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APPLICATIONS OF QUANTUM INVARIANTS IN LOW DIMENSIONAL TOPOLOGY

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In this short note we give lower bounds for the Heegaard genus of 3-manifolds using various TQFT in 2+1 dimensions. We also study the large k limit and the large G limit of our lower bounds, using a conjecture relating the various combinatorial and physical TQFTs. We also prove, assuming this conjecture, that the set of colored $SU(N)$ polynomials of a framed knot in S^3 distinguishes the knot from the unknot. © 1997 Elsevier Science Ltd

1. INTRODUCTION

In recent years a remarkable relation between physics and low-dimensional topology has emerged, under the name of *topological quantum field theory* (TQFT for short).

An axiomatic definition of a TQFT in $d+1$ dimensions has been provided by Atiyah–Segal in [1]. We briefly recall it:

- To an oriented d dimensional manifold X, one associates a complex vector space $Z(X)$.
- To an oriented $d+1$ dimensional manifold M with boundary ∂M, one associates an element $Z(M) \in Z(\partial M)$.

This (functor) Z usually satisfies extra compatibility conditions (depending on the dimension d), some of which are:

- For a disjoint union of d dimensional manifolds X, Y

$$Z(X \sqcup Y) = Z(X) \otimes Z(Y).$$

- For a change of orientation of a (unitary) TQFT we have:

$$Z(\overline{X}) = Z(X)^\star$$

(where V^* is the dual vector space of V.)

- For $M = M_1 \cup_X M_2$ where $\partial M_1 = X_1 \sqcup X$, $\partial M_2 = X_2 \sqcup \overline{X}$, one has

$$Z(M) = \langle Z(M_1), Z(M_2) \rangle \in Im(Z(X_1 \sqcup X_2 \sqcup X \sqcup \overline{X}) \to Z(X_1 \sqcup X_2))$$

The above-mentioned axioms for a TQFT in $d+1$ dimensions come from an attempt to axiomatize the path integral (nonperturbative) and the Hamiltonian approach to a quantum field theory.

An axiomatic definition of a *perturbative* TQFT in $d+1$ dimensions is still missing, but in the case of the Chern–Simons theory in $2+1$ dimensions there are some attempts [2–4].

From now on, we will concentrate on topological quantum field theories in $2+1$ dimensions. For a precise definition of them, the reader is refered to [6, 14].

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Any such theory gives invariants of closed 3-manifolds (with values in $\mathbb{C}$), invariants of framed (labeled) links in 3-manifolds (with values in $\mathbb{C}$), as well as finite-dimensional representations of the mapping class groups.

The first such theory was constructed using path integrals in the seminal paper of Witten [15]. We briefly recall the definition, fixing some notation:

Let G be a compact simple simply connected group, and k an integer. Let M be a (2-framed) closed 3-manifold with a framed colored link L. A coloring of the link is the assignment of a representation of the loop group ΩG at level k [7]. Let $G \hookrightarrow P \to M$ be the trivial principal G-bundle. We consider the space $\mathscr{A}$ of all G-connections on P. Let

$$CS : \mathscr{A} \to \mathbb{R}/\mathbb{Z}$$

be the Chern–Simons action. The gauge group $\mathscr{G} = \mathrm{Map}(M, G)$ of G-automorphisms of P acts on $\mathscr{A}$, and for any framed link L colored by λ, the holonomy around it gives

$$\mathscr{O}_{L,\lambda} : \mathscr{A} \to \mathbb{C}.$$

The invariant of the framed colored link L is the following partition function:

$$Z_{\mathrm{ph},G,k}(M, L, \lambda) = \int_{\mathscr{A}} \mathscr{D}A \mathrm{e}^{2\pi i k CS(A)} \mathscr{O}_{L,\lambda}(A).$$

The subscript ph stands for physics. Needless to say, the above path integrals have not yet been defined.

Shortly afterwards, a number of topological (combinatorial) definitions appeared in [11, 13]. They depended on a simple Lie group G and a primitive complex root of unity q, and will be denoted by $Z_{G,q}$.

The main conjecture is that:

CONJECTURE 1.1. *If h is the dual Coxeter number of G, then*

$$Z_{\mathrm{ph},G,k} = Z_{G,\exp(2\pi i/(k+h))}$$

The above conjecture seems ill-defined, as the left-hand side has not yet been defined. However, taking the large k limit (as $k \to \infty$) and using stationary phase approximation of the path integral, we arrive at the following conjecture:

CONJECTURE 1.2. *If M is a closed 3-manifold, as $k \to \infty$ we have:*

$$Z_{G,\exp(2\pi i/(k+h))}(M) \sim_{k\to\infty} k^{\theta_G(M)\dim(G)/2}$$

where $f(k) \sim_{k\to\infty} g(k)$ means $0 < a_1 \leqslant |f(k)/g(k)| \leqslant a_2$ as $k \to \infty$ and $\theta_G(M)$ is as in the following definition.

Definition 1.3. For a closed 3-manifold M, and a compact Lie group G, let

$$\theta_G(M) := \max_{\alpha \in \mathscr{R}_G^{\mathrm{sm}}(M)} \frac{h^1(M,\alpha) - h^0(M,\alpha)}{\dim(G)} + 1$$

where $\mathscr{R}_G^{\mathrm{sm}}(M)$ is the *smooth part* of the moduli space $\mathscr{R}_G(M) = \mathrm{Hom}(\pi_1(M), G)/G$ and $h^k(M, \alpha)$ is the dimension of the kth cohomology of M with twisted coefficients.

Remark 1.4. We will actually only use Conjecture 1.2 in the case of a subsequence of k approaching infinity. The normalization of $\theta_G(M)$ used in Conjecture 1.2 is chosen so that Corollary 2.3 has a simple form.

Let us give one more definition that we will need in the next section:

Definition 1.5. For a closed 3-manifold M let

$$\theta(M) = \overline{\lim}_{N\to\infty} \theta_{SU(N)}(M).$$

2. LOWER BOUNDS FOR THE HEEGAARD GENUS OF 3-MANIFOLDS

We first begin with a lemma:

LEMMA 2.1. *If Z is a TQFT in $2+1$ dimensions, and M,N are closed 3-manifolds, then*

- $Z(M\sharp N)Z(S^3) = Z(M)Z(N)$,
- $Z(S^2 \times S^1) = 1$.

Proof. It follows easily from the glueing axioms, as in [15]. □

Now we are ready to state the following theorem:

THEOREM 2.2. *If Z is any unitary TQFT in $2+1$ dimensions, and M is a closed 3-manifold, then*

$$|Z(M)| \leqslant Z(S^3)^{-g(M)+1}$$

where $g(M)$ is the Heegaard genus of M, i.e. the genus of a minimal Heegaard splitting. Furthermore, we have $0 < Z(S^3) < 1$, thus

$$g(M) - 1 \geqslant -\frac{\log|Z(M)|}{\log Z(S^3)}.$$

Proof. Let $M = H \cup_f H$ be a Heegaard splitting of M, where H is a handelbody of genus g ($\partial(H) = \Sigma_g$), and $f \in \mathrm{Diff}^+(\Sigma_g)$. Let $u := Z(H) \in Z(\Sigma_g)$. Then, we have

$$\begin{aligned}
|Z(M)| &= |\langle u, f_\star(u)\rangle| \\
&\leqslant \sqrt{\langle u,u\rangle\langle f_\star(u), f_\star(u)\rangle} \quad \text{(by Cauchy Schwarz)} \\
&= \langle u,u\rangle \quad \text{(since } Z \text{ is unitary)} \\
&= Z(\sharp_{i=1}^g S^2 \times S^1) \\
&= Z(S^2 \times S^1)^g Z(S^3)^{-g+1} \quad \text{(by Lemma 2.1)} \\
&= Z(S^3)^{-g+1}.
\end{aligned}$$

The above implies that $0 < Z(S^3) < 1$. Indeed, otherwise we necessarily have that $Z(S^3) > 1$. Then, for any 3-manifold M, by choosing a Heegaard splitting of large enough genus, the above implies that $Z(M) = 0$, which contradicts the fact that $Z(S^2 \times S^1) = 1$. Thus, we deduce that $0 < Z(S^3) < 1$, and taking a minimal genus Heegaard splitting concludes the proof of the theorem. □

Table 1

Manifold M	$g(M)$	$\theta_G(M)$	$\theta(M)$
S^3	0	0	0
$L_{p,q}$	1	$-l_G/d_G+1$	1
$S(a_1,\dots,a_n)$	$n-1$	$2n\mu_G/d_G-1$	$n-1$
$S^1\times\Sigma_g$	$2g+1$	$2g-1$	$2g-1$

Note: d_G, l_G, μ_G are the dimension, rank and number of positive roots of the Lie group G. $L_{p,q}$ is a Lens space with $\pi_1(L_{p,q})=\mathbb{Z}/p\mathbb{Z}$ and $S(a_1,\dots,a_n)$ is a Seifert fibered integral homology 3-sphere with singular fibers of orders $a_1,\dots,a_n$ (where a_i are coprime integers) [10].

We also have the following:

Corollary 2.3 (Depending on Conjecture 1.2). *For a closed* 3*-manifold* M*, and a compact simple simply connected group* G *we have*

- $g(M)\geqslant\theta_G(M)$,
- $\theta_G(M\# N)=\theta_G(M)+\theta_G(N)$.

Proof. For the first part use the previous corollary for the TQFT $Z=Z_{G,\exp(2\pi i/(k+h))}$ and the fact that $Z(S^3)$ is given by an explicit expression of [8]. For the second part use the TQFT $Z=Z_{G,\exp(2\pi i/(k+h))}$, and Lemma 2.1 and the fact that $Z(S^3)$ is given by [8].

Remark 2.4. In Table 1 we calculate a list of values of $\theta_G(M)$ for certain classes of 3-manifolds M for which Conjecture 1.2 has been verified by direct calculation [6].

3. DETECTING THE UNKNOT

In this section we use Conjecture 1.2 to show how the *TQFT* invariants might detect the unknot. Fix a framed knot K in S^3. It turns out [9] that given a simple Lie group G, and a representation λ of G, there is a rational function $Z_G(K,\lambda)(t)$ such that for every primitive complex root of unity q, we have

$$Z_G(K,\lambda)(q)=Z_{G,q}(S^3,K,\lambda)$$

Theorem 3.1 (Depending on Conjecture 1.2). *Let* $K\subseteq S^3$ *be a framed oriented knot, and* $\mathcal{O}$ *be the zero framed unknot in* S^3*. If*

$$Z_{SU(N)}(K,\lambda)=Z_{SU(N)}(\mathcal{O},\lambda)$$

for all colors λ *and all* $N\geqslant 2$ *then* $K=\mathcal{O}$.

Proof. Let $S^3_{K,a/b}$ denote the result of a/b *Dehn-surgery* on K, where a,b are coprime integers, with the convention $H_1(S^3_{K,a/b},\mathbb{Z})=\mathbb{Z}/a\mathbb{Z}$. Using the above-mentioned property of the colored $SU(N)$ polynomials together with the fact that for any coprime integers a,b and every primitive complex root of unity q, $Z_{SU(N),q}(S^3_{K,a/b})$ is a linear combination of $Z_{SU(N),q}(S^3,K,\lambda)$ (for suitable λ), we deduce that

$$Z_{SU(N),q}(S^3_{K,1/n})=Z_{SU(N),q}(S^3_{\mathcal{O},1/n})=Z_{SU(N),q}(S^3)$$

for all $n \in \mathbb{Z}, N \geqslant 2$ and all primitive complex roots of unity q. Now use $q = \exp(2\pi i/(k+N))$, Conjecture 1.2 and the value of $Z_{SU(N),q}(S^3)$ as in [8], to deduce

$$\theta_{SU(N)}(S^3_{K,1/n}) = 0 \quad \text{for all } N \geqslant 2, \quad n \in N.$$

Lemma 3.2 below implies that

$$\mathrm{Hom}(\pi_1(S^3_{K,1/n}), SU(N)) = \{0\}$$

for all $N \geqslant 2$ and $n \in \mathbb{N}$. Using the fact that $\pi_1(S^3_{K,1/n})$ is a residually finite group for $n \gg 0$ (as follows by Thurston [12]) we obtain

$$\pi_1(S^3_{K,1/n}) = 0$$

for all $n \gg 0$. Using the cyclic surgery theorem of Gordon–Luecke [5] the result follows. □

LEMMA 3.2. *If $\theta_G(M) = 0$ for a simple (simply connected) Lie group G and 3-manifold M, then $Hom(\pi_1(M), G) = 0$.*

Proof. Recall first that for an element $\alpha \in \mathscr{R}^{\mathrm{sm}}_G(M)$ we have that $h^0(M, \alpha)$ is the dimension of the stabilizer of the image of α in G. Thus

$$h^0(M, \alpha) \leqslant \dim(G)$$

with equality if and only if the stabilizer of α is G, in other words $\alpha \in \mathrm{Hom}(\pi_1(M), Z(G))$ where $Z(G)$ is the center of G. Since M is an integral homology 3-sphere, we have that $\mathrm{Hom}(\pi_1(M), Z(G)) = \{0\}$; thus $h^0(M, \alpha) = \dim(G)$ if and only if $\alpha = 0$, i.e., α is the trivial group homomorphism.

Recall further that $\mathscr{R}^{\mathrm{sm}}_G(M)$ is a smooth (possibly noncompact and nonconnected) manifold. For an element $\beta \in \mathscr{R}^{\mathrm{sm}}_G(M)$, the dimension of the component of $\mathscr{R}^{\mathrm{sm}}_G(M)$ that contains β is given by $h^1(M, \beta) - h^0(M, \beta)$. Using the assumption that $\theta_G(M) = 0$ and the above equation, we conclude that $h^1(M, \beta) = 0$ and $h^0(M, \beta) = \dim(G)$; thus $\beta = 0$. In other words, we have that $\mathscr{R}^{\mathrm{sm}}_G(M) = \{0\}$, and thus (since the singular points in $\mathscr{R}_G(M)$ are of codimension at least one, and since isolated points are smooth) the lemma follows. □

An equivalent formulation of the previous theorem is the following:

COROLLARY 3.3 (Depending on Conjecture 1.2). *If $K \subseteq S^3$ is a framed oriented knot, and $Z(K) = Z(\mathcal{O})$ for all TQFT Z in $2+1$ dimensions, then K is the unknot.*

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