, q)). \end{aligned} \quad (292)$$

Before we give the proof note the immediate symmetries

$$\begin{aligned} g^{(0, x^{-1})}(t, x^{-1}, q) &= g^{(0, x)}(t, x, q), & f^{(-1)}(t, x^{-1}, q) &= f^{(-1)}(t, x, q), \\ g^{(0, x)}(t, x^{-1}, q) &= g^{(0, x^{-1})}(t, x, q), & f^{(-1)}(t, x^{-1}, q) &= f^{(-1)}(t, x, q), \\ g^{(0, 1)}(t, x^{-1}, q) &= g^{(0, 1)}(t, x, q), & f^{(0)}(t, x^{-1}, q) &= f^{(0)}(t, x, q). \end{aligned} \quad (293)$$

*Proof of Theorem 1.9.* First we prove the existence of the elliptic function  $m_{2,1}$  with the properties listed in Theorem 1.9. Note that the only restriction to an existence of an elliptic function with prescribed poles and residues is the vanishing of the sum of the residues on a fundamental domain. Here, there are six residues, and their sum vanishing is equivalent to

$$\begin{aligned} 1 &= \frac{\theta(q^{3/4} \lambda_1^{-1}; q) \theta(q^{-1/4} x; q) \theta(q^{-3/4} x; q) \theta(q^{-3/4} \lambda_1; q)}{2\theta(x; q) \theta(x^{-1}; q) \theta(q^{-1} \lambda_1; q) \theta(q^{-3/2} \lambda_1; q)} \\ &+ \frac{\theta(-q^{3/4} \lambda_1^{-1}; q) \theta(-q^{-1/4} x; q) \theta(-q^{-3/4} x; q) \theta(-q^{-3/4} \lambda_1; q)}{2\theta(x; q) \theta(x^{-1}; q) \theta(q^{-1} \lambda_1; q) \theta(q^{-3/2} \lambda_1; q)} \\ &+ \frac{\theta(q^{1/4} \lambda_1^{-1}; q) \theta(q^{-1/4} x; q) \theta(q^{-3/4} x; q) \theta(q^{-1/4} \lambda_1; q)}{2\theta(x; q) \theta(x^{-1}; q) \theta(q^{-1/2} \lambda_1; q) \theta(q^{-1} \lambda_1; q)} \\ &+ \frac{\theta(-q^{1/4} \lambda_1^{-1}; q) \theta(-q^{-1/4} x; q) \theta(-q^{-3/4} x; q) \theta(-q^{-1/4} \lambda_1; q)}{2\theta(x; q) \theta(x^{-1}; q) \theta(q^{-1/2} \lambda_1; q) \theta(q^{-1} \lambda_1; q)} \\ &= \left( \frac{\theta(q^{3/4} \lambda_1^{-1}; q) \theta(q^{-3/4} \lambda_1; q)}{2\theta(q^{-1} \lambda_1; q) \theta(q^{-3/2} \lambda_1; q)} + \frac{\theta(q^{1/4} \lambda_1^{-1}; q) \theta(q^{-1/4} \lambda_1; q)}{2\theta(q^{-1/2} \lambda_1; q) \theta(q^{-1} \lambda_1; q)} \right) \frac{\theta(q^{-1/4} x; q) \theta(q^{-3/4} x; q)}{\theta(x; q) \theta(x^{-1}; q)} \\ &+ \left( \frac{\theta(-q^{3/4} \lambda_1^{-1}; q) \theta(-q^{-3/4} \lambda_1; q)}{2\theta(q^{-1} \lambda_1; q) \theta(q^{-3/2} \lambda_1; q)} + \frac{\theta(-q^{1/4} \lambda_1^{-1}; q) \theta(-q^{-1/4} \lambda_1; q)}{2\theta(q^{-1/2} \lambda_1; q) \theta(q^{-1} \lambda_1; q)} \right) \frac{\theta(-q^{-1/4} x; q) \theta(-q^{-3/4} x; q)}{\theta(x; q) \theta(x^{-1}; q)}. \end{aligned} \quad (294)$$

To prove this identity, notice that the right hand side is elliptic in  $\lambda_1 \mapsto q^{1/2} \lambda_1$  and  $x \mapsto qx$ . Moreover,  $\lambda_1$  has only one potential simple pole in its fundamental domain and therefore this is constant in  $\lambda_1$ . Then  $x$  has a potential double pole which cancels. Then specialising

the RHS to  $\lambda_1 = q^{1/4}$  and  $x = -q^{1/4}$  gives

$$\frac{\theta(q^{1/2}; q)\theta(q^{-1/2}; q)}{2\theta(q^{-3/4}; q)\theta(q^{-5/4}; q)} \frac{\theta(-1; q)\theta(-q^{-1/2}; q)}{\theta(-q^{1/4}; q)\theta(-q^{-1/4}; q)}. \quad (295)$$

This can be proven to equal one by elementary means using the Jacobi triple product identity. Therefore, such an  $m_{2,1}(t, x, \lambda_1, q)$  exists. Explicitly, we have

$$\begin{aligned} m_{2,1}(t, x, \lambda_1, q) &= \frac{-1}{2} \frac{\theta'(tx; q)}{\theta(tx; q)} + \frac{-1}{2} \frac{\theta'(tx^{-1}; q)}{\theta(tx^{-1}; q)} \\ &+ \frac{\theta(q^{3/4}\lambda_1^{-1}; q)\theta(q^{-1/4}x; q)\theta(q^{-3/4}x; q)\theta(q^{-3/4}\lambda_1; q)}{2\theta(x; q)\theta(x^{-1}; q)\theta(q^{-1}\lambda_1; q)\theta(q^{-3/2}\lambda_1; q)} \frac{\theta'(t\lambda_1 q^{1/4})}{\theta(t\lambda_1 q^{1/4})} + \frac{\theta(-q^{3/4}\lambda_1^{-1}; q)\theta(-q^{-1/4}x; q)\theta(-q^{-3/4}x; q)\theta(-q^{-3/4}\lambda_1; q)}{2\theta(x; q)\theta(x^{-1}; q)\theta(q^{-1}\lambda_1; q)\theta(q^{-3/2}\lambda_1; q)} \frac{\theta'(-t\lambda_1 q^{1/4})}{\theta(-t\lambda_1 q^{1/4})} \\ &+ \frac{\theta(q^{1/4}\lambda_1^{-1}; q)\theta(q^{-1/4}x; q)\theta(q^{-3/4}x; q)\theta(q^{-1/4}\lambda_1; q)}{2\theta(x; q)\theta(x^{-1}; q)\theta(q^{-1/2}\lambda_1; q)\theta(q^{-1}\lambda_1; q)} \frac{\theta'(t\lambda_1 q^{3/4})}{\theta(t\lambda_1 q^{3/4})} + \frac{\theta(-q^{1/4}\lambda_1^{-1}; q)\theta(-q^{-1/4}x; q)\theta(-q^{-3/4}x; q)\theta(-q^{-1/4}\lambda_1; q)}{2\theta(x; q)\theta(x^{-1}; q)\theta(q^{-1/2}\lambda_1; q)\theta(q^{-1}\lambda_1; q)} \frac{\theta'(-t\lambda_1 q^{3/4})}{\theta(-t\lambda_1 q^{3/4})} \\ &\quad - \frac{-1}{2} \frac{\theta'(x; q)}{\theta(x; q)} - \frac{-1}{2} \frac{\theta'(x^{-1}; q)}{\theta(x^{-1}; q)} - \frac{\theta(q^{3/4}\lambda_1^{-1}; q)\theta(q^{-1/4}x; q)\theta(q^{-3/4}x; q)\theta(q^{-3/4}\lambda_1; q)}{2\theta(x; q)\theta(x^{-1}; q)\theta(q^{-1}\lambda_1; q)\theta(q^{-3/2}\lambda_1; q)} \frac{\theta'(\lambda_1 q^{1/4})}{\theta(\lambda_1 q^{1/4})} \\ &- \frac{\theta(-q^{3/4}\lambda_1^{-1}; q)\theta(-q^{-1/4}x; q)\theta(-q^{-3/4}x; q)\theta(-q^{-3/4}\lambda_1; q)}{2\theta(x; q)\theta(x^{-1}; q)\theta(q^{-1}\lambda_1; q)\theta(q^{-3/2}\lambda_1; q)} \frac{\theta'(-\lambda_1 q^{1/4})}{\theta(-\lambda_1 q^{1/4})} - \frac{\theta(q^{1/4}\lambda_1^{-1}; q)\theta(q^{-1/4}x; q)\theta(q^{-3/4}x; q)\theta(q^{-1/4}\lambda_1; q)}{2\theta(x; q)\theta(x^{-1}; q)\theta(q^{-1/2}\lambda_1; q)\theta(q^{-1}\lambda_1; q)} \frac{\theta'(\lambda_1 q^{3/4})}{\theta(\lambda_1 q^{3/4})} \\ &\quad - \frac{\theta(-q^{1/4}\lambda_1^{-1}; q)\theta(-q^{-1/4}x; q)\theta(-q^{-3/4}x; q)\theta(-q^{-1/4}\lambda_1; q)}{2\theta(x; q)\theta(x^{-1}; q)\theta(q^{-1/2}\lambda_1; q)\theta(q^{-1}\lambda_1; q)} \frac{\theta'(-\lambda_1 q^{3/4})}{\theta(-\lambda_1 q^{3/4})}. \end{aligned} \quad (296)$$

Next we calculate the second column of the monodromy matrix using the same argument as Equation (210) from [54].

$$\begin{aligned} \hat{f}^{(-1)}(t, x; q) &= \oint_0 \mathcal{B}_{-1}(g)(\xi, x; q)\theta(t/\xi; q) \frac{d\xi}{2\pi i \xi} = \oint_0 \frac{\theta(t/\xi; q)}{(x\xi; q)_\infty (x^{-1}\xi; q)_\infty} \frac{d\xi}{2\pi i \xi} \\ &= - \sum_{k=0}^{\infty} (\text{Res}_{\xi=x^\pm q^{-k}}) \frac{\theta(t/\xi; q)}{(x\xi; q)_\infty (x^{-1}\xi; q)_\infty} \frac{d\xi}{2\pi i \xi} \\ &= - \sum_{k=0}^{\infty} (\text{Res}_{\varepsilon=0}) \frac{\theta(txq^k e^\varepsilon; q)}{(q^{-1}e^{-\varepsilon}; q^{-1})_k (e^{-\varepsilon}; q)_\infty (x^{-2}q^{-1}e^{-\varepsilon}; q^{-1})_k (x^{-2}e^{-\varepsilon}; q)_\infty} \frac{-d\varepsilon}{2\pi i} \\ &\quad - \sum_{k=0}^{\infty} (\text{Res}_{\varepsilon=0}) \frac{\theta(tx^{-1}q^k e^\varepsilon; q)}{(x^2q^{-1}e^{-\varepsilon}; q^{-1})_k (x^2e^{-\varepsilon}; q)_\infty (q^{-1}e^{-\varepsilon}; q^{-1})_k (e^{-\varepsilon}; q)_\infty} \frac{-d\varepsilon}{2\pi i} \\ &= \sum_{k=0}^{\infty} \frac{\theta(txq^k; q)(qx^2; q)_\infty}{(q^{-1}; q^{-1})_k (x^{-2}q^{-1}; q^{-1})_k \theta(x^2; q)} + \sum_{k=0}^{\infty} \frac{\theta(tx^{-1}q^k; q)(qx^{-2}; q)_\infty}{(x^2q^{-1}; q^{-1})_k \theta(x^{-2}; q)(q^{-1}; q^{-1})_k} \\ &= \frac{\theta(tx; q)(qx^2; q)_\infty}{\theta(x^2; q)} \sum_{k=0}^{\infty} (-1)^k \frac{q^{k(k+1)/2} x^k t^{-k}}{(q; q)_k (x^2 q; q)_k} + \frac{\theta(tx^{-1}; q)(qx^{-2}; q)_\infty}{\theta(x^{-2}; q)} \sum_{k=0}^{\infty} (-1)^k \frac{q^{k(k+1)/2} x^{-k} t^{-k}}{(q; q)_k (qx^{-2}; q)_k} \\ &= - \frac{\theta(q^{-1}t; q)(1-x)(q; q)_\infty}{\theta(q^{-1}x; q)} g^{(0, x^{-1})}(t, x, q) - \frac{\theta(q^{-1}t; q)(1-x^{-1})(q; q)_\infty}{\theta(q^{-1}x^{-1}; q)} g^{(0, x)}(t, x, q). \end{aligned} \quad (297)$$

Therefore, we see that

$$f^{(-1)}(t, x, q) = -g^{(0, x^{-1})}(t, x, q) - g^{(0, x)}(t, x, q). \quad (298)$$

Now note that the bottom row of  $\vec{M}$  is given by  $(0, 0, 1)$  simply from the fact the inhomogeneous module contains the homogeneous as a sub-module and then the inhomogeneity

normalises the last entry. Therefore, from Lemma 5.12, we see that

$$\begin{pmatrix} 1 & -1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} \vec{M}(t, x, \lambda, q) = \begin{pmatrix} \frac{\theta(x^{-2}; q)\theta(t; q)^2(q; q)_\infty^3}{2\theta(q^{-1}x; q)^2\theta(tx; q)\theta(tx^{-1}; q)} & 0 & * \\ * & 1 & * \\ 0 & 0 & 1 \end{pmatrix}. \quad (299)$$

Consider the function

$$\begin{aligned} \mathcal{E}_4 &:= f^{(1)}(t, x, \lambda_1, q) - \frac{\theta(x^{-2}; q)\theta(t; q)^2(q; q)_\infty^3}{2\theta(q^{-1}x; q)^2\theta(tx; q)\theta(tx^{-1}; q)} \left( g^{(0, x^{-1})}(t, x, q) - g^{(0, x)}(t, x, q) \right) \\ &\quad - m_{2,1}(t, x, \lambda_1, q) \left( -g^{(0, x^{-1})}(t, x, q) - g^{(0, x)}(t, x, q) \right) \\ &= f^{(1)}(t, x, \lambda_1, q) - \frac{\theta(x^{-2}; q)\theta(t; q)^2(q; q)_\infty^3}{2\theta(q^{-1}x; q)^2\theta(tx; q)\theta(tx^{-1}; q)} \left( g^{(0, x^{-1})}(t, x, q) + g^{(0, x)}(t, x, q) \right) \\ &\quad - m_{2,1}(t, x, \lambda_1, q) f^{(-1)}(t, x, q) \end{aligned} \quad (300)$$

where  $m_{2,1}$  has the properties stated in Theorem 1.9. Then by the definition of  $m_{2,1}$  along with Lemma 5.9 and Lemma 5.10 we see that  $\mathcal{E}_4$  is holomorphic and satisfies Equation (282a) and therefore must be zero from Lemma 5.8. This proves the first column. Now for the last column consider the function

$$\begin{aligned} \mathcal{E}_5 &:= f^{(0)}(t, \lambda_2, x, q) - g^{(0,1)}(t, x, q) \\ &\quad - \frac{\theta(x^{-2}; q)\theta(t; q)^2(q; q)_\infty^3}{2\theta(q^{-1}x; q)^2\theta(tx; q)\theta(tx^{-1}; q)} \left( g^{(0, x^{-1})}(t, x, q) - g^{(0, x)}(t, x, q) \right) \\ &\quad - \left( \frac{\theta'(t\lambda_2)}{\theta(t\lambda_2)} - \frac{\theta'(tx)}{2\theta(tx)} - \frac{\theta'(tx^{-1})}{2\theta(tx^{-1})} - \frac{\theta'(\lambda_2)}{\theta(\lambda_2)} - \frac{1}{2} \right) \\ &\quad \times \left( -g^{(0, x^{-1})}(t, x, q) - g^{(0, x)}(t, x, q) \right). \end{aligned} \quad (301)$$

This is holomorphic from Lemma 5.9, Lemma 5.10 and Lemma 5.11. Moreover,  $\mathcal{E}_5$  satisfies Equation (282a) and therefore vanishes from Lemma 5.8.

The entries of the RHS of Equation (41) have the following transformation properties under the  $S$  matrix

$$\begin{aligned} \frac{\theta(x^{-2}; \tilde{q})\theta(t; \tilde{q})^2(\tilde{q}; \tilde{q})_\infty^3}{2\theta(\tilde{q}^{-1}x; \tilde{q})^2\theta(tx; \tilde{q})\theta(tx^{-1}; \tilde{q})} &= \tau \frac{\theta(x^{-2}; q)\theta(t; q)^2(q; q)_\infty^3}{2\theta(q^{-1}x; q)^2\theta(tx; q)\theta(tx^{-1}; q)} \\ \left( \frac{\theta'(\tilde{t}\tilde{\lambda}_2)}{\theta(\tilde{t}\tilde{\lambda}_2)} - \frac{\theta'(\tilde{t}\tilde{x})}{2\theta(\tilde{t}\tilde{x})} - \frac{\theta'(\tilde{t}\tilde{x}^{-1})}{2\theta(\tilde{t}\tilde{x}^{-1})} - \frac{\theta'(\tilde{\lambda}_2)}{\theta(\tilde{\lambda}_2)} - \frac{1}{2} \right) &= \tau \left( \frac{\theta'(t\lambda_2)}{\theta(t\lambda_2)} - \frac{\theta'(tx)}{2\theta(tx)} - \frac{\theta'(tx^{-1})}{2\theta(tx^{-1})} - \frac{\theta'(\lambda_2)}{\theta(\lambda_2)} - \frac{1}{2} \right) \end{aligned} \quad (302)$$

while  $m_{2,1}$  has three elements in its  $\mathrm{SL}_2(\mathbb{Z})$  orbit  $m_{2,1}, m_{2,1}|_1 S, m_{2,1}|_1 T$ . To see this notice that  $m_{2,1}|_1 T^2$  simply permutes the terms in RHS of Equation (296). Then using Equations (49) (52) we can explicitly compute  $m_{2,1}|_1 S$  and  $m_{2,1}|_1 TS$  which can be used to show that  $m_{2,1}|_1 ST = m_{2,1}|_1 S$  and  $m_{2,1}|_1 TS = m_{2,1}|_1 T$ . Altogether, this shows the monodromy satisfies Equation (43).

**Lemma 5.8.** If  $h(t, x, q)$  is holomorphic for  $t \in \mathbb{C}^\times$  and satisfies Equation (282a) then it vanishes.



*Proof.* If  $h(t, x, q)$  is holomorphic for  $t \in \mathbb{C}^\times$  then it has a Laurent series expansion

$$h(t, x, q) = \sum_{k \in \mathbb{Z}} \alpha_k(x, q) t^k. \quad (303)$$

Therefore Equation (282a) determines the coefficients  $\alpha_k(x, q)$  from two initial conditions say  $\alpha_0$  and  $\alpha_{-1}$ . However for any non-zero choice of  $(\alpha_0, \alpha_{-1})$  we find that  $\alpha_k \sim O(q^{-k^2/2})$  for  $k \sim -\infty$  and therefore it is divergent unless  $\alpha_0 = \alpha_{-1} = 0$ . Notice the difference here with that of Corollary 5.5 comes from the fact that  $\alpha_{-1} = 0$  does not imply that  $\alpha_k = 0$  for  $k < 0$ .  $\square$

**Lemma 5.9.** We have

$$\begin{aligned} \operatorname{Res}_{t=q^m x} \frac{\theta(t; q)^2}{\theta(tx; q)\theta(tx^{-1}; q)} g^{(0, x^{-1})}(t, x, q) \frac{dt}{2\pi i t} &= \frac{-\theta(x; q)^2 f^{(-1)}(q^m x, x, q)}{\theta(x^2; q)(q; q)_\infty^3} \\ \operatorname{Res}_{t=q^m x^{-1}} \frac{\theta(t; q)^2}{\theta(tx; q)\theta(tx^{-1}; q)} g^{(0, x)}(t, x, q) \frac{dt}{2\pi i t} &= \frac{-\theta(x^{-1}; q)^2 f^{(-1)}(q^m x^{-1}, x, q)}{\theta(x^{-2}; q)(q; q)_\infty^3} \end{aligned} \quad (304)$$

while  $g^{(0,1)}(t, x, q)$  is holomorphic for  $t \in \mathbb{C}^\times \cup \{\infty\}$ .

*Proof.* This follows from Equation (298) and the fact that

$$\operatorname{Res}_{t=q^m x^\pm} \frac{\theta(t; q)^2}{\theta(tx; q)\theta(tx^{-1}; q)} \frac{dt}{2\pi i t} = \frac{\theta(x^\pm; q)^2}{\theta(x^{\pm 2}; q)(q; q)_\infty^3}. \quad (305)$$

$\square$

**Lemma 5.10.** (a) The function  $\mathcal{B}_{1/2} \hat{f}^{(1)}(\xi, x, q)$  has simple poles at  $\xi = \pm q^{-1/4 - \mathbb{Z}_{\geq 0}/2}$  with residue

$$\operatorname{Res}_{\xi = \pm q^{-1/4 - n/2}} \mathcal{B}_{1/2} \hat{f}^{(1)}(\xi, x, q) \frac{d\xi}{2\pi i \xi} = \frac{-\theta(\pm q^{-1/4} x; q^{1/2})}{2(q; q)_\infty^2} \sum_{\ell=0}^n (\pm 1)^n \frac{q^{\frac{n(n+2)}{4}} x^{2\ell-n}}{(q; q)_\ell (q; q)_{n-\ell}}. \quad (306)$$

(b) The function  $f^{(1)}(t, x, \lambda_1, q)$  has simple poles at  $t = \pm q^{-1/4 + \mathbb{Z}} \lambda_1^{-1}$  with residue

$$\begin{aligned} \operatorname{Res}_{t = \pm q^{-1/4 - n} \lambda_1^{-1}} f^{(1)}(t, x, \lambda_1, q) \frac{dt}{2\pi i t} \\ = \frac{\theta(\pm q^{3/4} \lambda_1^{-1}; q) \theta(\pm q^{-1/4} x; q) \theta(\pm q^{-3/4} x; q) \theta(\pm q^{-3/4} \lambda_1; q)}{2\theta(x; q) \theta(x^{-1}; q) \theta(q^{-1} \lambda_1; q) \theta(q^{-3/2} \lambda_1; q)} f^{(-1)}(\pm q^{-1/4 - n} \lambda_1^{-1}, x, q). \end{aligned} \quad (307)$$

*Proof.* Firstly note that the singularities are determined by the functional equation for  $\mathcal{B}_{1/2} \hat{f}^{(1)}(\xi, x, q)$ . For part (a), recall the auxiliary function  $H_r(\xi, q)$  defined in Equation (226). Note that from the power series at  $\xi = 0$ , we see that

$$\mathcal{B}_{1/2} \hat{f}^{(1)}(q^{-1/4} \xi, x, q) = \sum_{r \in \mathbb{Z}} (-1)^r q^{r^2/4} x^r \xi^r H_r(\xi, q). \quad (308)$$

Let

$$R_\pm(n, x, q) = \operatorname{Res}_{\xi = \pm q^{-1/4 - n/2}} \mathcal{B}_{1/2} \hat{f}^{(1)}(\xi, x, q) \frac{d\xi}{2\pi i \xi}. \quad (309)$$

Then, from Equation (231),

$$R_{\pm}(0, x, q) = \operatorname{Res}_{\xi=\pm 1} \sum_{r \in \mathbb{Z}} (-1)^r q^{r^2/4} \xi^r x^r H_r(\xi, q) \frac{d\xi}{2\pi i \xi} = \frac{-\theta(\pm q^{-1/4}x; q^{1/2})}{2(q; q)_{\infty}^2}. \quad (310)$$

Now from the functional equation of  $\mathcal{B}_{1/2} \hat{f}^{(1)}(\xi, x, q)$ , we can deduce that

$$\begin{aligned} & \operatorname{Res}_{\xi=\pm q^{-1/4-n/2}} (1 - q^{1/2} \xi^2) \mathcal{B}_{1/2} \hat{f}^{(1)}(\xi, q) + (x + x^{-1}) \xi \mathcal{B}_{1/2} \hat{f}^{(1)}(q^{1/2} \xi, q) - \mathcal{B}_{1/2} \hat{f}^{(1)}(q \xi, q) \frac{d\xi}{2\pi i \xi} \\ &= (1 - q^{-n}) R_{\pm}(n, x, q) \pm (x + x^{-1}) q^{-1/4-(n-1)/2} R_{\pm}(n-1, x, q) - R_{\pm}(n-2, x, q) = 0. \end{aligned} \quad (311)$$

Note that  $R_{\pm}(n, x, q) = 0$  for  $n < 0$  and that when  $R_{\pm}(-1, x, q)$  is zero we have a unique solution determined by  $R_{\pm}(0, x, q)$ . One can then check that

$$R_{\pm}(n, x, q) = \frac{-\theta(\pm q^{-1/4}x; q^{1/2})}{2(q; q)_{\infty}^2} \sum_{\ell=0}^n (\pm 1)^n \frac{q^{\frac{n(n+2)}{4}} x^{2\ell-n}}{(q; q)_{\ell} (q; q)_{n-\ell}}. \quad (312)$$

For part (b), we compute

$$\begin{aligned} & \operatorname{Res}_{t=\pm q^{-1/4-n} \lambda_1^{-1}} f^{(1)}(t, x, \lambda_1, q) \frac{dt}{2\pi i t} \\ &= \operatorname{Res}_{t=\pm q^{-1/4-n} \lambda_1^{-1}} \frac{\theta(t; q)(q; q)_{\infty}}{(1-x)\theta(x; q)} \mathcal{L}_{1/2} \mathcal{B}_{1/2} \hat{f}^{(1)}(t, x, \lambda_1, q) \frac{dt}{2\pi i t} \\ &= \frac{\theta(\pm q^{-1/4-n} \lambda_1^{-1}; q)(q; q)_{\infty}}{(1-x)\theta(x; q)\theta(\lambda_1; q^{1/2})} \sum_{k \in \mathbb{Z}} (-1)^k q^{(k-2n)(k-2n-1)/4} \lambda_1^{-k+2n} R_{\pm}(k, x, q) \\ &= \frac{\theta(\pm q^{-1/4-n} \lambda_1^{-1}; q)(q; q)_{\infty}}{(1-x)\theta(x; q)\theta(\lambda_1; q^{1/2})} \frac{-\theta(\pm q^{-1/4}x; q^{1/2})}{2(q; q)_{\infty}^2 q^{-n(2n+1)/2} \lambda_1^{-2n}} \sum_{k=0}^{\infty} \sum_{\ell=0}^k (-1)^k \frac{q^{\frac{k(k+1)}{2}} x^{2\ell-k}}{(q; q)_{\ell} (q; q)_{k-\ell}} (\pm q^{-1/4-n} \lambda_1^{-1})^k \\ &= -\frac{\theta(\pm q^{-1/4-n} \lambda_1^{-1}; q)\theta(\pm q^{-1/4}x; q^{1/2})}{2(q; q)_{\infty}(1-x)\theta(x; q)\theta(\lambda_1; q^{1/2})} q^{n(2n+1)/2} \lambda_1^{2n} \frac{(1-x)}{\theta(q^{-1}x; q)} \hat{f}^{(-1)}(\pm q^{-1/4-n/2} \lambda_1^{-1}, x, q) \\ &= \frac{\theta(\pm q^{-1/4-n} \lambda_1^{-1}; q)\theta(\pm q^{-1/4}x; q)\theta(\pm q^{-3/4}x; q)\theta(\pm q^{-5/4-n} \lambda_1^{-1}; q)}{2x\theta(x; q)^2 \theta(\lambda_1; q)\theta(q^{-1/2} \lambda_1; q) q^{-n(2n+1)/2} \lambda_1^{-2n}} f^{(-1)}(\pm q^{-1/4-n} \lambda_1^{-1}, x, q) \\ &= \frac{\theta(\pm q^{-1/4} \lambda_1^{-1}; q)\theta(\pm q^{-1/4}x; q)\theta(\pm q^{-3/4}x; q)\theta(\pm q^{-5/4} \lambda_1^{-1}; q)}{2x\theta(x; q)^2 \theta(\lambda_1; q)\theta(q^{-1/2} \lambda_1; q)} f^{(-1)}(\pm q^{-1/4-n} \lambda_1^{-1}, x, q) \\ &= \frac{\theta(\pm q^{3/4} \lambda_1^{-1}; q)\theta(\pm q^{-1/4}x; q)\theta(\pm q^{-3/4}x; q)\theta(\pm q^{-3/4} \lambda_1; q)}{2\theta(x; q)\theta(x^{-1}; q)\theta(q^{-1} \lambda_1; q)\theta(q^{-3/2} \lambda_1; q)} f^{(-1)}(\pm q^{-1/4-n} \lambda_1^{-1}, x, q). \end{aligned} \quad (313)$$

□

**Lemma 5.11.** (a) The function  $\mathcal{B}_1 \hat{f}^{(0)}(\xi, x, q)$  has simple poles at  $\xi \in q^{\mathbb{Z}}$  with residue

$$R(m, x, q) := \operatorname{Res}_{\xi=q^{-m}} \mathcal{B}_1 \hat{f}^{(0)}(\xi, x, q) \frac{d\xi}{2\pi i \xi} = -(qx; q)_{\infty} (qx^{-1}; q)_{\infty} \sum_{\ell=0}^m \frac{q^m x^{2\ell-m}}{(q; q)_{\ell} (q; q)_{m-\ell}}. \quad (314)$$

(b) The function  $f^{(0)}(t, x, \lambda_2, q)$  has simple poles at  $t \in \lambda_2^{-1}q^{\mathbb{Z}}$  with residue

$$\operatorname{Res}_{t=q^{-m}\lambda_2^{-1}} f^{(0)}(t, x, \lambda_2, q) \frac{dt}{2\pi it} = f^{(-1)}(q^{-m}\lambda_2^{-1}, x, q). \quad (315)$$

*Proof.* For part (a), notice that

$$\begin{aligned} \mathcal{B}_1 \hat{f}^{(0)}(\xi, x, q) &= \sum_{k=0}^{\infty} (qx; q)_k (qx^{-1}; q)_k \xi^k = (qx; q)_{\infty} (qx^{-1}; q)_{\infty} \sum_{k=0}^{\infty} \frac{1}{(q^{k+1}x; q)_{\infty} (q^{k+1}x^{-1}; q)_{\infty}} \xi^k \\ &= (qx; q)_{\infty} (qx^{-1}; q)_{\infty} \sum_{k=0}^{\infty} \sum_{n, \ell=0}^{\infty} \frac{q^{kn+k\ell+n+\ell} x^{\ell-n}}{(q; q)_n (q; q)_{\ell}} \xi^k \\ &= (qx; q)_{\infty} (qx^{-1}; q)_{\infty} \sum_{n, \ell=0}^{\infty} \frac{q^{n+\ell} x^{\ell-n}}{(q; q)_n (q; q)_{\ell} (1 - q^{n+\ell} \xi)} \\ &= (qx; q)_{\infty} (qx^{-1}; q)_{\infty} \sum_{n=0}^{\infty} \sum_{\ell=0}^n \frac{q^n x^{2\ell-n}}{(q; q)_{\ell} (q; q)_{n-\ell} (1 - q^n \xi)}. \end{aligned} \quad (316)$$

The residue then follows. For part (b), the pole structure is clear noting that Equation (290) is convergent if all the terms are and that  $\mathcal{B}_1 \hat{f}^{(0)}(q^k t \lambda_2, x, q)$  has poles at  $t \in \lambda_2^{-1}q^{\mathbb{Z} \leq -k}$ . We compute the residues  $\rho_m := \operatorname{Res}_{t=q^{-m}\lambda_2^{-1}} f^{(0)}(t, x, \lambda_2, q) \frac{dt}{2\pi it}$  as follows:

$$\begin{aligned} \rho_m &= \operatorname{Res}_{t=q^{-m}\lambda_2^{-1}} (\mathcal{L}_1 \mathcal{B}_1 \hat{f}^{(0)})(t, x, \lambda_2, q) \frac{dt}{2\pi it} \\ &= \operatorname{Res}_{t=q^{-m}\lambda_2^{-1}} \frac{1}{\theta(\lambda_2, q)} \sum_{k \in \mathbb{Z}} (-1)^k q^{k(k+1)/2} \lambda_2^k \mathcal{B}_1 \hat{f}^{(0)}(q^k \lambda_2 t, x, q) \frac{dt}{2\pi it} \\ &= \frac{1}{\theta(\lambda_2, q)} \sum_{k \in \mathbb{Z}} (-1)^k q^{k(k+1)/2} \lambda_2^k R(m-k, x, q) \\ &= \frac{1}{\theta(\lambda_2, q)} (-1)^m q^{m(m+1)/2} \lambda_2^m \sum_{k=0}^{\infty} (-1)^k q^{k(k-1)/2} (q^{-m}\lambda_2^{-1})^k R(k, x, q) \\ &= \frac{-(qx; q)_{\infty} (qx^{-1}; q)_{\infty}}{\theta(\lambda_2, q)} (-1)^m q^{m(m+1)/2} \lambda_2^m \sum_{k, \ell=0}^{\infty} (-1)^k \frac{q^{k(k+1)/2} x^{2\ell-k}}{(q; q)_{\ell} (q; q)_{k-\ell}} (q^{-m}\lambda_2^{-1})^k \\ &= f^{(-1)}(q^{-m}\lambda_2^{-1}, x, q). \end{aligned} \quad (317)$$

□

The functional equation (282a) for  $U, V$  defined in (292) can be written in the form

$$U(qt, x, \lambda_1, \lambda_2, q) = A(t, x, q)U(t, x, \lambda_1, \lambda_2, q) \quad \text{and} \quad V(qt, x, q) = A(t, x, q)V(t, x, q) \quad (318)$$

where

$$A(t, x, q) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ q^{-3} & q^{-3}t^{-1} - (1+x+x^{-1})q^{-2} & (1+x+x^{-1})q^{-1} - q^{-3}t^{-1} \end{pmatrix}. \quad (319)$$

Therefore, we see that

$$\det(U(qt, x, \lambda_1, \lambda_2, q)) = \det(U(t, x, \lambda_1, \lambda_2, q))q^{-3}, \quad \det(V(qt, x, q)) = \det(V(t, x, q))q^{-3}.$$

**Lemma 5.12.** We have:

$$\begin{aligned} \det(U(t, x, \lambda_1, \lambda_2, q)) &= -\frac{q^{-3}x}{(1-x)^2}t^{-3} \\ \det(V(t, x, q)) &= \frac{\theta(q^{-1}x; q)^2\theta(tx; q)\theta(tx^{-1}; q)}{\theta(x^{-2}; q)\theta(t; q)^2(q; q)_\infty^3} \frac{q^{-3}x}{(1-x)^2}t^{-3}. \end{aligned} \quad (320)$$

*Proof.* Firstly notice that

$$\begin{aligned} &U(t, x, \lambda_1, \lambda_2, q) \\ &= W(f^{(1)}(t, x, \lambda_1, q), f^{(-1)}(t, x, q), f^{(0)}(t, x, \lambda_2, q)) \\ &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ q^{-2}t^{-1} & -q^{-2} & q^{-2}t^{-1} - (x+x^{-1})q^{-1} \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ f^{(1)}(t, x, \lambda_1, q) & f^{(-1)}(t, x, q) & f^{(0)}(t, x, \lambda_2, q) \\ f^{(1)}(qt, x, \lambda_1, q) & f^{(-1)}(qt, x, \lambda_1, q) & f^{(0)}(qt, x, \lambda_2, q) \end{pmatrix} \end{aligned} \quad (321)$$

with a similar expression for  $V$ . Therefore,

$$\begin{aligned} \det(U(t, x, \lambda_1, \lambda_2, q)) &= q^{-2}t^{-1}(f^{(1)}(t, x, \lambda_1, q)f^{(-1)}(qt, x, q) - f^{(1)}(qt, x, \lambda_1, q)f^{(-1)}(t, x, q)) \\ \det(V(t, x, q)) &= q^{-2}t^{-1}(g^{(0, x^{-1})}(t, x, q)g^{(0, x)}(qt, x, q) - g^{(0, x^{-1})}(qt, x, q)g^{(0, x)}(t, x, q)). \end{aligned} \quad (322)$$

Now notice that both  $\det(U(t, x, \lambda_1, \lambda_2, q))t^3$  and  $\det(V(t, x, q))t^3$  are elliptic functions in  $t$ . Furthermore,  $\det(U(t, x, \lambda_1, q))$  has potentially simple poles in  $t$  at  $t \in \pm q^{-1/4 - \mathbb{Z}/2} \lambda^{-1}$  and  $t \in q^{\mathbb{Z}}$ . However, note that the poles of  $f^{(-1)}$  at  $t \in q^{\mathbb{Z}}$  cancel with zeros of  $f^{(1)}$ . Lemma 5.10 implies that

$$\begin{aligned} &\text{Res}_{t=\pm q^{-1/4-n}\lambda^{-1}} \det(U(t, x, \lambda_1, \lambda_2, q)q^2t) = \\ &\frac{\theta(\pm q^{3/4}\lambda_1^{-1}; q)\theta(\pm q^{-1/4}x; q)\theta(\pm q^{-3/4}x; q)\theta(\pm q^{-3/4}\lambda_1; q)}{2(q; q)_\infty^2\theta(x; q)\theta(x^{-1}; q)\theta(q^{-1}\lambda_1; q)\theta(q^{-3/2}\lambda_1; q)} \\ &\times f^{(-1)}(\pm q^{-1/4-n}\lambda_1^{-1}, x, q)f^{(-1)}(\pm q^{3/4-n}\lambda_1^{-1}, x, q) \\ &- \frac{\theta(\pm q^{3/4}\lambda_1^{-1}; q)\theta(\pm q^{-1/4}x; q)\theta(\pm q^{-3/4}x; q)\theta(\pm q^{-3/4}\lambda_1; q)}{2(q; q)_\infty^2\theta(x; q)\theta(x^{-1}; q)\theta(q^{-1}\lambda_1; q)\theta(q^{-3/2}\lambda_1; q)} \\ &\times f^{(-1)}(\pm q^{3/4-n}\lambda_1^{-1}, x, q)f^{(-1)}(\pm q^{-1/4-n}\lambda_1^{-1}, x, q) = 0. \end{aligned} \quad (323)$$

A similar calculation shows that

$$\text{Res}_{t=\pm q^{-3/4-n}\lambda_1^{-1}} \det(U(t, x, \lambda_1, \lambda_2, q)) = 0. \quad (324)$$

Therefore, we see that  $U(t, x, \lambda_1, q)t^2$  is elliptic and holomorphic in  $t \in \mathbb{C}^\times$ . Therefore, it is constant in  $t$ . Now considering the limit as  $t \rightarrow 0$ , using the definition of  $f^{(\pm 1)}$  and their asymptotic expansions (given by their formal power series expansions, by a version of

Watson's lemma), it follows that

$$\begin{aligned}
 \det(U(t, x, \lambda_1, \lambda_2, q))q^2t^3 &= \lim_{t \rightarrow 0} \det(U(t, x, \lambda_1, \lambda_2, q))q^2t^3 \\
 &= \lim_{t \rightarrow 0} \frac{\theta(t; q)\theta(q^{-1}x; q)}{\theta(t; q)\theta(x; q)(1-x)^2}t^2 - \lim_{t \rightarrow 0} \frac{\theta(qt; q)\theta(q^{-1}x; q)}{\theta(q^{-1}t; q)\theta(x; q)(1-x)^2}t^2 \\
 &= -\lim_{t \rightarrow 0} q^{-1} \frac{\theta(t; q)\theta(x; q)}{\theta(t; q)\theta(x; q)(1-x)^2} = -\frac{q^{-1}x}{(1-x)^2}.
 \end{aligned} \tag{325}$$

Now noting that  $\theta(t; q)^2/\theta(tx; q)\theta(tx^{-1}; q)$  is elliptic we see that

$$\det(V(t, x, q))t^3 \frac{\theta(t; q)^2}{\theta(tx; q)\theta(tx^{-1}; q)} \tag{326}$$

is elliptic and holomorphic in  $t$  and therefore constant in  $t$ . Then we see that

$$\begin{aligned}
 &\lim_{t \rightarrow \infty} \det(V(t, x, q))q^2t^3 \frac{\theta(t; q)^2}{\theta(tx; q)\theta(tx^{-1}; q)} \\
 &= \lim_{t \rightarrow \infty} (g^{(0, x^{-1})}(t, x, q)g^{(0, x)}(qt, x, q) - g^{(0, x^{-1})}(qt, x, q)g^{(0, x)}(t, x, q))t^2 \frac{\theta(t; q)^2}{\theta(tx; q)\theta(tx^{-1}; q)} \\
 &= q^{-1} \frac{(qx^2; q)_\infty (qx^{-2}; q)_\infty (1-x^2)\theta(q^{-1}x; q)\theta(q^{-1}x^{-1}; q)}{(1-x)^2\theta(x^2; q)\theta(x^{-2}; q)(q; q)_\infty^2} \\
 &= -q^{-1}x^2 \frac{\theta(x^2; q)\theta(q^{-1}x; q)\theta(q^{-1}x^{-1}; q)}{\theta(x^2; q)\theta(x^{-2}; q)(1-x)^2(q; q)_\infty^3}.
 \end{aligned} \tag{327}$$

□

To finish the proof of Theorem 1.9 we will use the state integrals introduced in [28, Equ.(139)] for  $w$  in a neighbourhood of zero.

$$\int_{\mathbb{R}+i\varepsilon} \Phi_b(x + ib^{-1}u)\Phi_b(x - ib^{-1}u) \frac{\exp(-\pi ix^2 - 2\pi \frac{zx}{b})}{1 + \tilde{q}^{1/2} \exp(-\frac{2\pi x}{b})} dx. \tag{328}$$

The factorisation of this integral was done in [28, Equ.(149)]. This module is again not self dual (see Proposition 2.6) and means that additional functions arise in the factorisation. It was shown, using Equation (49), that Equation (328) factors as combinations of elementary functions holomorphic in  $\mathbb{C}'$  times

$$\mathcal{I}(z, w, \tau) = g^{(0,1)}(\tilde{t}, \tilde{x}, \tilde{q}) + \tau g^{(0,x)}(\tilde{t}, \tilde{x}, \tilde{q})L^{(x^{-1})}(t, x, q) - \tau g^{(0,x^{-1})}(\tilde{t}, \tilde{x}, \tilde{q})L^{(x)}(t, x, q) \tag{329}$$

where

$$L^{(x^\mp)}(t, x, q) = \frac{\theta(t; q)(1-x^\mp)(q; q)_\infty^2}{\theta(q^{-1}x^\pm; q)\theta(tx^\mp; q)} (qx^{\pm 2}; q)_\infty \sum_{k=0}^{\infty} (-1)^k \frac{q^{k(k+1)/2} x^{\pm k}}{(q; q)_k (qx^{\pm 2}; q)_k (1-q^k x^\pm)} t^{-k}. \tag{330}$$

These functions  $L^{(x^\mp)}$  can then be shown to satisfy

$$L^{(x^\mp)}(t, x, q) - L^{(x^\mp)}(qt, x, q) = \frac{\theta(x^{-2}; q)\theta(t; q)^2(q; q)_\infty^3}{\theta(q^{-1}x; q)^2\theta(tx; q)\theta(tx^{-1}; q)} \frac{(1-x)^2}{x} qt g^{(0, x^\pm)}(qt, x, q). \tag{331}$$

Again this determines  $L^{(x^\mp)}$  up to the addition of an elliptic function so checking the principal parts

$$V(t, x, q)^{-1} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ q^{-2}t^{-1} & -q^{-2} & q^{-2}t^{-1} - (x + x^{-1})q^{-1} \end{pmatrix} = \begin{pmatrix} -L^{(x)}(t, x, q) & * & * \\ L^{(x^{-1})}(t, x, q) & * & * \\ 1 & 0 & 0 \end{pmatrix}. \quad (332)$$

where the  $*$  are given by  $\pm \det(V(t, x, q))^{-1} g^{(0, x^\pm)}(q^m t, x, q)$  where  $m = 0, 1$ . Finally, one can use the functional equations to take  $w$  away from 0 which gives the analytic continuation or can alter the contour depending on  $w$ . Noting that the entries of  $\Omega_{V, S}$  are combinations of elementary functions times  $\mathcal{I}(z + n + m\tau, w, \tau)$ , we see that  $\Omega_{V, S}$  extends for  $\tau \in \mathbb{C}'$  and using the modularity of the monodromy and part (c) of Theorem 1.2 completes the proof.  $\square$

**5.8. An analytic lift of the colored Jones polynomial.** We finish this section by giving a proof of Theorem 1.10. The main observation is that when  $x = q^N$ , the series  $\hat{f}^{(0)}(t, q^N, q)$  terminates to a polynomial of  $t$ , in which case the  $q$ -Borel transform, followed by a  $q$ -Laplace transform is the identity. Explicitly, when  $x = q^N$  for  $N \in \mathbb{Z}_{\geq 1}$  we have

$$\begin{aligned} \hat{f}^{(0)}(t, q^N, q) &= \sum_{k=0}^{\infty} (-1)^k q^{-k(k+1)/2} (q^{1+N}; q)_k (q^{1-N}; q)_k t^k \\ &= \sum_{k=0}^{N-1} (-1)^k q^{-k(k+1)/2} (q^{N+1}; q)_k (q^{1-N}; q)_k t^k. \end{aligned} \quad (333)$$

The theorem then follows from Equation (58).  $\square$

**5.9. Specialisation  $t = q^m$ .** In this short subsection, included for completeness, we briefly comment how our analytic functions of  $t$ , specialised to  $t = q^m$ , become the known sequences of  $q$ -series and  $(x, q)$ -series that have appeared in the literature. In [36] and [26, 27, 28] the  $q$ -holonomic modules are discrete versions of what we have considered in Section 5. Here we describe how the solutions can be constructed from the ones presented here. Consider a solution  $f(t, q)$  of a  $q$ -difference equation

$$a_r(t, q)f(q^r t, q) + a_{r-1}(t, q)f(q^{r-1} t, q) + \cdots + a_0(t, q)f(t, q) = 0$$

corresponding to an edge of slope  $\kappa$  on the Newton polygon, and let

$$f_m(q) = (-1)^{\kappa m} q^{-\kappa m(m+1)/2} \text{Res}_{z=0} \theta(q^m \mathbf{e}(z), q)^{-\kappa} f(q^m \mathbf{e}(z), q) \frac{dz}{2\pi i z}. \quad (334)$$

Then we find that  $f_m(q)$  satisfies the linear  $q$ -difference equation

$$a_r(q^m, q)f_{r+m}(q) + a_{r-1}(q^m, q)f_{r+m-1}(q) + \cdots + a_0(q^m, q)f_m(q) = 0.$$

This follows from

$$\begin{aligned} 0 &= \text{Res}_{z=0} (a_r(q^m \mathbf{e}(z), q)f(q^{r+m} \mathbf{e}(z), q) + a_{r-1}(q^m \mathbf{e}(z), q)f(q^{r+m-1} \mathbf{e}(z), q) + \cdots \\ &\quad + a_0(q^m \mathbf{e}(z), q)f(q^m \mathbf{e}(z), q)) \frac{\mathbf{e}(\kappa z) dz}{\theta(\mathbf{e}(z); q)^{\kappa} 2\pi i z} \\ &= a_r(q^m, q)f_{r+m}(q) + a_{r-1}(q^m, q)f_{r+m-1}(q) + \cdots + a_0(q^m, q)f_m(q) \end{aligned} \quad (335)$$

where the equality follows from holomorphicity of  $\theta^{-\kappa} f$  or the higher order of vanishing of the indicial polynomial.

For example, for the Equation (28), and its solution solution  $g^{(0,1)}$  defined in Equation (209), we obtain that

$$\operatorname{Res}_{z=0} g^{(0,1)}(q^m \mathbf{e}(z), q) \frac{dz}{2\pi iz} = \sum_{k=0}^{\infty} \left( k + \frac{1}{2} - m - 2E_1^{(k)}(q) \right) (-1)^k \frac{q^{k(k+1)/2 - km - m}}{(q; q)_k^2}, \quad (336)$$

a  $q$ -series that appears in [27, Equ.13b] and [28, Equ.6].

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