THE NEWTON POLYTOPE OF A RECURRENT SEQUENCE OF POLYNOMIALS

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Abstract. A recurrent sequence of polynomials is a sequence of polynomials that satisfies a linear recursion with fixed polynomial coefficients. Our paper proves that the sequence of Newton polytopes of a recurrent sequence of polynomials is quasi-linear. Our proof uses the Lech-Mahler-Skolem theorem of p-adic analytic number theory with recent results in tropical geometry. A subsequent paper lists some applications of our result to TQFT.

Contents

1. Introduction

1.1. **Recurrent sequences of polynomials.** Consider the ring $R = \mathbb{Q}[x_1^{\pm 1}, \dots, x_r^{\pm 1}]$ of Laurent polynomials in r variables $x = (x_1, \dots, x_r)$. A sequence $p_n \in R$ is *recurrent* if it satisfies a linear recursion with coefficients in R. In other words, there exists a natural number d and $c_k \in R$ for $k = 0, \dots, d$ with $c_d \neq 0$ such that for all integers n we have:

$$\sum_{k=0}^{d} c_k p_{n+k} = 0 \tag{1}$$

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For a polynomial $p \in R$, let $N(p)$ denote its *Newton polytope*, i.e., the convex hull of the exponents of the nonzero monomials of p. Polytopes can be described in several ways. One popular descriptions is as the convex hull $\mathrm{conv}(S)$ of a finite set S of vectors in $\mathbb{R}^r$. Another description is as a solution to a linear system of inequalities $Ax \leq b$ for suitable matrices A and vectors b.

Our goal is to describe the structure of the sequence of Newton polytopes $N(p_n)$ of a recurrent sequence (p_n) of polynomials. This requires to introduce quasi-linear vectors/matrices. A *quasi-linear vector* is a sequence $v : \mathbb{N} \to \mathbb{Q}^r$ of vectors of the form

$$v(n) = v_1(n)n + v_0(n)$$

for all but finitely many n, where $v_0, v_1 : \mathbb{N} \to \mathbb{Q}^r$ are periodic sequences.

Definition 1.1. We say that a sequence (P_n) of polytopes is eventually *quasi-linear* if there exists a finite set S of quasi-linear vectors such that for all but finitely many n we have:

$$P_n = \mathrm{conv}(\{v(n)\,|v \in S\})$$

Equivalently Lemma 2.1 shows that (P_n) is eventually quasi-linear if there exist a matrix A and quasi-linear vector b such that for all but finitely many n we have:

$$P_n = \{x \in \mathbb{R}^r\,|Ax \leq b(n)\}$$

1.2. **Our results.**

Theorem 1.1. *If (p_n) is a recurrent sequence of polynomials, then the set $\{n \in \mathbb{N} : p_n(x) = 0\}$ differs from a finite union of full arithmetic progressions by a finite set. Moreover, if $p_n(x) \neq 0$ for all n, then $N(p_n)$ is quasi-linear.*

The proof of Theorem 1.1 uses the Lech-Mahler-Skolem theorem of p-adic analytic number theory together with some recent results in tropical geometry.

Recurrent sequences of polynomials occur naturally in classical and quantum topology; see for example [HS04, GM11, Gar13]. Quasi-linear sequences of polytopes occur in recent work of Calegari-Walker [CW13] and in lattice point counting problems, old [Ehr62] and new [CLS12]. Quasi-polynomials appear in lattice point counting problems [Ehr62, CLS12] and also in Quantum Topology [Gar11b, Gar11a]. The next corollary of Theorem 1.1 follows from some recent results of Chen-Li-Sam which generalize the Ehrhart theory; see [CLS12].

Corollary 1.2. If (P_n) is a quasi-linear sequence of polytopes, the volume and the number of lattice points of P_n is a quasi-polynomial function of n.

We end this section with an example.

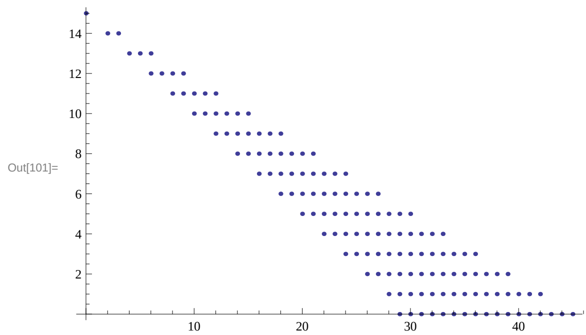
Example 1.3. Consider the sequence of polynomials $p_n \in \mathbb{Q}[x_1, x_2]$ that satisfy the linear recursion

$$p_{n+2} + x_1 p_{n+1} - (x_1^3 + x_2)p_n = 0$$

with initial conditions $p_0 = p_1 = 1$. The Newton polytope P_n of p_n is a triangle given by

$$P_n = \mathrm{conv}\left(\{(0, \lfloor n/2 \rfloor), (n-1, 0), (n + \lfloor n/2 \rfloor, 0)\}\right)$$

where $\lfloor x \rfloor$ is the biggest integer which is less than or equal to x. For example, the lattice points of P_{30} are shown here



The number of lattice points $|P_n \cap \mathbb{Z}^2|$ and the area $A(P_n)$ are given by

$$|P_n \cap \mathbb{Z}^2| = \frac{n^2}{8} + \begin{cases} \frac{3n}{4} + 2 & \text{if } n \text{ is even} \\ \frac{n}{2} + \frac{11}{8} & \text{if } n \text{ is odd} \end{cases} \qquad A(P_n) = \frac{n^2}{8} + \begin{cases} \frac{n}{4} & \text{if } n \text{ is even} \\ -\frac{1}{8} & \text{if } n \text{ is odd} \end{cases}$$

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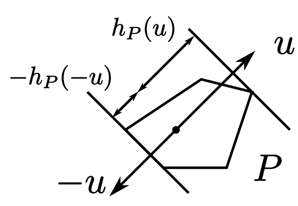
2. The support function of a polytope

2.1. **Properties of the support function.** Let us review some standard facts of *polyhedral geometry* regarding the *support function* h_P of a convex body P in $\mathbb{R}^r$. For a detailed discussion, see [Sch93, Sec.1.7] and also [Grü03, Zie95]. The support function is defined by

$$h_P : \mathbb{R}^r \setminus \{0\} \longrightarrow \mathbb{R}, \qquad h_P(u) = \sup\{u \cdot x \,|x \in P\}$$

where $u \cdot v$ denotes the standard inner product of two vectors u and v of $\mathbb{R}^r$. Given a unit vector u, there is a unique hyperplane with outer normal vector u that touches P, and entirely contains P in the left-half space. The value $h_P(u)$ of the support function is the signed distance from the origin to the above hyperplane. This is illustrated in the following figure:



Let us list some useful properties of the support function:

- h_P uniquely determines the convex body P. This is the famous *Minkowski reconstruction theorem*. For a detailed proof, see [Sch93, Thm.1.7.1] and also [Kla04]. Moreover,

$$P = \{x \in \mathbb{R}^r \,|\, x \cdot u \leq h_P(u) \text{ for all } u \in \mathbb{R}^r \setminus \{0\}\} \tag{2}$$

- h_P is homogeneous and subadditive.

- When P is a convex polytope with vertex set V_P, then

$$h_P(u) = \max\{u \cdot v \,|v \in V_P\} \tag{3}$$

In particular, h_P is a piece-wise linear function.
- The support function recovers the vertices of the polytope. Indeed, if

$$h_P(u) = \max\{u \cdot v \,|v \in V\}$$

then $P = \mathrm{conv}(V)$.
- The support function recovers the normals to the facets of the polytope. Indeed, the corner locus of h_P (i.e., the locus of points where h_p is not differentiable) is a fan $\mathcal{F}$ whose rays are outer pointing normals to the facets of P. The maximal cones of $\mathcal{F}$ are in 1-1 correspondence with the vertices of P.
- The projection of P to the line $\mathbb{R}u$ is the line segment $[-h_P(-u), h_P(u)]$. See the above figure for an illustration.

Given a sequence (P_n) of polytopes, we say that their support function h_{P_n} is *piece-wise quasi-linear* if there exists a a finite set S of quasi-linear vectors such that for all but finitely many n and all $u \in \mathbb{R}^r$ we have:

$$h_{P_n}(u) = \max\{u \cdot v(n) \,|v \in S\} \tag{4}$$

We say that a vector $\omega \in \mathbb{R}^r$ is *generic* if it has $\mathbb{Q}$-linearly independent coordinates.

Lemma 2.1. The following are equivalent for a sequence (P_n) of polytopes:
(a) There exists a finite set S of quasi-linear vectors such that for all but finitely many n we have:

$$P_n = \mathrm{conv}(\{v(n) \,|v \in S\}) \tag{5}$$

(b) There exists a matrix A and a quasi-linear vector b such that for all but finitely many n we have:

$$P_n = \{x \in \mathbb{R}^r \,|Ax \leq b(n)\} \tag{6}$$

(c) (h_{P_n}) is piece-wise quasi-linear.
(d) There exists a rational fan $\mathcal{F}$ in $\mathbb{R}^r$ and a quasi-linear vector δ_σ for each maximal cone σ of $\mathcal{F}$ such that for all but finitely many n and all $\omega \in \sigma$ generic, we have:

$$h_{P_n}(\omega) = \delta_C(n) \cdot \omega \tag{7}$$

Proof. (a) implies (c) by Equation (3). (c) implies (b) by Equation (2). (b) implies (a) by the description of the vertices of a polytope from inequalities. If (7) holds for ω generic, then it follows by the continuity of h_{P_n} that it holds for all ω. Equation (4) defines a constant coefficient tropical hypersurface, i.e., a fan which implies that (c) is equivalent to (d). □

Remark 2.2. Note that if Equation (4) (resp., (7)) holds for u (resp., ω) generic, then by the continuity of h_{P_n}, it holds for all u (resp., ω).

3. Generalized power sums and their zeros

Generalized power sums play a key role to the Lech-Mahler-Skolem (in short, LMS) theorem. For a detailed discussion, see [vdP89] and also [EvdPSW03]. Recall that a *generalized power sum* a_n for $n = 0, 1, 2, \dots$ is an expression of the form

$$a_n = \sum_{i=1}^{m} A_i(n)\alpha_i^n \tag{8}$$

with *roots* α_i, $1 \leq i \leq m$, distinct nonzero quantities, and coefficients $A_i(n)$ polynomials of degree $m_i - 1$ for positive integers m_i, $1 \leq i \leq m$. The generalized power sum a_n is said to have *order*

$$d = \sum_{i=1}^{m} m_i$$

and satisfies a linear recursion with constant coefficients of the form

$$a_{n+d} = s_1 a_{n+d-1} + \cdots + s_d a_n$$

where

$$s(x) = \prod_{i=1}^{m} (1 - \alpha_i x)^{m_i} = 1 - s_1 x - \cdots - s_d x^d.$$

It is well-known that a sequence is *recurrent* i.e., satisfies a linear recursion with constant coefficients if and only if it is a generalized power sum. Observe that the monic polynomial polynomial $s(x)$ of smallest possible degree is uniquely determined by (a_n).

The LMS theorem concerns the zeros of a generalized power sum.

Theorem 3.1. [Sko35, Mah35, Lec53] *The zero set of a generalized power sum is the union of a finite set and a finite set of arithmetic progressions.*

A detailed proof of this important theorem is discussed in [vdP89], for recurrent sequences with values in an arbitrary field of characteristic zero. In the next section we will need a slightly stronger form of the LMS theorem. We say that a recurrent sequence (a_n) is *non-degenerate* if the ratio of two distinct roots of (a_n) is not a root of unity; see [EvdPSW03, Sec.1.1.9].

The LMS theorem in the case of number fields follows from the following two theorems.

Theorem 3.2. [EvdPSW03, Thm.1.2] *If (a_n) is recurrent sequence there exists $M \in \mathbb{N}$ such that for every r with $0 \leq r \leq M-1$, the subsequence (a_{nM+r}) is either zero or non-degenerate.*

Although we will not need this fact, if (a_n) takes values in a number field K, there are absolute bounds for M in terms of the degree of $K/\mathbb{Q}$ and the order of (a_n).

Theorem 3.3. [EvdPSW03, Cor.1.20] *If (a_n) is non-degenerate recurrent sequence with values in a number field K, then it has finitely many zero terms.*

In fact, the number of zeros is bounded above by the degree of $K/\mathbb{Q}$ and the order of (a_n); see [ESS02, Eqn.1.18].

4. Fatou's Lemma

Recurrent sequences (a_n) of rational numbers are well-known, they satisfy linear recursion of the form

$$\sum_{k=0}^{d} c_k a_{n+k} = 0 \tag{9}$$

for all n where c_k are rational numbers with $c_d \neq 0$. In [Fat06, p.369-370] Fatou proved that if (a_n) is a recurrent sequence of *integers*, then it satisfies a *monic* linear recursion, i.e., one of the form (9) where c_k are integers for $k = 0, \dots, d$ and $c_d = 1$. More precisely, Fatou proved the following lemma, quoted by several authors, e.g. [Sta97, Exerc.4.2(a)].

Lemma 4.1. [Fat06] Consider a power series $G(y) = \sum_{n=0}^{\infty} a_n y^n \in \mathbb{Z}[[y]] \cap \mathbb{Q}(y)$. Then, there exist $A(y), B(y) \in \mathbb{Z}[y]$ polynomials with $B(0) = 1$ such that

$$G(y) = \frac{A(y)}{B(y)}$$

Moreover, if $B(y) = 1 + \sum_{k=1}^{d} b_k y^k$, then (a_n) satisfies the monic linear recursion

$$a_{n+d} + \sum_{k=1}^{d} b_k a_{n+d-k} = 0$$

for all n.

Let $R = \mathbb{Q}[x_1^{\pm 1}, \dots, x_r^{\pm 1}]$ and $K = \mathbb{Q}(x_1, \dots, x_r)$. Fatou's proof also proves the following.

Lemma 4.2. Consider a power series $G(y) = \sum_{n=0}^{\infty} p_n y^n \in R[[y]] \cap K(y)$. Then, there exist $A(z), B(z) \in R[z]$ polynomials with $B(0) = 1$ such that

$$G(y) = \frac{A(y)}{B(y)}$$

Moreover, if $B(y) = 1 + \sum_{k=1}^{d} b_k y^k$, then (p_n) satisfies the monic linear recursion

$$p_{n+d} + \sum_{k=1}^{d} b_k p_{n+d-k} = 0 \tag{10}$$

for all n.

5. Laurent series solutions to polynomial equations

In this section we recall some results regarding Laurent series solutions of polynomial equations whose coefficients are polynomials in several variables. These results are a generalization of the Newton-Puiseux algorithm.

Laurent power series in one variable form a field, whereas they only form a ring in the case of several variables. McDonald [McD95] constructed multivariate Laurent series solutions to a polynomial equation $p(x, y)$ where $x = (x_1, \dots, x_r)$. Unlike the univariate (i.e., $r = 1$) case, McDonald's solutions depend on a generic weight vector $\omega \in \mathbb{R}^r$. Aroca-Ilardi [AI09] extended McDonald's results and constructed an algebraically closed field $K_\omega((x))$ which depends on ω. In [GY14], Yu and the author free the above Laurent series from their dependence on a weight vector ω. Let us recall the necessary definitions and notation to state the results of [GY14].

If $x = (x_1, \dots, x_r)$ and $\alpha = (\alpha_1, \dots, \alpha_r)$, we denote $x^\alpha = x_1^{\alpha_1} \dots x_r^{\alpha_r}$. For a field K, let $K\mathcal{P}(x)$ denote the set of series ϕ of the form $\phi = \sum_{\alpha \in \mathbb{Q}^r} c_\alpha x^\alpha$ where $c_\alpha \in K$ for all α. For such a series ϕ, its support $\mathcal{E}(\phi)$ is the set of $\alpha \in \mathbb{Q}^r$ such that $c_\alpha \neq 0$. $K\mathcal{P}(x)$ is not a ring. However, if C is a line-free cone (i.e., it does not contain a linear subspace) and $x = (x_1, \dots, x_r)$, then $K_C[[x]]$ and $K_C((x))$ defined by

$$K_C[[x]] = \{\phi \in K\mathcal{P}(x) \,|\mathcal{E}(\phi) \subset C \cap \frac{1}{N}\mathbb{Z}^r \text{ for some } N \in \mathbb{N}\}$$
$$K_C((x)) = \cup_{\gamma \in \mathbb{Q}^r} x^\gamma K_C[[x]]$$

are rings.

We say that $\omega \in \mathbb{R}^r$ is generic if its coordinates are $\mathbb{Q}$-linearly independent. From now on, ω stands for a generic vector. We say that a cone C in $\mathbb{R}^r$ is ω-positive if $\omega \in C^\vee$, where the dual cone is defined by

$$\sigma^\vee = \{x \in \mathbb{R}^r | x \cdot y \geq 0, \text{ for all}\, y \in C\}$$

Let

$$K_\omega((x)) = \cup_C K_C((x))$$

where the union is over all ω-positive cones C. Aroca-Illardi [AI09] show that $K_\omega((x))$ is algebraically closed for all generic ω. Thus, $K_\omega((x))$ is an algebraically closed field which contains multivariable Laurent series rings $K_C((x))$.

Fix a polynomial

$$p(x, y) = a_d(x)y^d + \cdots + a_0(x) \in K[x_1^{\pm 1}, \dots, x_r^{\pm 1}][y]$$

of $r + 1$ variables (x, y) (where $x = (x_1, \dots, x_r)$ and an algebraically closed field K of characteristic zero. Let $N(p)$ denote the Newton polytope of p in $\mathbb{R}^{r+1}$ and $\Sigma(p)$ denote the fiber polytope of p with respect to the projection $\mathbb{R}^{r+1} \to \mathbb{R}$, where $(x, y) \mapsto y$ [BS92]. Let $\mathcal{F}$ denote the normal fan of $\Sigma(p)$ in $\mathbb{R}^r$. If σ is a maximal cone of $\mathcal{F}$, the dual cone $\sigma^\vee = \{x \in \mathbb{R}^r | x \cdot y \geq 0, \text{ for all}\, y \in C\}$ is line-free, i.e., contains no linear subspace.

Theorem 5.1. [GY14, Thm.1] *For every cone σ as above, there exist $y_1(x), \dots, y_d(x) \in K_{\sigma^\vee}((x))$ such that*

$$p(x, y) = a_r(x)(y - y_1(x)) \dots (y - y_d(x))$$

Corollary 5.1. With the notation of Theorem 1.1, choose $\omega \in \sigma$ generic. Then $y_j(x) \in K_\omega((x))$ for $j = 1, \dots, r$ are the roots of the polynomial $p(x, y)$ in the algebraically closed field $K_\omega((x))$. This works even if $p(x, y) \in K_\omega((x))[y]$.

Corollary 5.2. With the notation of Theorem 5.1, if $R(y_1(x), \dots, y_d(x))$ is a rational function of $y_j(x)$, then after possible refinement of σ, it follows that $R(y_1(x), \dots, y_d(x)) \in K_{\sigma^\vee}((x))$.

6. Proof of Theorem 1.1

Fix a recurrent sequence $p_n(x_1,\dots,x_r) \in R = \mathbb{Q}[x_1^{\pm 1},\dots,x_r^{\pm 1}]$. By Fatou's Lemma 4.2, we can find a monic recursion relation (10) with coefficients $b_k \in R$, and, without loss of generality, assume $b_d \neq 0$. Consider the characteristic polynomial $p(x,y)$ of (10) and its factorization from Theorem 5.1

$$p(x,y) = y^d + \sum_{k=1}^{d} b_k(x) y^{d-k} = \prod_j (y - y_j(x))^{m_j} \tag{11}$$

for a fixed maximal cone σ of the normal fan $\mathcal{F}$ of the fiber polytope $\Sigma(p)$ of p. Here, $y_j(x) \in K_{\sigma^\vee}((x))$ for $j=1,\dots,d$. Thus we can write

$$y_j(x) = \alpha_j x^{\beta_j} \sum_{\gamma \in \sigma \cap \frac{1}{N}\mathbb{Z}^r} c_{j,\gamma} x^\gamma, \qquad c_{j,0} = 1, \qquad \alpha_j \neq 0$$

for $j=1,\dots,d$, where $c_{j,\gamma}, \alpha_j \in K$ for a number field K. This partitions the j-indexing set $\{1,\dots,d\}$ into a disjoint union $J_1 \sqcup J_2 \dots \sqcup J_s$ such that $v(y_j(x)) = \beta_i$ for all $j \in J_i$ where $\beta_i \neq \beta_{i'}$ for $i \neq i'$. Let $\mathcal{X} = \{\alpha_1,\dots,\alpha_d\} \subset K^*$. Let

$$S = \{0,1,\dots,d\} \times \left(\mathbb{Q}^r + \left(\frac{1}{N}\sigma \cap \mathbb{Z}^r\right)\right)$$

Step 1: Laurent series presentation of $p_n(x)$ by generalized power sums.

Lemma 6.1. After possibly refining σ, there exist a collection $(a_{i,\gamma}(n))$ of generalized power sums (indexed by $(i,\gamma) \in S$) with roots a subset of $\mathcal{X}$ such that for all n we have:

$$p_n(x) = \sum_{(i,\gamma) \in S} a_{i,\gamma}(n) x^{n\beta_i + \gamma} \tag{12}$$

Proof. The general solution $p_n(x)$ of a linear recurrence equation with constant coefficients has the form

$$p_n(x) = \sum_j c_j(x,n) y_j(x)^n \tag{13}$$

where $c_j(x,n)$ are polynomials in n with coefficients rational functions of $y_1(x),\dots,y_d(x)$. Using Corollary 5.2, and after possibly refining σ, it follows that $c_j(x,n) \in K_{\sigma^\vee}((x))[n]$ for all $j=1,\dots,d$. Using the identity

$$\begin{aligned}\left(1 + \sum_{k=1}^{\infty} c_k x^k\right)^n &= 1 + nc_1 x + \left(nc_2 + \frac{n(n-1)}{2}c_1^2\right)x^2 + \\ &\left(nc_3 + n(n-1)c_1c_2 + \frac{n(n-1)(n-2)}{6}c_1^3\right)x^3 + \dots\end{aligned} \tag{14}$$

where the coefficients of each power of x are polynomials in n, and Equation (13), it follows that

$$p_n(x) = \sum_{(i,\gamma) \in S} a_{i,\gamma}(n) x^{n\beta_i + \gamma}$$

where

$$a_{i,\gamma}(n) = \sum_{j\in J_i} \alpha_j^n \operatorname{coeff}\left(c_j(x,n)\left(\frac{y_j(x)}{\alpha_j x^{\beta_j}}\right)^n, x^\gamma\right)$$

□

Step 2: Reduction to the non-degenerate case.

Let G denotes the subgroup of the abelian group K^* generated by the finite set $\mathcal{X}$. G is a finitely generated abelian group, and its torsion subgroup is finite of order, say, M. It follows that the subset $\{\alpha_1^M, \dots, \alpha_d^M\}$ of K^*, after removing any repetitions, consists of non-degenerate roots. Therefore, for every $(i,\gamma) \in S$ and every r with $0 \leq r \leq M-1$, the generalized power sum $(a_{i,\gamma}(Mn+r))$ is either zero or non-degenerate.

Let us now fix a generic weight $\omega \in \sigma^\vee$. It gives a total ordering $<_\omega$ of the set S as follows: $(i,\gamma) <_\omega (i',\gamma')$ if and only if $\gamma_i \cdot \omega < \gamma_{i'} \cdot \omega$ or $i = i'$ and $\gamma \cdot \omega < \gamma' \cdot \omega$. Since σ is ω-positive, it follows that S is well-ordered.Let $p_{\omega,n}(t) = p_n(t^{\omega_1}, \dots, t^{\omega_r})$, and let v denote the valuation at $t = 0$: in other words $v(\sum_k c_k t^{b_k}) = \min\{b_k \,|c_k \neq 0\}$.

Step 3: $v(p_{\omega,nM+r}(t))$ is a linear function of n with coefficients piece-wise linear functions of ω for all but finitely many n.

Indeed we have

$$p_{\omega,nM+r}(t) = \sum_{(i,\gamma)\in S} a_{i,\gamma}(nM+r) x^{(nM+r)\beta_i\cdot\omega+\gamma\cdot\omega} \tag{15}$$

If $p_{\omega,nM+r}(t) = 0$ for infinitely many n, it follows that for all $(i,\gamma) \in S$, $a_{i,\gamma}(nM+r) = 0$ for infinitely many n. By non-degeneracy and Theorem 3.3, it follows that $a_{i,\gamma}(nM+r) = 0$ for all $(i,\gamma) \in S$ and all n. Thus, $p_{\omega,nM+r}(t) = 0$ for all n. In that case, $v(p_{\omega,nM+r}(t)) = -\infty$ is a constant function of n.

Otherwise, $p_{\omega,nM+r}(t)$ is nonzero for all but finitely many n. Since S is well-ordered by $<_\omega$, it follows that there is a smallest $(i,\gamma) \in S$ such that $(a_{i,\gamma}(nM+r))$ is not identically zero as a function of n. Since $(a_{i,\gamma}(nM+r))$ is non-degenerate, Theorem 3.3 implies that $\{n \in \mathbb{N} \,|a_{i,\gamma}(nM+r) = 0\}$ is a finite set, and for all n in its complement, Equation (15) implies that

$$v(p_{\omega,nM+r}(t)) = (nM+r)\beta_i \cdot \omega + \gamma \cdot \omega$$

Although (i,γ) depends on ω, it is easy to see that they are locally constant functions of ω and after possibly refining σ further, the result of step 3 follows.

Since $-h_{p_n}(-\omega) = v(p_{\omega,n}(t))$, it follows that the restriction of $h_{p_n}(\omega)$ to each arithmetic progression $M\mathbb{N} + r$ is a linear function of n (for all but finitely many n) with coefficients piece-wise linear functions of ω. Lemma 2.1 implies that $N(p_n)$ is quasi-linear. This concludes the proof of Theorem 1.1. □

References

[AI09] Fuensanta Aroca and Giovanna Ilardi, *A family of algebraically closed fields containing polynomials in several variables*, Comm. Algebra **37** (2009), no. 4, 1284–1296.

[BS92] Louis J. Billera and Bernd Sturmfels, *Fiber polytopes*, Ann. of Math. (2) **135** (1992), no. 3, 527–549.

[CLS12] Sheng Chen, Nan Li, and Steven V. Sam, *Generalized Ehrhart polynomials*, Trans. Amer. Math. Soc. **364** (2012), no. 1, 551–569.

[CW13] Danny Calegari and Alden Walker, *Integer hulls of linear polyhedra and scl in families*, Trans. Amer. Math. Soc. **365** (2013), no. 10, 5085–5102.

[Ehr62] Eugène Ehrhart, *Sur les polyèdres homothétiques bordés à n dimensions*, C. R. Acad. Sci. Paris **254** (1962), 988–990.

[ESS02] J.-H. Evertse, H. P. Schlickewei, and W. M. Schmidt, *Linear equations in variables which lie in a multiplicative group*, Ann. of Math. (2) **155** (2002), no. 3, 807–836.

[EvdPSW03] Graham Everest, Alf van der Poorten, Igor Shparlinski, and Thomas Ward, *Recurrence sequences*, Mathematical Surveys and Monographs, vol. 104, American Mathematical Society, Providence, RI, 2003.

[Fat06] P. Fatou, *Séries trigonométriques et séries de Taylor*, Acta Math. **30** (1906), no. 1, 335–400.

[Gar11a] Stavros Garoufalidis, *The degree of a q-holonomic sequence is a quadratic quasi-polynomial*, Electron. J. Combin. **18** (2011), no. 2, Paper 4, 23.

[Gar11b] ———, *The Jones slopes of a knot*, Quantum Topol. **2** (2011), no. 1, 43–69.

[Gar13] ———, *Applications of recurrent sequences to TQFT*, 2013, Preprint.

[GM11] Stavros Garoufalidis and Thomas W. Mattman, *The A-polynomial of the $(-2,3,n)$ pretzel knots*, New York J. Math. **17** (2011), 269–279.

[Grü03] Branko Grünbaum, *Convex polytopes*, second ed., Graduate Texts in Mathematics, vol. 221, Springer-Verlag, New York, 2003, Prepared and with a preface by Volker Kaibel, Victor Klee and Günter M. Ziegler.

[GY14] Stavros Garoufalidis and Josephine Yu, *Laurent solutions to polynomial equations*, 2014, Preprint.

[HS04] Jim Hoste and Patrick D. Shanahan, *A formula for the A-polynomial of twist knots*, J. Knot Theory Ramifications **13** (2004), no. 2, 193–209.

[Kla04] Daniel A. Klain, *The Minkowski problem for polytopes*, Adv. Math. **185** (2004), no. 2, 270–288.

[Lec53] Christer Lech, *A note on recurring series*, Ark. Mat. **2** (1953), 417–421.

[Mah35] Kurt Mahler, *Eine arithmetische eigenschaft der taylor koeffizienten rationaler funktionen*, Tech. Report 38, Proc. Akad. Wet. Amsterdam, 1935, 51–60.

[McD95] John McDonald, *Fiber polytopes and fractional power series*, J. Pure Appl. Algebra **104** (1995), no. 2, 213–233.

[Sch93] Rolf Schneider, *Convex bodies: the Brunn-Minkowski theory*, Encyclopedia of Mathematics and its Applications, vol. 44, Cambridge University Press, Cambridge, 1993.

[Sko35] Thoralf Skolem, *Ein verfahren zur behandlung gewisser exponentialer gleichungen*, Tech. report, Lund Hakan Ohlssons Boktryckeri, 1935, 163–188.

[Sta97] Richard P. Stanley, *Enumerative combinatorics. Vol. 1*, Cambridge Studies in Advanced Mathematics, vol. 49, Cambridge University Press, Cambridge, 1997, With a foreword by Gian-Carlo Rota, Corrected reprint of the 1986 original.

[vdP89] Alf J. van der Poorten, *Some facts that should be better known, especially about rational functions*, Number theory and applications (Banff, AB, 1988), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., vol. 265, Kluwer Acad. Publ., Dordrecht, 1989, pp. 497–528.

[Zie95] Günter M. Ziegler, *Lectures on polytopes*, Graduate Texts in Mathematics, vol. 152, Springer-Verlag, New York, 1995.

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