BLOCH GROUPS, ALGEBRAIC K-THEORY, UNITS, AND NAHM'S CONJECTURE

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Abstract. Given an element of the Bloch group of a number field F and a natural number n, we construct an explicit unit in the field $F_n = F(e^{2\pi i/n})$, well-defined up to n-th powers of nonzero elements of F_n. The construction uses the cyclic quantum dilogarithm, and under the identification of the Bloch group of F with the K-group $K_3(F)$ gives an explicit formula for a certain abstract Chern class from $K_3(F)$. The units we define are conjectured to coincide with numbers appearing in the quantum modularity conjecture for the Kashaev invariant of knots (which was the original motivation for our investigation), and also appear in the radial asymptotics of Nahm sums near roots of unity. This latter connection is used to prove one direction of Nahm's conjecture relating the modularity of certain q-hypergeometric series to the vanishing of the associated elements in the Bloch group of $\overline{\mathbf{Q}}$.

Contents

1. Introduction

The purpose of the paper is to associate to an element ξ of the Bloch group of a number field F and a primitive nth root of unity ζ an explicit unit or near unit $R_\zeta(\xi)$ in the field $F_n = F(\zeta)$, well-defined up to n-th powers of nonzero elements of F_n. Our construction uses the cyclic quantum dilogarithm and is shown to give an explicit formula for an abstract Chern class map on $K_3(F)$. The near unit is conjectured (and checked numerically in many cases) to coincide with a specific number that appears in the Quantum Modularity Conjecture of the Kashaev invariant of a knot. This was in fact the starting point of our investigation [13], [39].

As a surprising consequence of our main theorem we were able to prove one direction of Werner Nahm's famous conjecture, namely that the modularity of certain q-hypergeometric series ("Nahm sums") implies the vanishing of certain explicit elements in the Bloch group of $\overline{\mathbf{Q}}$. A precise statement will be given in Section 1.3 of this introduction.

1.1. Bloch groups and associated units.

We first recall the definition of the classical Bloch group, as introduced in [2]. Let $Z(F)$ denote the free abelian group on $\mathbf{P}^1(F) = F \cup \{\infty\}$, i.e. the group of formal finite combinations $\xi = \sum_i n_i[X_i]$ with $n_i \in \mathbf{Z}$ and $X_i \in \mathbf{P}^1(F)$.

Definition 1.1. The *Bloch group* of a field F is the quotient

$$B(F) = A(F)/C(F)\,, \tag{1}$$

where $A(F)$ is the kernel of the map

$$d:\ Z(F) \longrightarrow \textstyle\bigwedge^2 F^\times\,, \qquad [X] \mapsto (X) \wedge (1-X) \tag{2}$$

(and $[0]$, $[1]$, $[\infty] \mapsto 0$) and $C(F) \subseteq A(F)$ the group generated by the *five-term relation*

$$\xi_{X,Y} \;=\; [X] - [Y] + \left[\frac{Y}{X}\right] - \left[\frac{1-X^{-1}}{1-Y^{-1}}\right] + \left[\frac{1-X}{1-Y}\right] \tag{3}$$

with X and Y ranging over $\mathbf{P}^1(F)$ (but forbidding arguments $\frac{0}{0}$ or $\frac{\infty}{\infty}$ on the right-hand side).

In this paper, we will study an invariant of the Bloch group whose values are units in F_n modulo nth powers of units, where n is a natural number and F_n the field obtained by adjoining to F a primitive n-th root of unity $\zeta = \zeta_n$. The extension F_n/F is Galois with Galois group $G = \mathrm{Gal}(F_n/F)$, and G admits a canonical map

$$\chi:\ G \longrightarrow (\mathbf{Z}/n\mathbf{Z})^\times \tag{4}$$

determined by $\sigma\zeta = \zeta^{\chi(\sigma)}$. The powers χ^j ($j \in \mathbf{Z}/n\mathbf{Z}$) of this character define eigenspaces $\bigl(F_n^\times/F_n^{\times n}\bigr)^{\chi^j}$ in the obvious way as the set of $x \in F_n^\times/F_n^{\times n}$ such that $\sigma(x) = x^{\chi^j(\sigma)}$ for all $\sigma \in G$, and similarly for $(\mathcal{O}_n^\times/\mathcal{O}_n^{\times n})^{\chi^j}$ or $(\mathcal{O}_{S,n}^\times/\mathcal{O}_{S,n}^{\times n})^{\chi^j}$, where $\mathcal{O}_n$ (resp. $\mathcal{O}_{S,n}$) is the ring of integers (resp. S-integers) of F_n. Then our main result is the following theorem.

Theorem 1.2. *Suppose that F does not contain any non-trivial nth root of unity. Then there is a canonical map*

$$R_\zeta :\, B(F)/nB(F) \;\longrightarrow\; \bigl(\mathcal{O}_{S,n}^\times/\mathcal{O}_{S,n}^{\times n}\bigr)^{\chi^{-1}} \;\subset\; \bigl(F_n^\times/F_n^{\times n}\bigr)^{\chi^{-1}} \tag{5}$$

for some finite set S of primes depending only on F. If n is prime to a certain integer M_F depending on F, then the map R_ζ is injective and its image is contained in $\bigl(\mathcal{O}_n^\times/\mathcal{O}_n^{\times n}\bigr)^{\chi^{-1}}$, and equal to this if n is prime.

Remark 1.3. Note that the field F_n and the character χ of (4) do not depend on the primitive nth root of unity ζ. The map R_ζ from $B(F)$ to $F_n^\times/F_n^{\times n}$ does depend on ζ, but in a very simple way, described by either of the formulas

$$\sigma\bigl(R_\zeta(\xi)\bigr) \;=\; R_{\sigma(\zeta)}(\xi) \quad (\sigma \in G), \qquad R_\zeta(\xi) \;=\; R_{\zeta^k}(\xi)^k \quad \bigl(k \in (\mathbf{Z}/n\mathbf{Z})^\times\bigr)\,, \tag{6}$$

where the simultaneous validity of these two formulas explains why the image of each map R_ζ lies in the χ^{-1} eigenspace of $F_n^\times/F_n^{\times n}$.

Remark 1.4. The optimal definition of M_F is somewhat complicated to state. However, one may take it to be $6\,\Delta_F\,|K_2(\mathcal{O}_F)|$.

The detailed construction of the map R_ζ will be given in Section 2. A rough description is as follows. Let $\xi = \sum n_i[X_i]$ be an element of $Z(F)$ whose image in $\wedge^2(F^\times/F^{\times n})$ under the map induced by d vanishes. We define an algebraic number $P_\zeta(\xi)$ by the formula

$$P_\zeta(\xi) \;=\; \prod_i D_\zeta(x_i)^{n_i}\,, \tag{7}$$

where x_i is some nth root of X_i and $D_\zeta(x)$ is the *cyclic quantum dilogarithm function*

$$D_\zeta(x) \;=\; \prod_{k=1}^{n-1}(1 \,-\, \zeta^k x)^k\,. \tag{8}$$

The number $P_\zeta(\xi)$ belongs to the Kummer extension H_ξ of F defined by adjoining all of the x_i to F_n and is well-defined modulo H_ξ^n. We show that for n prime to some M_F it has the form ab^n with b in $H_\xi^\times$ and $a \in F_n^\times$ (or even $a \in \mathcal{O}_n^\times$ under a sufficiently strong coprimality assumption about n). Then $R_\zeta(\xi)$ is defined as the image of a modulo nth powers.

1.2. **Algebraic K-groups and associated units.** A second main theme of the paper concerns the relation to the algebraic K-theory of fields. The group $B(F)$ was introduced by Bloch as a concrete model for the abstract K-group $K_3(F)$. It was proved by Suslin [31] that, if F is a number field, then (up to 2-torsion) $K_3(F)$ is an extension of $B(F)$ by the roots of unity in F, and in this case one also knows by results of Borel and Suslin-Merkurjev [30], [21], [36] that $K_3(F)$ has the structure

$$K_3(F) \;\cong\; \mathbf{Z}^{r_2(F)} \;\oplus\; \begin{cases} \mathbf{Z}/w_2(F)\mathbf{Z} & \text{if } r_1(F) = 0, \\ \mathbf{Z}/2w_2(F)\mathbf{Z} \,\oplus\, (\mathbf{Z}/2\mathbf{Z})^{r_1(F)-1} & \text{if } r_1(F) \geq 1, \end{cases} \tag{9}$$

where $(r_1(F), r_2(F))$ is the signature of F and $w_2(F)$ is the integer

$$w_2(F) \;=\; 2 \prod_p p^{\nu_p}\,, \qquad \nu_p \;:=\; \max\bigl\{\nu \in \mathbf{Z} \mid \zeta_{p^\nu} + \zeta_{p^\nu}^{-1} \in F\bigr\}\,. \tag{10}$$

For a detailed introduction to the the algebraic K-theory of number fields, see [36].

Theorem 1.2 is then a companion of the following result concerning $K_3(F)$:

Theorem 1.5. *Let F be a number field. Then there is a canonical map*

$$c_\zeta \,:\, K_3(F)/nK_3(F) \;\longrightarrow\; \bigl(\mathcal{O}_{S,n}^\times/\mathcal{O}_{S,n}^{\times n}\bigr)^{\chi^{-1}} \;\subset\; \bigl(F_n^\times/F_n^{\times n}\bigr)^{\chi^{-1}} \tag{11}$$

defined using the theory of Chern classes for some finite set S of primes depending only on F. If n is prime to a certain integer M_F depending on F, then the map R_ζ is injective and its image is contained in $\bigl(\mathcal{O}_n^\times/\mathcal{O}_n^{\times n}\bigr)^{\chi^{-1}}$, and equal to this if n is prime.

We note that the proof of Theorem 1.2 relies upon the precise computation of $K_3(F)$ and the properties of c_ζ given above. Finally, in view of the near isomorphism between $B(F)$ and $K_3(F)$, one might guess that the two maps P_ζ and c_ζ are the same, at least up to a simple scalar. This is the content of our next theorem.

Theorem 1.6. *For n prime to M_F, the map R_ζ equals c_ζ^γ for some $\gamma \in (\mathbf{Z}/n\mathbf{Z})^\times$.*

The constant γ does not depend on the underlying field — both our construction and the Chern class map are well behaved in finite extensions, so we can compare the maps over any two fields with the maps in their compositum. We conjecture that the constant γ is, up to sign, a small power of 2 that is independent of n. To justify our conjecture, and to determine γ, it suffices to compute the image under both maps R_ζ and c_ζ of some element of $K_3(F)/nK_3(F)$ of exact order n. For each root of unity ζ of order n, there is a specific element η_ζ (eq. (23)) of the finite Bloch group $B(\mathbf{Q}(\zeta+\zeta^{-1}))$ that is of exact order n. Using the relation of the map R_ζ to the radial asymptotics of certain q-series called Nahm sums discussed in Section 8, we will prove

$$R_\zeta(\eta_\zeta)^4 \;=\; \zeta \tag{12}$$

(Theorem 8.5). On the other hand, certain expected functorial properties of the map c_ζ, discussed in Section 5.3 indicate that up to sign and a small power of 2, we have:

$$c_\zeta(\eta_\zeta) \;\overset{?}{=}\; \zeta\,, \tag{13}$$

and in combination with (12) this justifies our conjecture concerning γ.

The above-mentioned relation between our *mod n regulator map* on Bloch groups and the asyptotics of Nahm sums near roots of unity is also an ingredient of our proof of one direction of Nahm's conjecture (under some restrictions) relating the modularity of his sums to torsion in the Bloch group. The argument, described in Section 8.3, uses the full strength of Theorem 1.2 and gives a nice demonstration of the usefulness, despite its somewhat abstract statement, of that theorem.

Theorem 1.2 motivates a mod n (or *étale*) version of the Bloch group of a number field F, defined by

$$B(F;\mathbf{Z}/n\mathbf{Z}) \;=\; A(F;\mathbf{Z}/n\mathbf{Z})/(nZ(F)+C(F))\,, \tag{14}$$

where $A(F;\mathbf{Z}/n\mathbf{Z})$ is the kernel of the map $d: Z(F)\to \wedge^2(F^\times/F^{\times\, n})$ induced by d. This is studied in Section 6, where we establish the following relation to $K_2(F)$.

Theorem 1.7. *The étale Bloch group is related to the original Bloch group by an exact sequence*

$$0 \;\longrightarrow\; B(F)/nB(F) \;\longrightarrow\; B(F;\mathbf{Z}/n\mathbf{Z}) \;\longrightarrow\; K_2(F)[n] \;\longrightarrow\; 0\,, \tag{15}$$

where $K_2(F)[n]$ is the n-torsion in the K-group $K_2(F)$.

There is a corresponding exact sequence (equation (27)) with $B(F)/nB(F)$ replaced by $K_3(F)/nK_3(F)$ and $B(F;\mathbf{Z}/n\mathbf{Z})$ replaced by a Galois cohomology group.

A large part of the story that we have told here for the Bloch group $B(F)$ and the third K-group $K_3(F)$ can be generalized to higher Bloch groups $B_m(F)$ and $K_{2m-1}(F)$ with $m\geq 2$, and here the étale version really comes into its own, because the higher Bloch groups as originally introduced in [37] have several alternative definitions that are only conjecturally isomorphic and are difficult or impossible to compute rigorously, whereas their étale versions turn out to have a canonial definition and be amenable to rigorous computations. The study of the higher cases has many proofs in common with the $m=2$ case studied here, but there are also many new aspects, and the discussion will therefore be given in a separate paper [3] which is work in progress.

1.3. **Nahm's Conjecture.**

The near unit constructed in Section 1.1 also appears in connection with the asymptotics near roots of unity of certain q-hypergeometric series called Nahm sums. These series are defined by

$$f_{A,B,C}(q) \;=\; \sum_{m\in\mathbf{Z}^r_{\geq 0}} \frac{q^{\frac12 m^tAm+Bm+C}}{(q)_{m_1}\cdots(q)_{m_r}}\,,$$

where $A\in M_r(\mathbf{Q})$ is a positive definite symmetric matrix, B an element of $\mathbf{Q}^r$, and C a rational number. Based on ideas coming from characters of rational conformal field theories, Nahm conjectured a relation between the modularity of the associated holomorphic function $\widetilde{f}_{A,B,C}(\tau) = f_{A,B,C}(e^{2\pi i\tau})$ in the complex upper half-plane and the vanishing of a certain element or elements in the Bloch group of $\overline{\mathbf{Q}}$. (See [24], [38], and Section 8 for more details.) This relation conjecturally goes in both directions, but with the implication from the vanishing of the Bloch elements to the modularity of certain Nahm sums not yet having a sufficiently precise formulation to be studied. The conjectural implication from modularity to vanishing of Bloch elements, on the other hand, had a completely precise formulation, as follows. Let A be as above and $(X_1,\dots,X_r)$ the unique solution in $(0,1)^r$ of *Nahm's equation*

$$1 \,-\, X_i \;=\; \prod_{j=1}^r X_j^{a_{ij}} \qquad (i=1,\dots,r)\,.$$

Then Nahm shows that the element $\xi_A = \sum_{i=1}^{r}[X_i]$ belongs to $B(\mathbf{R} \cap \overline{\mathbf{Q}})$, and his assertion is the following theorem, which we will prove as a consequence of the injectivity statement in Theorem 1.2.

Theorem 1.8 (One direction of Nahm's Conjecture). *If the function $\widetilde{f}_{A,B,C}(\tau)$ is modular for some A, B and C as above, then ξ_A vanishes in the Bloch group of $\overline{\mathbf{Q}}$.*

We remark that the vanishing condition can be (and often is) stated by saying that ξ_A is a torsion element in the Bloch group of the smallest real (but in general not totally real) number field containing all the X_i, but when we take the image of this Bloch group in the Bloch group of $\overline{\mathbf{Q}}$ or $\mathbf{C}$, then the torsion vanishes.

1.4. **Plan of the paper.** In Section 2 we recall the cyclic quantum dilogarithm and use it, together with some basic facts about Kummer extensions, to define the map R_ζ. The fact that the map R_ζ satisfies the 5-term relation follows from some state-sum identities of Kashaev-Mangazeev-Stroganov [17], reviewed in Section 2.4. The remaining statements of Theorem 1.2 are deduced from Theorems 1.5 and 1.6.

In Section 3 we recall the basic properties of Chern classes and use them to define the map c_ζ and prove Theorem 1.5. Its proof follows from Lemmas 3.1 and 3.5.

The comparison of the maps c_ζ and R_ζ is done via reduction to the case of finite fields. This reduction is discussed in Section 4, and the proof of Theorem 1.6 is given in Section 5.

In Section 6, we discuss the connection of our map R_ζ with Tate's results on $K_2(\mathcal{O}_F)$.

The units produced by our map R_ζ have also appeared in two related contexts, namely in the Quantum Modularity Conjecture concerning the asymptotics of the Kashaev invariant at roots of unity, and in the asymptotics of Nahm sums at roots of unity. In Section 7, we give examples of the units produced by our map R_ζ and compare them with those that appear in the Quantum Modularity Conjecture. In Section 8, we state the connection of our map R_ζ with the radial asymptotics of Nahm sums at roots of unity and give two applications: a proof equation (12) (as a consequence of a special modular Nahm sum, the Andrews-Gordon identity), and a proof of Theorem 1.8.

Remark. During the writing of this paper, we learned that Gangl and Kontsevich in unpublished work also proposed the map P_ζ as an explicit realization of the Chern class map. Although they did not check in general that the image of P_ζ could be lifted to a suitable element $R_\zeta \in (F_n^\times/F_n^{\times n})^{\chi^{-1}}$, they did propose an alternate proof of the 5-term identity using cyclic algebras. Goncharov also informs us that he was aware many years ago that the function P_ζ should be an explicit realization of the Chern class map.

2. The maps P_ζ and R_ζ

Let n be a positive integer, and let F be a field of characteristic prime to n. Let $F_n = F(\zeta_n)$, and let $\zeta = \zeta_n \in F_n$ denote a primitive nth root of unity, which we usually consider as fixed and omit from the notations. For convenience, we will always assume that n prime to 6.

2.1. **The map P_ζ.** Let $\mu = \langle \zeta \rangle$ denote the G_F-module of nth roots of unity. Recall that $F_n = F(\zeta)$. The *universal Kummer extension* is by definition the extension H/F_n obtained by adjoining nth roots of every element in F. Let $\Phi = \mathrm{Gal}(H/F_n)$. We have [20, Chpt.VI]:

Lemma 2.1. The extension H/F is Galois. There is a natural isomorphism

$$\phi : F^\times/F^{\times n} \;\simeq\; \mathrm{Hom}(\Phi,\mu) \;\simeq\; H^1(\Phi,\mu)$$

given by $X \mapsto (\sigma \in \Phi \mapsto \sigma x/x)$, where $x \in H^\times$ is any element that satisfies $x^n = X$.

Consider the function

$$P_\zeta(X) := D_\zeta(x) \;\in\; H^\times/H^{\times n} \qquad (X \in F^\times \smallsetminus \{0,1\},\; x^n = X)\,, \tag{16}$$

where $D_\zeta(x)$ is the cyclic quantum dilogarithm defined in (8). (We previously defined $P_\zeta(X)$, in equation (8) of the induction, as an element of $H^\times$, but only its image modulo nth powers was ever used, and it is more canonical to define it in the manner above.)

Lemma 2.2. The function $P_\zeta : F^\times \to H^\times/H^{\times n}$ has the following properties.
(a) $P_\zeta(X)$ is independent of the choice of nth root x of X.
(b) $P_\zeta(1) = 1$, and more generally $P_\zeta(X)P_\zeta(1/X) = 1$ for any $X \in F_n^\times$.
(c) $P_\zeta(X) \in\; H^\times/H^{\times n}$ is invariant under the action of $\Phi = \mathrm{Gal}(H/F_n)$.
(d) $\sigma(P_\zeta(X)) = P_\zeta(X)^{\chi^{-1}(\sigma)}$ for all $\sigma \in G$.

Proof. First note that, because $P_\zeta(X)$ is defined only up to nth powers, we can replace the definition (16) by

$$P_\zeta(X) \;=\; \prod_{k \bmod n} (1 \,-\, \zeta^k x)^k \quad \bmod H^{\times n} \qquad (x^n = X), \tag{17}$$

where we can now even include the $k = 0$ term that was omitted in (8). Part (a) then follows from the calculation

$$\frac{\prod_{k \bmod n}(1-\zeta^k x)^k}{\prod_{k \bmod n}(1-\zeta^{k+1}x)^k} \;=\; \prod_{k \bmod n}(1-\zeta^k x) \;=\; 1 - X \;\in\; F^\times \;\subset\; H^{\times n}.$$

Similarly, replacing k by $-k$ in the definition of $P_\zeta(1/X)$, gives

$$P_z(X)P_z(1/X) \;=\; \prod_{k \bmod n}(1-\zeta^k x)^k(1-\zeta^{-k}x^{-1})^{-k} \;=\; \prod_{k \bmod n}(-\zeta^k x)^k \;=\; 1 \quad \in\; H^\times/H^{\times n}\,,$$

proving the second statement of (b), and the first statement follows because an element killed by both 2 and the odd number n in any group must be trivial. (It can also be proved more explicitly by evaluating $D_n(1)^n$ itself for $(n,6) = 1$ as the $(-1)^{n(n-1)/2}n^n$, which is an nth power because $(-1)^{(n-1)/2}n$ is a square in $\mathbf{Q}(\zeta_n)$.) For part (c), we note that the effect of an element $\sigma \in \Phi$ on $D_\zeta(x)$ is to replace x by $\zeta^i x$ for some i, so the result follows from part (a). For part (d), we first observe that the statement makes sense because $\Phi = \mathrm{Gal}(H/F_n)$ is a normal subgroup of $\mathrm{Gal}(H/F)$ and hence acts trivially on $P_\zeta(X) \in H^\times/H^{\times n}$ by virtue of (c), so that the quotient $G = \mathrm{Gal}(F_n/F)$ acts on $P_\zeta(X)$. For the proof, we choose a lift of $\sigma \in G$ to $\mathrm{Gal}(H/F)$ that fixes x. Then

$$\sigma P_\zeta(X) \;=\; \prod_k \bigl(1-\sigma(\zeta)^k x\bigr)^k \;=\; \prod_k \bigl(1-\zeta^{k\chi(\sigma)}x\bigr)^k \;=\; \prod_k \bigl(1-\zeta^k x\bigr)^{k\chi(\sigma)^{-1}} \;=\; P_\zeta(X)^{\chi(\sigma)^{-1}}\,,$$

where all products are over $k \pmod n$ and all calculations are modulo $H^{\times n}$. $\square$

Remark 2.3. When n is not prime to 6, then we could also make the calculations above work after replacing the right-hand side of (16) by $P_\zeta(X) = \frac{D_\zeta(x)}{D_\zeta(1)}$. (When $(n,6)=1$ this is not necessary since an elementary calculation shows that then $D_\zeta(1) \in \mathbf{Q}(\zeta)^n$.)

We extend the map P_ζ to the free abelian group $Z(F) = \mathbf{Z}[\mathbf{P}^1(F)]$ by linearity as in (7), with $P_\zeta(0) = P_z(1) = P_\zeta(\infty) = 0$.

2.2. **The map R_ζ.** The next proposition associates an element $R_\zeta(\xi) \in (F_n^\times/F_n^{\times n})^{\chi^{-1}}$ to every element of $B(F)/nB(F)$ as long as $(n, w_F) = 1$. Recall the group $A(F;\mathbf{Z}/n\mathbf{Z})$ from subsection 1.1.

Proposition 2.4. (a) For $\xi \in A(F;\mathbf{Z}/n\mathbf{Z})$, the image of $P_\zeta(\xi)^{w_F}$ lifts to $F_n^\times/F_n^{\times n}$.
(b) The image of $P_\zeta(\xi)^{w_F}$ admits a unique lift to $F_n^\times/F_n^{\times n}$ on which G acts by χ^{-1}. If n is prime to w_F, then $P_\zeta(\xi)$ itself admits a unique lift $R_\zeta(\xi) \in (F_n^\times/F_n^{\times n})^{\chi^{-1}}$.

Proof. For part (a), by Hilbert 90 and inflation-restriction, there is a commutative diagram:

$$\begin{tikzcd}
H^1(\Phi,\mu) \arrow[r] & H^1(F_n,\mu) \arrow[r] \arrow[d, equal] & H^1(H,\mu)^\Phi \arrow[r, "\delta"] \arrow[d, equal] & H^2(\Phi,\mu) \\
& F_n^\times/F_n^{\times n} \arrow[r] & \left(H^\times/H^{\times n}\right)^\Phi &
\end{tikzcd}$$

That is, there is an obstruction to descending from $\left(H^\times/H^{\times n}\right)^\Phi$ to $F_n^\times/F_n^{\times n}$ which lands in $H^2(\Phi,\mu)$.

We now claim that there is a commutative diagram as follows:

$$\begin{tikzcd}
Z(F) \arrow[r, "P_\zeta"] \arrow[d, "d"'] & (H^\times/H^{\times n})^\Phi \arrow[d, "\delta"] \\
\bigwedge^2(F^\times/F^{\times n}) \arrow[r, hook, "\cup"] & H^2(\Phi,\mu)\ ,
\end{tikzcd}$$

where the left vertical map is the one defined in (2) and the bottom horizontal map is the map induced by the cup product from the isomorphism $F^\times/F^{\times n} \to H^1(\Phi,\mu)$ of Lemma 2.1. Note that the cup product is more naturally a map $\bigwedge^2 H^1(\Phi,\mu) \to H^2(\Phi,\mu^{\otimes 2})$, but can be interpreted as in the theorem by using the trivialization $\mu \simeq \mathbf{Z}/n\mathbf{Z} \simeq \mu^{\otimes 2}$ defined by the choice of the root of unity ζ.

We now show that the above diagram commutes. By linearity, it suffices to prove this for elements ξ of the form $[X]$. Write $X = x^n$ and $1 - X = y^n$. For $Z \in F^\times/F^{\times n}$ and $z^n = Z$, let (following Lemma 2.1),

$$\sigma(z) = \zeta^{\phi(z,\sigma)} z.$$

By definition, we have $P_\zeta([X]) = D_\zeta(x)$ modulo nth powers. The obstruction to lifting $D(x)$ amounts to finding an element $u \in H^\times$ such that $D_\zeta(x)/u^n \in F_n^\times$. Such a u would necessarily satisfy

$$\Big(\frac{\sigma u}{u}\Big)^n = \frac{\sigma D_\zeta(x)}{D_\zeta(x)} \;=\; \frac{D_\zeta(\zeta^{\phi(x,\sigma)}x)}{D_\zeta(x)} \;=\; \Bigg(\prod_{k=0}^{\phi(x,\sigma)-1} \frac{1-\zeta^k x}{y}\Bigg)^n.$$

The expression inside the nth power is determined exactly modulo $\langle\zeta\rangle$. Hence we may define a cocycle

$$h = h_X : \Phi \;\rightarrow\; H^\times/\mu\,, \qquad h(\sigma) \;:=\; \prod_{k=0}^{\phi(x,\sigma)-1} \frac{1-\zeta^k x}{y}\,.$$

This gives an element of $H^1(\Phi, H^\times/\mu)$, which by consideration of the exact sequence

$$H^1(\Phi, H^\times) \longrightarrow H^1(\Phi, H^\times/\langle\zeta\rangle) \longrightarrow H^2(\Phi, \mu)$$

maps to $H^2(\Phi,\mu)$. This is actually an injection, because the first term vanishes by Hilbert 90. This is the image of δ; explicitly, the class $\delta(h) \in H^2(\Phi,\mu)$ is given by

$$\begin{aligned}
\delta(h)(\sigma,\tau) &= \frac{h(\sigma\tau)}{h(\sigma)\sigma h(\tau)} \\
&= \frac{1}{h(\sigma)\sigma h(\tau)} \prod_{k=0}^{\phi(x,\sigma)+\phi(x,\tau)-1} \frac{1-\zeta^k x}{y} \\
&= \frac{1}{h(\sigma)\sigma h(\tau)} \prod_{k=0}^{\phi(x,\sigma)-1} \frac{1-\zeta^k x}{y} \prod_{k=0}^{\phi(x,\tau)-1} \frac{1-\zeta^k \zeta^{\phi(x,\sigma)} x}{y} \\
&= \frac{1}{h(\sigma)\sigma h(\tau)} \prod_{k=0}^{\phi(x,\sigma)-1} \frac{1-\zeta^k x}{y} \prod_{k=0}^{\phi(x,\tau)-1} \frac{1-\zeta^k \zeta^{\phi(x,\sigma)} x}{\zeta^{\phi(y,\sigma)} y} \cdot \zeta^{\phi(y,\sigma)} \\
&= \zeta^{\phi(x,\tau)\phi(y,\sigma)}
\end{aligned}$$

On the other hand, the class in $H^1(\Phi,\mu)$ associated to $X = x^n$ is the map $\tau \mapsto \zeta^{\phi(x,\tau)}$, and the class associated to $1 - X = y^n$ is the map $\sigma \mapsto \zeta^{\phi(y,\sigma)}$, and the exterior product of these two classes in $H^2(\Phi,\zeta)$ is precisely $\delta(h)$. The fact that the cup product gives an injection is an easy fact about the cohomology of abelian groups of exponent n. This concludes the proof of part (a).

For part (b), suppose that $\xi \in A(F;\mathbf{Z}/n\mathbf{Z})$. By the argument above, there certainly exists an element in $F_n^\times/F_n^{\times n}$ which maps to $P_\zeta(\xi)$. Let M denote the image of $F_n^\times/F_n^{\times n}$ in $(H^\times/H^{\times n})^\Phi$, and let $S = F^\times/F^{\times n}$. We have a short exact sequence as follows:

$$0 \longrightarrow S \longrightarrow F_n^\times/F_n^{\times n} \longrightarrow M \longrightarrow 0.$$

Taking χ^{-1}-invariants is the same as tensoring with $\mathbf{Z}/n\mathbf{Z}(1)$ and taking invariants. Hence there is an exact sequence

$$\big(F_n^\times/F_n^{\times n}\big)^{\chi^{-1}} \longrightarrow M^{\chi^{-1}} \longrightarrow H^1(G, S(1)).$$

In particular, the obstruction to lifting to a χ^{-1}-invariant element lies in $H^1(G,S(1))$, and it suffices to prove that this group is annihilated by w_F. By construction, the module S is trivial as a G-module, and hence the action of G on $S(1)$ is via the character χ. Sah's

Lemma ([19, Lem.8.8.1]) implies that the self-map of $H^1(G, S(1))$ induced by $g-1$ for any $g \in Z(G) = G$ is the zero map. On the other hand, since $\chi : G \to (\mathbf{Z}/n\mathbf{Z})^\times$ is the cyclotomic character, the greatest common divisor of $\chi(g) - 1$ for $g \in G$ is $w_F \mathbf{Z}/n\mathbf{Z}$. In particular, the group is annihilated by w_F. The result follows. □

Remark 2.5. Suppose $(w_F, n) = 1$, and let $P \in H^\times$ be a representative of $P_\zeta(\xi) \in H^\times / H^{\times n}$. Then the construction of the element $R_\zeta(\xi)$ whose existence is asserted by Proposition 2.4 reduces to the problem of finding $S \in H^\times$ such that

(a) $P/S^n \in F_n^\times$, and
(b) the image of P/S^n in $F_n^\times / F_n^{\times n}$ lies in the χ^{-1}-eigenspace,

since then $R_\zeta(\xi) = P/S^n \in (F_n^\times / F_n^{\times n})^{\chi^{-1}}$. In practice, S will be constructed via a Hilbert 90 argument as an additive Galois average, and the difficulty is ensuring that $S \neq 0$. See Section 8, where this is done for a particular P constructed as a radial limit of a Nahm sum.

2.3. **Reduction to the case of prime powers.** In this section, we discuss the compatibility of the map R_ζ with the prime factorization of n. This will be important in Section 5, where we consider the relation of our map and the Chern class in K-theory.

Lemma 2.6. Let $(n, w_F) = 1$ and $\zeta = \zeta_n$ as usual. Then the following compatibilities hold:

(1) If $(n, k) = 1$, then $R_{\zeta^k}(X) = R_\zeta(X)^{k^{-1}}$.
(2) Let $n = qr$, and let $\zeta_r = \zeta_n^q$. Then the image of R_{ζ_n} modulo rth powers is equal to the image of $R_{\zeta_r}(X)$ under the map

$$\left(F_r^\times / F_r^{\times r}\right)^{\chi^{-1}} \to \left(F_n^\times / F_n^{\times r}\right)^{\chi^{-1}}$$

induced by the inclusion.

Proof. The first statement reflects the fact that $gR_\zeta = R_{g(\zeta)}$ for $g \in G = \mathrm{Gal}(F_n/F)$. For the second claim, we calculate

$$\begin{aligned} P_{\zeta_n}(X) &= \prod_{k \bmod n} \left(1 - \zeta_n^k x\right)^k = \prod_{\substack{i \bmod q \\ j \bmod r}} \left(1 - \zeta_n^{ri+j} x\right)^{ri+j} \\ &\equiv \prod_{\substack{i \bmod q \\ j \bmod r}} \left(1 - \zeta_q^i \zeta_n^j x\right)^j = \prod_{j \bmod r} \left(1 - \zeta_r^j x^q\right)^j = P_{\zeta_r}(X), \end{aligned}$$

where the congruence is modulo rth powers. □

Next, we discuss a reduction of the map P_{ζ_n} to the case that n is a prime power.

Lemma 2.7. Let $n = ab$ with $(a, b) = 1$ and ζ a primitive nth root of unity. If $X \in A(F; \mathbf{Z}/n\mathbf{Z})$, let $u_n = R_\zeta(X)$, $u_a = R_{\zeta^b}(X)$ and $u_b = R_{\zeta^a}(X)$. Then u_n determines and is uniquely determined by u_a and u_b.

Proof. Part (2) of Lemma 2.6 shows that the image of u_n in $F_n^\times / F_n^{\times a}$ is the image of u_a under the natural map

$$F_a^\times / F_a^{\times a} \to F_n^\times / F_n^{\times a}.$$

Equivalently, u_a determines u_n up to an ath power, and similarly u_b determines u_n up to a bth power. This is enough to determine u_n completely since a and b are coprime. The converse is already shown. $\square$

Remark 2.8. Both lemmas hold also for $(n, w_F) > 1$ if we replace R_ζ by $R_\zeta^{w_F}$.

2.4. **The 5-term relation.** In this section, we use a result of Kashaev, Mangazeev and Stroganov to show that the map R_ζ satisfies the 5-term relation, and consequently descends to a map of the group $B(F; \mathbf{Z}/n\mathbf{Z})$.

Theorem 2.9. *Let F be a field and $F_n = F(\zeta)$, where ζ is a root of unity of order n prime to w_F and to the characteristic prime of F. Then the map R_ζ vanishes on the subgroup $C(F) \subset A(F; \mathbf{Z}/n\mathbf{Z}) \subset Z(F)$ generated by the 5-term relation, and therefore induces a map*

$$B(F) \longrightarrow B(F)/nB(F) \longrightarrow B(F; \mathbf{Z}/n\mathbf{Z}) \xrightarrow{R_\zeta} \left(F_n^\times/F_n^{\times n}\right)^{\chi^{-1}}.$$

Proof. Denote by H the universal Kummer extension as before. Then it suffices to show that the appropriate product of the functions D_ζ is a perfect nth power in H.

Let X, Y, $Z \in F^\times$ be related by $Z = (1-X)/(1-Y)$, and choose nth roots x, y, z of X, Y, Z. Using the standard notation $(x;q)_k = (1-x)(1-qx)\cdots(1-q^{k-1}x)$ (q-Pochhammer symbol) and following the notation of [17] (except that they use $w(x|k)$ for $(x\zeta;\zeta)_k^{-1}$), we set

$$f(x,\, y\,|\,z) = \sum_{k=0}^{n-1} \frac{(\zeta y;\zeta)_k}{(\zeta x;\zeta)_k} = \sum_{k \bmod n} \frac{(\zeta y;\zeta)_k}{(\zeta x;\zeta)_k}\, z^k \;\in H,$$

where the second equality follows from the relation between x, y, and z. By equation (C.7) of [17], we have

$$(\zeta y)^{n(1-n)/2}\, f(x,\, y\,|\,z)^n \;=\; \frac{D_\zeta(1) D_\zeta(y\zeta/x) D_\zeta(x/yz)}{D_\zeta(1/x) D_\zeta(y\zeta) D_\zeta(\zeta/z)}.$$

Considering this modulo nth powers, and using Lemma 2.1, we find

$$1 \;=\; P_\zeta(X)\, P_\zeta(Y)^{-1}\, P_\zeta(Y/X)\, P_\zeta(YZ/X)^{-1}\, P_\zeta(Z)\,.$$

This is precisely the 5-term relation for the map P_ζ, and from the uniqueness clause in Proposition 2.4 implies the same 5-term relation for the map R_ζ. $\square$

2.5. **An eigenspace computation.** As in Section 1.1, we write $G = \mathrm{Gal}(F_n/F)$, identified with a subgroup of $(\mathbf{Z}/n\mathbf{Z})^\times$ via the map χ of Equation (4). Since $F_n^\times/F_n^{\times n}$ is an n-torsion G-module, the χ^{-1} eigenspace makes sense and is given by

$$\left(F_n^\times/F_n^{\times n}\right)^{\chi^{-1}} = \{x \in F_n^\times/F_n^{\times n}\,|\sigma x = x^{\chi(\sigma^{-1})}\,, \text{for all}\, \sigma \in G\}\,,$$

where $x^{\chi(\sigma^{-1})}$ is computed using any lift of $\chi(\sigma^{-1}) \in (\mathbf{Z}/n\mathbf{Z})^\times$ to $\mathbf{Z}$.

In characteristic zero, one can also consider the the action of G on $M \otimes_{\mathbf{Z}} R$, where R is a $\mathbf{Z}[G]$ module that contains the eigenvalues of $\sigma \in G$. For example, one can take $M = \mathcal{O}_n^\times$ and $R = \mathbf{C}$. If $n = p$ is prime, then one can take $R = \mathbf{Z}_p$, which contains the $(p-1)$th roots of unity. In particular, if $n = p$, then one can define $(M \otimes_{\mathbf{Z}} \mathbf{Z}_p)^{\chi^{-1}}$, which will have the property that

$$(M \otimes_{\mathbf{Z}} \mathbf{Z}_p)^{\chi^{-1}} \otimes \mathbf{Z}/p\mathbf{Z} = (M/pM)^{\chi^{-1}}\,.$$

Proposition 2.10. (a) Suppose that F is disjoint from $\mathbf{Q}(\zeta_n)$. Then there exists an isomorphism of G-modules

$$\left(\mathcal{O}_n^\times \otimes \mathbf{C}\right)^{\chi^{-1}} = \mathbf{C}^{r_2(F)}\,. \tag{18}$$

(b) If, furthermore, $n = p$ is prime, so that $\chi : G \to (\mathbf{Z}/p\mathbf{Z})^\times$ admits a natural Teichmüller lift to $\mathbf{Z}_p^\times$, then

$$\mathrm{rank}_{\mathbf{Z}_p}\left(\mathcal{O}_n^\times \otimes \mathbf{Z}_p\right)^{\chi^{-1}} = r_2(F)\,.$$

If in addition χ and χ^{-1} are distinct characters of G, then

$$\left(\mathcal{O}_p^\times/\mathcal{O}_p^{\times p}\right)^{\chi^{-1}} = (\mathbf{Z}/p\mathbf{Z})^{r_2(F)}\,.$$

Proof. Part (b) follows easily from part (a) and the above discussion, together with the fact that if $\chi \neq \chi^{-1}$ then the torsion in the unit group (which just comprises roots of unity) is in the χ-eigenspace and not the χ^{-1}-eigenspace.

For (a), let $\widetilde{F}$ be the Galois closure of F over $\mathbf{Q}$ and let $\Gamma = \mathrm{Gal}(\widetilde{F}/\mathbf{Q})$. By assumption, with $\widetilde{F}_n = \widetilde{F}(\zeta_n)$, we have $\mathrm{Gal}(\widetilde{F}_n/\mathbf{Q}) = \Gamma \times G = \Gamma \times (\mathbf{Z}/n\mathbf{Z})^\times$. From the proof of Dirichlet's unit theorem, the unit group of $\widetilde{F}_n$, tensored with $\mathbf{C}$, decomposes equivariantly as

$$\bigoplus_W W^{\dim(W|c=1)}\,,$$

where W runs over all the non-trivial irreducible representations of $\Gamma \times G$ and $c \in \Gamma$ is any complex conjugation, which we may take to be $(c, -1) \in G \times (\mathbf{Z}/n\mathbf{Z})^\times$ for a complex conjugation $c \in \Gamma$. The irreducible representations of W are of the form $U \otimes V$ for irreducible representations U of Γ and V of $G = (\mathbf{Z}/n\mathbf{Z})^\times$. Note that

$$\dim(U \otimes V|(c,-1) = 1) = \dim(U|c=1)\dim(V|c=1) + \dim(U|c=-1)\dim(V|c=-1)\,.$$

If we take the χ^{-1}-eigenspace under the action of the second factor, the only representation V of G which occurs is χ^{-1}, on which -1 acts by -1, and hence we are left with

$$\left(\mathcal{O}_{\widetilde{F}_n}^\times \otimes \mathbf{C}\right)^{\chi^{-1}} \;=\; \bigoplus_V V^{\dim(V|c=-1)}\,,$$

where the sum runs over all representations V of Γ. In particular, there is an isomorphism in the Grothendieck group of G-modules

$$\left[\,\left(\mathcal{O}_{\widetilde{F}_n}^\times \otimes \mathbf{C}\right)^{\chi^{-1}}\,\right] + \left[\mathcal{O}_{\widetilde{F}}^\times \otimes \mathbf{C}\right] + \left[\mathbf{C}\right] \;=\; \left[\mathbf{C}[G]\right].$$

Now take the $\Delta = \mathrm{Gal}(\widetilde{F}/F) = \mathrm{Gal}(\widetilde{F}_n/F_n)$-invariant part and take dimensions, we obtain the equality

$$\dim_{\mathbf{C}}\Big(\left(\mathcal{O}_{F_n}^\times \otimes \mathbf{C}\right)^{\chi^{-1}}\Big) + (r_1 + r_2 - 1) + 1 \;=\; r_1 + 2r_2\,,$$

where (r_1, r_2) is the signature of F. The result follows. $\square$

3. Chern Classes for algebraic K-theory

In this section, we will define the Chern class map (11).

3.1. **Definitions.** In the following discussion, it will be important to carefully distinguish canonical isomorphisms from mere isomorphisms. To this end, let $\simeq$ denote an isomorphism and $=$ a canonical isomorphism. Let F be a number field, and let $\mathcal{O} := \mathcal{O}_F$ denote the ring of integers of F. The Tate twist $\mathbf{Z}_p(m)$ is the free $\mathbf{Z}_p$ module on which the Galois group G_F acts via the mth power χ^m of the cyclotomic character. For all $m \geq 1$, there exists a Chern class map:

$$c : K_{2m-1}(F) \to H^1(F, \mathbf{Z}_p(m))\,.$$

These Chern class maps arise as the boundary map of a spectral sequence, specifically, the Atiyah–Hirzebruch spectral sequence for étale K-theory. These maps were originally constructed by Soulé [28], Section II. We may compose this map with reduction mod p^i to obtain a map:

$$c : K_{2m-1}(F) \to H^1(F, \mathbf{Z}/p^i\mathbf{Z}(m)).$$

By the Chinese remainder theorem, we may also piece these maps together to obtain a map:

$$c : K_{2m-1}(F) \to H^1(F, \mathbf{Z}/n\mathbf{Z}(m))$$

for any integer n. Let ζ be a primitive nth root of unity, $F_n = F(\zeta)$ and write G for the (possibly trivial) Galois group $\mathrm{Gal}(F_n/F)$. Let μ denote the module of n roots of unity. There is a canonical injection

$$\chi : G \to \mathrm{Aut}(\mu_n) = (\mathbf{Z}/n\mathbf{Z})^\times\,.$$

By inflation–restriction, there is a canonical map:

$$H^1(F, \mathbf{Z}/n\mathbf{Z}(m)) \to H^1(F_n, \mathbf{Z}/n\mathbf{Z}(m))^G = H^1(F_n, \mathbf{Z}/n\mathbf{Z}(1))^{\chi^{1-m}}. \tag{19}$$

For $i \geq 1$, there is an invariant $w_i(F) \in \mathbf{N}$ that we will need. It is defined in terms of Galois cohomology by

$$w_i(F) \;=\; \prod_p \big| H^0(F, \mathbf{Q}_p/\mathbf{Z}_p(m)) \big|\,,$$

Note that $w_1(F)$ is equal to w_F, the number of roots of unity in F, and $w_2(F)$ agrees with (10). We also define

$$\widetilde{w}_F \;=\; \prod_p \big| H^0(\widetilde{F}(\zeta_p + \zeta_p^{-1}), \mathbf{Q}_p/\mathbf{Z}_p(1)) \big|\,. \tag{20}$$

where $\widetilde{F}$ is the Galois closure of F over $\mathbf{Q}$. Thus $\widetilde{w}_F$ is divisible only by the finitely many primes p such ζ_p belongs to $\widetilde{F}(\zeta_p + \zeta_p^{-1})$. If $p|\widetilde{w}_F$ and $p > 2$, then p necessarily ramifies in F. Note that $\widetilde{w}_F$ is always divisible by w_F.

Lemma 3.1. The map (19) is injective for integers n prime to $w_i(F)$.

Proof. The kernel of this map is $H^1(F_n/F, \mathbf{Z}/n\mathbf{Z}(m))$. Assume that this is non-zero. By Sah's lemma, this group is annihilated by $\chi^i(g) - 1$ for any $g \in G$. Equivalently, the kernel has order divisible by $p|n$ if and only if the elements $a^i - 1$ are divisible by p for all $(a, p) = 1$. Yet this is equivalent to saying that $H^0(F, \mathbf{Z}/p\mathbf{Z}(m)) \subset H^0(F, \mathbf{Q}_p/\mathbf{Z}_p(m))$ is non-zero, and hence $p|w_i(F)$. ∎

There is an isomorphism $\mathbf{Z}/n\mathbf{Z}(1) = \mu$ coming from the choice of a given nth root of unity ζ. By Hilbert 90, for a number field L, there is a canonical isomorphism $H^1(L,\mu) = L^\times/L^{\times n}$, and hence c and ζ give rise to a map:

$$c_\zeta : K_{2m-1}(F) \to \left(F_n^\times/F_n^{\times n}\right)^{\chi^{1-i}} . \tag{21}$$

3.2. **The relation between étale cohomology and Galois cohomology.** There are isomorphisms that can be found in Sections 5.2 and 5.4 of [36]

$$K_{2m-1}(F) \otimes \mathbf{Z}_p \simeq K_{2m-1}(\mathcal{O}_F[1/p]) \simeq K_{2m-1}(\mathcal{O}_F) \otimes \mathbf{Z}_p$$

for $m > 1$. These isomorphisms are also reflected in the following isomorphism between étale cohomology groups and Galois cohomology groups:

$$H^1_{\text{ét}}(\mathcal{O}_F[1/p], \mathbf{Z}_p(m)) \simeq H^1(F, \mathbf{Z}_p(m))$$

for $i \geq 2$. In particular, we may also view the Chern class maps considered above as morphisms

$$c : K_{2m-1}(F) \otimes \mathbf{Z}_p \simeq K_{2m-1}(\mathcal{O}_F) \otimes \mathbf{Z}_p \to H^1_{\text{ét}}(\mathcal{O}_F[1/p], \mathbf{Z}_p(m)).$$

Theorem 3.2. *For $p > 2$, there is an isomorphism*

$$c : K_3(F) \otimes \mathbf{Z}_p \simeq K_{2m-1}(\mathcal{O}_F) \otimes \mathbf{Z}_p \to H^1(\mathcal{O}_F[1/p], \mathbf{Z}_p(2)).$$

The rank of $K_3(F)$ is r_2.

Sketch. This follows from the Quillen–Lichtenbaum conjecture, as proven by Voevodsky and Rost (see [36], [35]). In this case, it can also be deduced from the description of torsion in $K_3(F)$ by Merkur'ev and Suslin [21] (described in terms of $w_2(F)$ above) combined with Borel's theorem for the rank (see also Theorem 6.5 of [32]), and the result of Soulé that the Chern class map is surjective. $\square$

Lemma 3.3. Suppose that $p \nmid w_2(F)$. Then the map

$$c_\zeta : K_3(F) \to K_3(F)/nK_3(F) \to \left(F_n^\times/F_n^{\times n}\right)^{\chi^{-1}} \tag{22}$$

is injective.

Proof. By the Chinese Remainder Theorem, it suffices to consider the case $n = p^m$. In light of Theorem 3.2, it suffices to show that the map

$$H^1(F, \mathbf{Z}_p(2))/n \to H^1(F, \mathbf{Z}/n\mathbf{Z}(2)) \to F_n^\times/F_n^{\times n}$$

is injective. The kernel of the first map is $H^0(F, \mathbf{Z}_p(2))/n = 0$. The kernel of the second map is, via inflation–restriction, the group $H^1(\mathrm{Gal}(F_n/F), H^0(F, \mathbf{Z}/n\mathbf{Z}(2)))$. This group is certainly zero unless

$$H^0(F, \mathbf{Z}/n\mathbf{Z}(2)) \subset H^0(F, \mathbf{Q}_p/\mathbf{Z}_p(2))$$

is non-zero, or in other words, unless p divides $w_2(F)$. $\square$

3.3. **Upgrading from $F_n^\times$ to $\mathcal{O}_{F_n}[1/S]^\times$.** The following is a consequence of the finite generation of $K_3(F)$:

Lemma 3.4. For any field F, there exists a finite set S of primes which avoids any given finite set of primes not dividing n such that the image of c_ζ on $K_3(F)/nK_3(F)$ may be realized by an element of $\mathcal{O}_{F(\zeta)}^\times[1/S]$.

Proof. Note that without the requirement that S avoids any given finite set of primes not dividing n, the result is a trivial consequence of the fact that $K_3(F)$ is finitely generated. The construction of c as a map to units in $F_n^\times$ proceeded via Hilbert 90. In light of Theorem 3.2 above, it suffices to do the same with $H^1(F_n,\mu)$ replaced by $H^1_{\text{ét}}(\mathcal{O}_{F_n}[1/S],\mu)$ for some set S containing $p|n$. However, in this case, the class group intervenes, as there is an exact sequence ([22], p.125):

$$\mathcal{O}_{F_n}[1/S]^\times/\mathcal{O}_{F_n}[1/S]^{\times n} \to H^1_{\text{ét}}(\mathcal{O}_{F_n}[1/S],\mu) \to \operatorname{Pic}(\mathcal{O}_{F_n}[1/S])[n]$$

where $M[n]$ denotes the n-torsion of M and Pic is the Picard group, which may be identified with the class group of $\mathcal{O}_{F_n}[1/S]$. On the other hand, it is well known that one can represent generators in the class group by a set of primes avoiding any given finite set of primes, and hence for a set S including primes for each generator of the class group, the last term vanishes. $\square$

3.4. **Upgrading from S-units to units.** We give the following slight improvement on Lemma 3.4.

Lemma 3.5. Suppose that any prime divisor p of n is odd and divides neither the discriminant of F nor the order of $K_2(\mathcal{O}_F)$. Then the image of c_ζ on $K_3(F)/nK_3(F)$ may be realized by an element of $\mathcal{O}_n^\times$.

Proof. By Lemma 2.7, it suffices to consider the case when $n=n$ is a power of p. Let $\zeta=\zeta_n$. The fact that p is prime to the discriminant of F implies that $F(\zeta)/F$ is totally ramified at p. The image of c_ζ factors through $H^1_{\text{ét}}(\mathcal{O}[1/p],\mathbf{Z}/n\mathbf{Z}(2))$, and, via inflation–restriction, through $H^1_{\text{ét}}(\mathcal{O}_{F(\zeta)}[1/p],\mathbf{Z}/n\mathbf{Z}(1))$. The Kummer sequence for étale cohomology gives a short exact sequence:

$$\mathcal{O}_{F(\zeta)}[1/p]^\times/\mathcal{O}_{F(\zeta)}[1/p]^{\times n} \to H^1_{\text{ét}}(\mathcal{O}_{F(\zeta)}[1/p],\mathbf{Z}/n(1)) \to \operatorname{Pic}(\mathcal{O}_F(\zeta)[1/p])[n]\,.$$

The image of c_ζ lands in the χ^{-1}-invariant part of the second group. The χ^{-1}-invariant part $M^{\chi^{-1}}$ of a G-module M is non-zero if and only if the largest χ^{-1}-invariant quotient $M_{\chi^{-1}}$ is non-zero. However, by results of Keune [18], there is an injection

$$\left(\operatorname{Pic}(\mathcal{O}_F(\zeta)[1/p])/p^m\right)_{\chi^{-1}} \to K_2(\mathcal{O}_F)/p^m\,.$$

In particular, the pushforward of the image of c_ζ to the Picard group is trivial whenever $K_2(\mathcal{O}_F)\otimes\mathbf{Z}_p$ is trivial. Since we are assuming that p does not divide the order of $K_2(\mathcal{O}_F)$, we deduce that the image of c_ζ is realized by p-units. We now upgrade this to actual units. There is an exact sequence:

$$(\mathcal{O}_{F(\zeta)})^\times/(\mathcal{O}_{F(\zeta)})^{\times n} \to (\mathcal{O}_{F(\zeta)}[1/p])^\times/(\mathcal{O}_{F(\zeta)}[1/p])^{\times n} \to \bigoplus_{v|p}\mathbf{Z}/n\mathbf{Z}\,,$$

where the last map is the valuation map. Since p is totally ramified in $F(\zeta)/F$, the action of G on the final term is trivial. By assumption, the quotient $\mathrm{Gal}(F(\zeta_p)/F)$ is non-trivial, and hence the χ^{-1}-invariants of the final term are zero. Hence, after taking χ^{-1}-invariants, we see that the image of c_ζ comes from a unit. □

3.5. **Proof of Theorem 1.5.** We have all the ingredients to give a proof of Theorem 1.5. Fix a natural number n and a primitive nth root of unity ζ. Consider the Chern class map

$$c_\zeta : K_3(F)/nK_3(F) \to \left(F_n^\times/F_n^{\times n}\right)^{\chi^{-1}} .$$

from (22). When n is coprime to $w_2(F)$, the above map is injective by Lemma 3.3. When n is coprime to the discriminant Δ_F of F and the order of $K_2(\mathcal{O}_F)$, Lemma 3.5 implies the above map factors through a map

$$c_\zeta : K_3(F)/nK_3(F) \to \left(\mathcal{O}_n^\times/\mathcal{O}_n^{\times n}\right)^{\chi^{-1}} ,$$

where $\mathcal{O}_n$ is the ring of integers of F_n. When n is square-free and coprime to $w_2(F)$, then (9) and Proposition 2.10 imply that both sides of the above equation are abelian groups isomorphic to $(\mathbf{Z}/n\mathbf{Z})^{r_2(F)}$. It follows that when n is square-free and coprime to $w_2(F)\,\Delta_F|K_2(\mathcal{O}_F)|$, then the above map is an injection of finite abelian groups of the same order, and hence an isomorphism. This concludes the proof of Theorem 1.5. □

4. Reduction to finite fields

As we will see in Section 5, the comparison of the maps c_ζ and R_ζ and the proof of Theorem 1.6 require a reduction of both maps to the case of finite fields. In this section, we review the local Chern classes and the Bloch groups of finite fields, and introduce local (finite field) versions of the maps c_ζ and R_ζ. We will be considering the case that n is a prime power p^m, and will denote by ζ a primitive nth root of unity.

4.1. **Local Chern class maps.** Let $\mathfrak{q}$ be a prime of norm $q \equiv -1 \bmod n$ in $\mathcal{O}_F$. The residue field of $\mathcal{O}_F$ at $\mathfrak{q}$ is $\mathbf{F}_q$, and the residue field of $\mathcal{O}_{F(\zeta)}$ at a prime $\mathfrak{Q}$ above $\mathfrak{q}$ is $\mathbf{F}_{q^2} = \mathbf{F}_q(\zeta)$. Following Lemma 3.4, suppose that S does not contain any primes dividing q.

Lemma 4.1. There exists a commutative diagram of Chern class maps as follows:

$$\begin{tikzcd} K_3(F)/nK_3(F) \arrow[r, "c_\zeta"] \arrow[d] & (\mathcal{O}[1/S]^\times_{F(\zeta)}/\mathcal{O}[1/S]^{\times n}_{F(\zeta)})^{\chi^{-1}} \arrow[d] \\ K_3(\mathbf{F}_q)/nK_3(\mathbf{F}_q) \arrow[r, equal, "c_{\zeta,\mathfrak{q}}"] & \mathbf{F}_{q^2}^\times/\mathbf{F}_{q^2}^{\times n}. \end{tikzcd}$$

Proof. By the Chinese Reminder Theorem, we may reduce to the case when $n = p^m$. There is an isomorphism $K_3(F)\otimes\mathbf{Z}_p \simeq K_3(\mathcal{O}_F)\otimes\mathbf{Z}_p$ (see Theorem 3.2). Let $\mathcal{O}_{F,\mathfrak{q}}$ be the completion of $\mathcal{O}_F$ at $\mathfrak{q}$. We have a more general diagram as follows:

$$\begin{tikzcd}
K_3(\mathcal{O}_F)/nK_3(\mathcal{O}_F) \arrow[r, "c_\zeta"] \arrow[d] & H^1_{\text{ét}}(\mathcal{O}_F[1/p], \mathbf{Z}/n\mathbf{Z}(2)) \arrow[r] \arrow[d] & (\mathcal{O}[1/S]^\times_{F(\zeta)}/\mathcal{O}[1/S]^{\times n}_{F(\zeta)})^{\chi^{-1}} \arrow[d] \\
K_3(\mathcal{O}_{F,\mathfrak{q}}; \mathbf{Z}_p)/nK_3(\mathcal{O}_{F,\mathfrak{q}}; \mathbf{Z}_p) \arrow[r, equal, "c_{\zeta,\mathfrak{q}}"] \arrow[d, equal] & H^1_{\text{ur}}(\mathcal{O}_{F,\mathfrak{q}}, \mathbf{Z}/n\mathbf{Z}(2)) \arrow[r, equal] \arrow[d, equal] & \mathbf{F}^\times_{q^2}/\mathbf{F}^{\times n}_{q^2} \arrow[d, equal] \\
K_3(\mathbf{F}_q)/nK_3(\mathbf{F}_q) \arrow[r, equal, "c_{\zeta,\mathfrak{q}}"] & H^1(\mathbf{F}_q, \mathbf{Z}/n\mathbf{Z}(2)) \arrow[r, equal] & \mathbf{F}^\times_{q^2}/\mathbf{F}^{\times n}_{q^2}.
\end{tikzcd}$$

The image of $H^1_{\text{ét}}(\mathcal{O}_F[1/p])$ in the cohomology of $\mathcal{O}_{F,\mathfrak{q}}$ for $\mathfrak{q}$ prime to p lands in the subgroup H^1_{ur} of unramified classes. This subgroup is precisely the image of $H^1(\mathcal{O}/\mathfrak{q}, \mathbf{Z}/n\mathbf{Z}(2))$ under inflation. The maps on the right hand side of the diagram are just what one gets when unwinding the application of Hilbert's Theorem 90. The identification of the two lower horizontal lines is a reflection of Gabber rigidity, which implies that $K_3(\mathcal{O}_{F,\mathfrak{q}}; \mathbf{Z}_p) \simeq K_3(\mathbf{F}_\mathfrak{q}) \otimes \mathbf{Z}_p$. □

Proposition 4.2. Let $\widetilde{F}$ denote the Galois closure of F, and suppose that $\zeta \notin \widetilde{F}(\zeta+\zeta^{-1})$. (Equivalently, suppose that n is prime to $\widetilde{w}_F$ of equation 20.)
(a) There is a map:

$$K_3(F)/nK_3(F) \xrightarrow{\bigoplus c_{\zeta,\mathfrak{q}}} \bigoplus \mathbf{F}^\times_{q^2}/\mathbf{F}^{\times n}_{q^2}$$

where the sum ranges over all primes $\mathfrak{q}$ of prime norm $q \equiv -1 \bmod n$ which split completely in F, or alternatively runs over all but finitely many primes $q \equiv -1 \bmod n$ which split completely in F.
(b) The image of this map is isomorphic to the image of the global map c_ζ, which is injective if $(n, w_2(F)) = 1$.
(c) For $\xi \in K_3(F)$, the set

$$\{\mathfrak{q} \subset \mathcal{O}_F[1/S] \mid c_{\zeta,\mathfrak{q}}(\xi) = 0\}$$

(for any finite S) determines the image of ξ up to a scalar.

Proof. It suffices to consider the case when $n = p^m$. Let $\xi \in K_3(F)$, and let the class of $c_\zeta(\xi)$ be represented by an S-unit ϵ. Because of the Galois action, this gives rise via Kummer theory to a $\mathbf{Z}/n\mathbf{Z}$-extension H of $F(\zeta+\zeta^{-1})$, and such that the reduction mod $\mathfrak{q}$ of ϵ determines the element $\mathrm{Frob}_\mathfrak{q} \in \mathrm{Gal}(H/F(\zeta+\zeta^{-1}))$. (Explicitly, we have $H(\zeta) = F(\zeta, \epsilon^{1/n})$.) Hence our assumptions imply that any prime q which splits completely in $F(\zeta+\zeta^{-1})$ (which forces $q \equiv \pm 1 \bmod n$) and is additionally congruent to $-1 \bmod p$ must split in H. Let $\widetilde{H}$ denote the Galois closure of H over $\mathbf{Q}$, and $\widetilde{F}$ the Galois closure of F over $\mathbf{Q}$. Note that the Galois closure of $F(\zeta+\zeta^{-1})$ is $\widetilde{F}(\zeta+\zeta^{-1})$. A prime q splits completely in H if and only if it splits completely in $\widetilde{H}$, and splits completely in $F(\zeta+\zeta^{-1})$ if and only if it splits completely in $\widetilde{F}(\zeta+\zeta^{-1})$. We have a diagram of fields as follows:

$$\begin{tikzcd} \widetilde{H} \arrow[r, no head] \arrow[d, no head] & \widetilde{H}(\zeta) \arrow[d, no head] \\ \widetilde{F}(\zeta+\zeta^{-1}) \arrow[r, no head] & \widetilde{F}(\zeta) \end{tikzcd}$$

By assumption, we have $\zeta \notin \widetilde{F}(\zeta+\zeta^{-1})$. Since $H/F(\zeta+\zeta^{-1})$ is cyclic of degree n, it follows that $\mathrm{Gal}(\widetilde{H}/\widetilde{F}(\zeta+\zeta^{-1}))$ is an abelian p-group. On the other hand, $\mathrm{Gal}(\widetilde{F}(\zeta)/\widetilde{F}(\zeta+\zeta^{-1})) = \mathbf{Z}/2\mathbf{Z}$. so $\mathrm{Gal}(\widetilde{H}(\zeta)/\widetilde{F}(\zeta+\zeta^{-1}))$ is the direct sum of $\mathbf{Z}/2\mathbf{Z}$ with a p-group, Let $\sigma \in \mathrm{Gal}(\widetilde{H}(\zeta)/\widetilde{F}(\zeta+\zeta^{-1})) \subset \mathrm{Gal}(\widetilde{H}/\mathbf{Q})$ denote an element of order $2p$. By the Cebotarev density theorem, there exist infinitely many primes $q \in \mathbf{Q}$ with Frobenius element in $\mathrm{Gal}(\widetilde{H}/\mathbf{Q})$ corresponding to σ. By construction, the prime q splits completely in $\widetilde{F}(\zeta+\zeta^{-1})$ because the corresponding Frobenius element is trivial in $\mathrm{Gal}(\widetilde{F}(\zeta+\zeta^{-1})/\mathbf{Q})$. On the other hand, since σ has order divisible by 2 and by p, it is non-trivial in both $\mathrm{Gal}(\widetilde{F}(\zeta)/\widetilde{F}(\zeta+\zeta^{-1})) = \mathrm{Gal}(\mathbf{Q}(\zeta)/\mathbf{Q}(\zeta+\zeta^{-1}))$ and $\mathrm{Gal}(\widetilde{H}/\widetilde{F}(\zeta+\zeta^{-1}))$. The first condition implies that $q \equiv -1 \bmod n$, and the second condition implies that q does not split completely in H, a contradiction. The injectivity (under the stated hypothesis) follows from Lemma 3.1. ☐

Remark 4.3. The condition that $\zeta \notin \widetilde{F}(\zeta+\zeta^{-1})$ is automatic if p is unramified in F, because then the ramification degree of $\mathbf{Q}(\zeta)$ is $p-1$ whereas the ramification degree of $\widetilde{F}(\zeta+\zeta^{-1})$ is $(p-1)/2$ for p odd. If $\zeta \in \widetilde{F}(\zeta+\zeta^{-1})$, then there are no primes q which split completely in F and have norm $-1 \bmod n$. In particular, when $\zeta \in \widetilde{F}(\zeta+\zeta^{-1})$, we have $B(\mathbf{F}_q)\otimes\mathbf{F}_p = 0$ for every prime q which splits completely in F.

4.2. **The Bloch group of $\mathbf{F}_q$.** In order to make our maps explicit, we must relate the Chern class map to the Bloch group. Let $p > 2$ and $q > 2$ be odd primes such that $q \equiv -1 \bmod n$, where $n = p^m$. For a finite field $\mathbf{F}_q$, the group $\mathbf{F}_q^\times$ is cyclic, so $\bigwedge^2 \mathbf{F}_q^\times$ is a 2-torsion group. Hence the Bloch group $B(\mathbf{F}_q)$ coincides with the pre-Bloch group after tensoring with $\mathbf{F}_p$, where the pre-Bloch group is defined as the quotient of the free abelian group on $\mathbf{F}_q \smallsetminus \{0,1\}$ by the 5-term relation. By [16], the Bloch group $B(\mathbf{F}_q)$ is a cyclic group of order $q+1$ up to 2-torsion. Moreover, following [16], one may relate $B(\mathbf{F}_q)$ to the cohomology of $\mathrm{SL}_2(\mathbf{F}_q)$ in degree three, as we now discuss.

There is an isomorphism

$$H_3(\mathrm{SL}_2(\mathbf{F}_q),\mathbf{Z})\otimes\mathbf{Z}/n\mathbf{Z} \;\simeq\; \mathbf{Z}/n\mathbf{Z}.$$

Let us describe this isomorphism more carefully. By a computation of Quillen, we know that $H^3(\mathrm{SL}_2(\mathbf{F}_q),\mathbf{Z})$ is cyclic of order q^2-1. It follows that the p-part of this group comes from the p-Sylow subgroup. If one chooses an isomorphism

$$\mathbf{F}_{q^2} \simeq (\mathbf{F}_q)^2$$

of abelian groups, one gets a well defined map:

$$\mathbf{F}_{q^2}^\times \;=\; C \;=\; \mathrm{Aut}_{\mathbf{F}_q}(\mathbf{F}_{q^2}) \to \mathrm{GL}_2(\mathbf{F}_q)$$

which is well defined up to conjugation. There is, correspondingly, a map $C^1 \to \mathrm{SL}_2(\mathbf{F}_q)$, where

$$C^1 \;=\; \mathrm{Ker}\big(N : \mathbf{F}_{q^2}^{\times} \to \mathbf{F}_q^{\times}\big)\,.$$

We refer to both C and C^1 as the non-split Cartan subgroup. By Quillen's computation, we deduce that there is a canonical map:

$$C^1 \;=\; H^3(C^1, \mathbf{Z}) \to H^3(\mathrm{SL}_2(\mathbf{F}_q), \mathbf{Z})$$

which is an isomorphism after tensoring with $\mathbf{Z}/n\mathbf{Z}$. There is a canonical isomorphism $C^1[n] = \mu$, where μ denotes the nth roots of unity. Hence to give an element of order p in $H^3(\mathrm{SL}_2(\mathbf{F}_q), \mathbf{Z})$ up to conjugation is equivalent to giving a primitive nth root of unity $\zeta \in C^1 \subset C = \mathbf{F}_{q^2}^{\times}$. From [16], there is a canonical map:

$$H^3(\mathrm{SL}_2(\mathbf{F}_q), \mathbf{Z}) \to B(\mathbf{F}_q)\,,$$

at least away from 2-power torsion, which is an isomorphism after tensoring with $\mathbf{Z}/n\mathbf{Z}$. Given a root of unity ζ, let t denote the corresponding element of $\mathrm{SL}_2(\mathbf{F}_q)$. The corresponding element of $B(\mathbf{F}_q)$, up to six-torsion, is given (see [16], p.36) by:

$$\sum_{k=1}^{n-1} \left[\frac{t(\infty) - t^{k+1}(\infty)}{t(\infty) - t^{k+2}(\infty)}\right]\,.$$

This construction yields the same element for ζ and ζ^{-1}. We may represent t by its conjugacy class in $\mathrm{GL}_2(\mathbf{F}_q)$, which has determinant one and trace $\zeta + \zeta^{-1} \in \mathbf{F}_q$. The choice of ζ up to (multiplicative) sign is given by this trace. Note that the congruence condition on q ensures that the Chebyshev polynomial with roots $\zeta + \zeta^{-1}$ has distinct roots which split completely over $\mathbf{F}_q$. Explicitly, we may choose

$$t \;=\; \begin{pmatrix} 0 & 1 \\ -1 & \zeta + \zeta^{-1} \end{pmatrix} = A \begin{pmatrix} \zeta^{-1} & 0 \\ 0 & \zeta \end{pmatrix} A^{-1}, \qquad A \;=\; \begin{pmatrix} \zeta & \zeta^{-1} \\ 1 & 1 \end{pmatrix}.$$

Let F_k be the Chebyshev polynomials, so $F_k(2\cos\phi) = \dfrac{\sin k\phi}{\sin\phi}$. Then

$$t^k(\infty) \;=\; \frac{F_{k-1}(\zeta + \zeta^{-1})}{F_k(\zeta + \zeta^{-1})}\,,$$

and an elementary computation then shows that the corresponding element in $B(\mathbf{F}_q) \otimes \mathbf{Z}_p$ is given by

$$\sum_{k=1}^{n-1} \left[1 - \frac{1}{F_k(\zeta + \zeta^{-1})^2}\right] \;\sim\; \sum_{k=1}^{n-1} \left[\left(\frac{\zeta^k - \zeta^{-k}}{\zeta - \zeta^{-1}}\right)^2\right]\,,$$

where $\sim$ denotes equality in $B(\mathbf{F}_q) \otimes \mathbf{Z}_p$, since $[x] = [1 - 1/x]$ up to 3-torsion. (When $p = 3$, one may verify directly that the latter term is also a 3-torsion element.)

4.3. **The local Chern class map c_ζ.** In this section, q will denote a prime with $q \equiv -1 \bmod p^m$ which splits completely in F. Let $\mathfrak{q}$ be a prime above q. There is a natural map $B(F) \to B(\mathcal{O}_F/\mathfrak{q}) = B(\mathbf{F}_q)$. The elements $[0]$, $[1]$, and $[\infty]$ are trivial elements of $B(F)$ and $B(\mathbf{F}_q)$; the reduction map then sends $[x]$ to $[\overline{x}]$ under the natural reduction map $\mathbf{P}^1(F) \to \mathbf{P}^1(\mathbf{F}_q)$.

Lemma 4.4. Let $p > 2$. There is a commutative diagram as follows:

$$\begin{tikzcd} B(F)/nB(F) \arrow[r] \arrow[d, equal] & \bigoplus B(\mathbf{F}_q) \otimes \mathbf{Z}/n\mathbf{Z} \arrow[d, dash, "\simeq"] & \\ K_3(F)/nK_3(F) \arrow[r] & \bigoplus K_3(\mathbf{F}_q) \otimes \mathbf{Z}/n\mathbf{Z} \arrow[r, equal, "\bigoplus c_{\zeta,\mathfrak{q}}"] & \bigoplus \mathbf{F}_{q^2}^\times / \mathbf{F}_{q^2}^{\times n}, \end{tikzcd}$$

where the product runs over all primes $\mathfrak{q}$ of norm $q \equiv -1 \bmod n$ which split completely in F, or alternatively all but finitely many such primes.

Proof. The isomorphism of the left vertical map is a theorem of Suslin [31], and the isomorphism of the right vertical map follows from [16]. The fact that the diagram commutes is a consequence of the fact that both constructions are compatible (and can be seen in group cohomology). ☐

Recall that an element x of an abelian group G is p-saturated if $x \notin [p]G$, where $[p] : G \to G$ is the multiplication by p map.

Corollary 4.5. There is an algorithm to prove that a set of generators of $B(F)$ is p-saturated for $p > 2$.

Proof. Computing $B(\mathbf{F}_q)$ is clearly algorithmically possible. Moreover, we can *a priori* compute $B(F) \otimes \mathbf{Z}_p$ as an abstract $\mathbf{Z}_p$-module. Hence it suffices to find sufficiently many distinct primes $\mathfrak{q}$ such that the image of a given set of generators has the same order as $B(F)/nB(F)$. ☐

In light of the commutative diagram of Lemma 4.4, we also use c_ζ to denote the Chern class map on $B(F)/nB(F)$.

4.4. **The local R_ζ map.** Suppose that $q \equiv -1 \bmod p$. It follows that the field $\mathbf{F}_q$ does not contain ζ_p, and so Proposition 2.4 applies to give maps P_ζ and R_ζ which are well defined over this field. In particular, since $(p, q-1) = 1$, all elements of $\mathbf{F}_q$ are p-th powers, and hence the Kummer extension H is given by $H = F_n$ and R_ζ and P_ζ coincide.

5. Comparison between the maps c_ζ and R_ζ

The main goal of this section, carried out in the first subsection, is to prove Theorem 1.6. The main result here is Theorem 5.2, which says that that our mod n local regulator map $R_{\zeta,q}$ gives an isomorphism from $B(\mathbf{F}_q) \otimes \mathbf{Z}/n\mathbf{Z}$ to $\mathbf{F}_{q^2}^\times \otimes \mathbf{Z}/n\mathbf{Z}$ for any prime power n and prime $q \equiv -1 \pmod n$. This implies in particular the existence of a curious "mod-p-q dilogarithm map" from $\mathbf{F}_q$ to $\mathbf{Z}/n\mathbf{Z}$, and in Section 5.2, we digress briefly to give an explicit formula for this map. In the final subsection, we describe the expected properties of the Chern class map that would imply the conjectural equality (13) and hence, in conjunction with (12), the evaluation $\gamma = 2$ of the comparison constant γ occurring in Theorem 1.6.

5.1. **Proof of Theorem 1.6.** Throughout this section, we set $n = p^m$, and let ζ denotes a primitive nth root of unity. For a prime $q \equiv -1 \bmod n$ that splits completely in F, and for a corresponding prime $\mathfrak{q}$ above q, let $R_{\zeta,\mathfrak{q}}$ denote the map $B(\mathcal{O}_F/\mathfrak{q}) = B(\mathbf{F}_q) \to \mathbf{F}^{\times}_{q^2}/\mathbf{F}^{\times n}_{q^2}$.

We have two maps we wish to compare. One of them is

$$c_\zeta : \; B(F)/nB(F) \;\to\; \left(F_n^{\times}/F_n^{\times n}\right)^{\chi^{-1}}.$$

Because $B(F)$ is a finitely generated abelian group, we may represent the generators of the image by S-units for some fixed S (at this point possibly depending on n) and consider the map

$$c_\zeta : \; B(F)/nB(F) \to (\mathcal{O}_{F(\zeta)}[1/S]^{\times}/\mathcal{O}_{F(\zeta)}[1/S]^{\times n})^{\chi^{-1}} \hookrightarrow \bigoplus \mathbf{F}^{\times}_{q^2}/\mathbf{F}^{\times n}_{q^2} \simeq \bigoplus B(\mathbf{F}_q),$$

where the final sum is over all but finitely may primes $\mathfrak{q}$ of norm $q \equiv -1 \bmod n$ which split completely in F. We have the diagram

$$\begin{tikzcd} B(F)/nB(F) \arrow[r, "R_\zeta"] \arrow[d] & (\mathcal{O}_{F(\zeta)}[1/S]^{\times}/\mathcal{O}_{F(\zeta)}[1/S]^{\times n})^{\chi^{-1}} \arrow[d] \\ \bigoplus B(\mathbf{F}_q) \otimes \mathbf{Z}/n\mathbf{Z} \arrow[r, "R_{\zeta,\mathfrak{q}}"] & \bigoplus \mathbf{F}^{\times}_{q^2}/\mathbf{F}^{\times n}_{q^2}. \end{tikzcd}$$

We have already shown, by Cebotarev (Proposition 4.2(b)), that $c_\zeta(\xi)$ for $\xi \in K_3(F)$ is determined up to scalar by the set of primes for which $c_{\zeta,\mathfrak{q}}(\xi) = 0$. Hence the result is a formal consequence of knowing that the maps $R_{\zeta,\mathfrak{q}}$ are isomorphisms for all $\mathfrak{q}$ of norm $q \equiv -1 \bmod n$. This is exactly Theorem 5.2 below. □

By (9), the p-torsion subgroup of $K_3(\mathbf{Q}(\zeta+\zeta^{-1}))$ is isomorphic to $\mathbf{Z}/n\mathbf{Z}$. On the other hand, since $\mathbf{Q}(\zeta+\zeta^{-1})$ is totally real, we have an isomorphism:

$$K_3(\mathbf{Q}(\zeta+\zeta^{-1})) \otimes \mathbf{Z}_p \simeq \mathbf{Z}/n\mathbf{Z}.$$

Lemma 5.1. Let $p > 2$ and $n = p^m$. Suppose that $q \equiv -1 \bmod n$ and $q \not\equiv -1 \bmod pn$. The prime q splits completely in $\mathbf{Q}(\zeta+\zeta^{-1})$. Let $\mathbf{F}_q$ denote the residue field at one of the primes above q. Then the map

$$K_3(\mathbf{Q}(\zeta+\zeta^{-1})) \otimes \mathbf{Z}_p \to B(\mathbf{F}_q) \otimes \mathbf{Z}_p$$

is an isomorphism.

Proof. A generator of $B(\mathbf{Q}(\zeta+\zeta^{-1}))[n] \simeq K_3(\mathbf{Q}(\zeta+\zeta^{-1})) \otimes \mathbf{Z}_p$ is given explicitly by the element

$$\eta_\zeta \;:=\; \sum_{\ell=1}^{n-1} \left[\left(\frac{\zeta^\ell - \zeta^{-\ell}}{\zeta - \zeta^{-1}} \right)^2 \right] \tag{23}$$

This follows from Theorem 1.3 of [40]. On the other hand, the reduction modulo any prime above q generates the latter group, as follows from the discussion in Section 4.2. $\square$

We now prove Theorem 5.2 as mentioned above:

Theorem 5.2. *Let n be a prime power and $q \equiv -1 \bmod n$. Then the map*

$$R_{\zeta,q} : B(\mathbf{F}_q) \otimes \mathbf{Z}/n\mathbf{Z} \to \mathbf{F}_{q^2}^{\times} \otimes \mathbf{Z}/n\mathbf{Z}$$

is an isomorphism, where ζ is an nth root of unity.

Proof. Note that $B(\mathbf{F}_q)$ is cyclic of order $q+1$ up to 2-torsion, and $\mathbf{F}_{q^2}^{\times}$ is cyclic of order q^2-1. In particular, for odd primes p with $q \equiv -1 \bmod p$, the groups $B(\mathbf{F}_q) \otimes \mathbf{Z}_p$ and $\mathbf{F}_{q^2}^{\times} \otimes \mathbf{Z}_p$ are isomorphic to each other and to $\mathbf{Z}_p/(q+1)\mathbf{Z}_p$. We begin with the following:

Lemma 5.3. $R_\zeta(\eta_\zeta) = \zeta^\gamma \in (\mathbf{Q}(\zeta)^\times/\mathbf{Q}(\zeta)^{\times n})^{\chi^{-1}}$ for some $\gamma \in \mathbf{Z}_p$.

Proof. Write $\zeta_n = \zeta$ and let ζ' be an n^2th root of unity. Consider the image of $R_{\zeta'}(\eta_{\zeta'})$. Because η_ζ is divisible by n in $B(\mathbf{Q}(\zeta')^+)$, the image is a nth power. Hence, by the compatibility of the maps R for varying n (Lemma 2.6 (2)), it follows that $R_\zeta(\eta_\zeta)$ lies in the kernel of the map

$$\big(\mathbf{Q}(\zeta)^\times/\mathbf{Q}(\zeta)^{\times n}\big)^{\chi^{-1}} \to \big(\mathbf{Q}(\zeta')^\times/\mathbf{Q}(\zeta')^{\times n}\big)^{\chi^{-1}}.$$

But this kernel consists precisely of nth roots of unity. $\square$

Let $\eta_{\zeta,q} \in B(\mathbf{F}_q)$ denote the reduction of η_ζ in $B(\mathbf{F}_q)$. By Lemma 5.1, the image also generates $B(\mathbf{F}_q) \otimes \mathbf{Z}/n\mathbf{Z}$. Since all primes $q \equiv -1 \bmod n$ split completely in $\mathbf{Q}(\zeta)^+$, if $\gamma \not\equiv 0 \bmod p$, the result above follows by specialization. We proceed by contradiction and assume that $\gamma \equiv 0 \bmod p$, which means that the image of the map $P_{\zeta,\mathfrak{q}}$ is divisible by p for all $\mathfrak{q}$ of norm q satisfying $q \equiv -1 \bmod n$. In particular, to prove the result, it suffices to find a *single* such $\mathfrak{q}$ for which $R_{\zeta,\mathfrak{q}}$ is an isomorphism.

Choose a completely split prime $\mathfrak{r}$ in $\mathbf{Q}(\zeta)$. Assume that

$$\zeta \equiv a^{-1} \bmod \mathfrak{r}, \qquad \zeta \not\equiv a^{-1} \bmod \mathfrak{r}^2$$

for some integer $a \neq 1$. The splitting assumption means that an a satisfying the first condition exists, replacing a^{-1} by $(a+N(\mathfrak{r}))^{-1}$ if necessary implies the second, because

$$\frac{1}{a} - \frac{1}{a+N(\mathfrak{r})} = \frac{N(\mathfrak{r})}{a(a+N(\mathfrak{r}))} \not\equiv 1 \bmod \mathfrak{r}^2 .$$

Let

$$\tau = \prod_{k=0}^{n-1}(1-\zeta^k a)^k \in \mathbf{Q}(\zeta)^\times .$$

Lemma 5.4. $\tau \cdot \zeta^i$ is not a perfect pth power for any i.

Proof. The assumption on $\mathfrak{r}$ implies that all the pth roots of unity are distinct modulo $\mathfrak{r}$, and hence the only factor of τ divisible by $\mathfrak{r}$ is $(1-a\zeta)$, which has valuation one. $\square$

The element τ gives rise, via Kummer theory, to a $\mathbf{Z}/n\mathbf{Z}$-extension $F/\mathbf{Q}(\zeta)^+$. By the Lemma above, it is non-trivial. Let $q \equiv -1 \bmod n$ be prime. Then, for a prime $\mathfrak{q}$ above q, the element $\mathrm{Frob}_\mathfrak{q} \in \mathrm{Gal}(F/\mathbf{Q}(\zeta)^+)$ fails to generate $\mathbf{Z}/n\mathbf{Z}$ if and only if τ is a perfect pth power modulo $\mathfrak{q}$. This is equivalent to saying that $\mathrm{Frob}_\mathfrak{q}$ generates $\mathrm{Gal}(F/\mathbf{Q}(\zeta)^+)$ if and only if

$$R_{\zeta,\mathfrak{q}}([a^n]) = P_{\zeta,\mathfrak{q}}([a^n]) = \prod_{k=0}^{n-1}(1-a\zeta^k)^k \in \mathbf{F}_{q^2}^\times \otimes \mathbf{Z}/n\mathbf{Z}$$

is a generator. Hence it suffices to find a single $q \equiv -1 \bmod n$ and $q \not\equiv -1 \bmod np$ with the desired Frobenius. Such a q exists by Cebotarev density unless $\langle\tau\rangle = \langle\zeta\rangle \bmod \mathbf{Q}(\zeta)^{\times p}$. However, this cannot happen by Lemma 5.4. ☐

Proof of Theorem 1.2. Assume that n is prime to $w_2(F)$. It follows that the Chern class map gives an injection

$$K_3(F)/nK_3(F) \to \mathcal{O}_{F_n}[1/S]^\times/\mathcal{O}_{F_n}[1/S]^{\times n}$$

for some finite set of primes S. If, in addition, we assume that p does not divide $\widetilde{w}_F$, then we deduce from Proposition 4.2 that this map can be extended to an injection into the group $\bigoplus_\mathfrak{q} B(\mathbf{F}_q)/nB(\mathbf{F}_q)$. By Theorem 1.5, this agrees with the map R_ζ defined on $B(F)$, which is thus injective. If one additionally assumes that n is prime to $|\Delta_F||K_2(\mathcal{O}_F)|$, then by Lemma 3.4 one may additionally assume that the image is precisely the χ^{-1}-invariants of $\mathcal{O}_{F_n}^\times/\mathcal{O}_{F_n}^{\times n}$. ☐

5.2. **Digression: the mod-p-q dilogarithm.** Let q be prime, and $q+1 \equiv 0 \bmod n$ with n a power of p as before. Fix an nth root of unity ζ in $\mathbf{F}_{q^2}$. Then there is a trivialization $\log_\zeta : \mathbf{F}_{q^2}^\times \otimes \mathbf{Z}/n\mathbf{Z} \simeq \mathbf{Z}/n\mathbf{Z}$ sending ζ to 1. The isomorphism $B(\mathbf{F}_q)\otimes\mathbf{Z}_p \simeq \mathbf{Z}/n\mathbf{Z}$ of Theorem 5.2 now gives a curious function, the p-q *dilogarithm*, which is a function

$$L:\ \mathbf{F}_q \ \to\ \mathbf{F}_{q^2}^\times \otimes \mathbf{Z}/n\mathbf{Z} \ \overset{\log_\zeta}{\to}\ \mathbf{Z}/n\mathbf{Z}$$

satisfying the 5-term relation. What is perhaps surprising is that the quantum *logarithm* suffices to give an explicit formula, as follows.

Proposition 5.5. The function L is given by the formula

$$L(a) \ = \ \sum_{b^n=a} \log_\zeta(b)\log_\zeta(1-b) \qquad (a \in \mathbf{F}_q^\times),$$

where the sum is over the nth roots b of a in $\mathbf{F}_{q^2}^\times$.

Proof. Since $\mathbf{F}_q^\times$ has order prime to n, the element a has a unique nth power $c \in \mathbf{F}_q^\times$. Then (17) can be rewritten as $L(a) = \sum_{k \bmod n} k\log_\zeta(1-\zeta^k c)$. (Note that $R_\zeta = P_\zeta$ for finite fields.) The elements $b = \zeta^k c$ are the nth roots of a in $\mathbf{F}_{q^2}^\times$, and $\log_\zeta(b) = k$ because c has order prime to n and thus $\log_\zeta(c) = 0$. ☐

5.3. **The Chern class map on n-torsion in $\mathbf{Q}(\zeta)^+$.** (The following section contains a speculative digression and is not used elsewhere in the paper.) We have proved that the maps c_ζ and R_ζ agree up to an invertible element of $\mathbf{Z}_p^\times$. To determine the value of this ratio, whose conjectural value is 2, we need to compute the images of specific elements of the Bloch group. More specifically, as explained in the introduction, we need the two statements (12) and (13). The first of these will be proved below (Theorem 8.5). Here we want to show that the second is not pure fancy. We shall give a heuristic justification of why the image of the Chern class map on η_ζ should be ζ — at least up to a sign and a small power of 2 in the exponent. We hope that the arguments of this section could, with care, be made into a precise argument. However, since the main conjecture of this section is somewhat orthogonal to the main purpose of this paper, and correctly proving everything would (at the very least) involve establishing that several diagrams relating the cohomology of SL_2 and PSL_2 and GL_2 and PGL_2 commute up to precise signs and factors of 2. Thus we content ourselves with a sketch, and enter the happy land where all diagrams commute.

The first subtle point is that the relation between $K_3(F)$ and $B(F)$ as established by Suslin is not an isomorphism. There is always an issue with 2-torsion coming from the image of Milnor K_3. However, even for primes p away from 2, there is an exact sequence of Suslin ([31], Theorem 5.2; here F is a number field so certainly infinite):

$$0 \to \mathrm{Tor}_1(\mu_F, \mu_F) \otimes \mathbf{Z}[1/2] \to K_3(F) \otimes \mathbf{Z}[1/2] \to B(F) \otimes \mathbf{Z}[1/2] \to 0,$$

and hence when $p | w_F = |\mu_F|$, the comparison map is not an isomorphism. (This is one of the headaches which implicitly us to assume that $\zeta \notin F$ when computing the Chern class map on $B(F)$.) This issue arises in the following way. Over the field $\mathbf{Q}(\zeta)$, the Bott element provides a direct relationship between $K_1(F, \mathbf{Z}/n\mathbf{Z})$ and $K_3(\mathbf{Z}, \mathbf{Z}/n\mathbf{Z})$. This suggests we should push forward η_ζ to $\mathbf{Q}(\zeta)$ and compute the Chern class there. However, since in $B(\mathbf{Q}(\zeta))$, the class η_ζ may (and indeed does) become trivial, we instead consider η_ζ as an element of $K_3(\mathbf{Q}(\zeta))$, and then compute the Chern class map directly in K-theory.

By Theorem 4.10 of Dupont–Sah [7], the diagonal map

$$x \to \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}$$

induces an injection

$$\mu_{\mathbf{C}} \simeq H_3(\mu_{\mathbf{C}}, \mathbf{Z}) \to H_3(\mathrm{SL}_2(\mathbf{C}), \mathbf{Z})$$

whose image is precisely the torsion subgroup. (We shall be more precise about this first isomorphism below.) Let n be odd, and let ζ be a primitive nth root of unity, let $E = \mathbf{Q}(\zeta)$, and let $E^+ = \mathbf{Q}(\zeta)^+$. If μ_E is the group of nth roots of unity, the map $\mu_E \to \mathrm{SL}_2(E)$ is conjugate to a map

$$\mu_E \to \mathrm{SL}_2(E^+)$$

as follows; send ζ to

$$t = A \begin{pmatrix} \zeta^{-1} & 0 \\ 0 & \zeta \end{pmatrix} A^{-1}, \quad \text{where } A = \begin{pmatrix} \zeta & \zeta^{-1} \\ 1 & 1 \end{pmatrix}.$$

The cohomology of μ_E with coefficients in $\mathbf{Z}/n\mathbf{Z}$ is (non-canonically) isomorphic to $\mathbf{Z}/n\mathbf{Z}$ in all degrees. More precisely, there is a canonical isomorphism

$$H_1(\mu_E, \mathbf{Z}) = H_1(\mu_E, \mathbf{Z}/n\mathbf{Z}) = \mu_E\,,$$

we have $H_2(\mu_E, \mathbf{Z}) = 0$, and thus via the Bockstein map $H_2(\mu_E, \mathbf{Z}/n\mathbf{Z}) = H_1(\mu_E, \mathbf{Z})[n] = \mu_E$. A choice of ζ leads to a choice of element $\beta \in H_2(\mu_E, \mathbf{Z}/n\mathbf{Z}) = \mu_E$, and hence to an isomorphism

$$\mu_E = H_1(\mu_E, \mathbf{Z}/n\mathbf{Z}) \xrightarrow{*\beta} H_3(\mu_E, \mathbf{Z}/n\mathbf{Z}) = H_3(\mu_E, \mathbf{Z})$$

where the isomorphism is given by the Pontryagin product of μ_E with $\beta \in H_2(\mu_E, \mathbf{Z}/n\mathbf{Z})$. These choices induce a map

$$\mu_E \to H_3(\mu_E, \mathbf{Z}) \to H_3(\mathrm{SL}_2(E^+), \mathbf{Z}) \to K_3(E^+) \to B(E^+)$$

which sends ζ to η_ζ. That the image of ζ is η_ζ follows (for example) by §8.1 of [40]). Implicit in this statement also is that the Pontryagin product of $1 \in \mathbf{Z}/n\mathbf{Z} = H_1(\mathbf{Z}/n\mathbf{Z}, \mathbf{Z}/n\mathbf{Z})$ with $1 \in H_2(\mathbf{Z}/n\mathbf{Z}, \mathbf{Z}/n\mathbf{Z})$ is exactly the class constructed in Proposition 3.25 of Parry and Sah [27]. (The maps above are only properly defined modulo 2-torsion, since μ has odd order this issue can safely be ignored). Denote by η_{E^+} the corresponding element in $K_3(E^+)$. The Chern class maps are compatible with base change, so to compute $c(\eta_{E^+})$ it suffices to compute $c(\eta_E)$ where $\eta_E \in K_3(E)$ is the image of η_{E^+} under the map $K_3(E^+) \to K_3(E)$. The Chern class map on $K_1(E) = E^\times$ canonically sends $\zeta \in E^\times$ to ζ; we would like to directly connect the Chern class map on K_1 with the one on K_3 using the Bott element. The Bott element $\beta \in K_2(E; \mathbf{Z}/n\mathbf{Z})$ is defined as follows. There is an isomorphism:

$$\mu_E = \ker\left(E^\times \xrightarrow{n} E^\times\right) = \pi_2(E^\times; \mathbf{Z}/n\mathbf{Z})\,.$$

The element β is defined as the image of ζ under the composition

$$\pi_2(\mathrm{BGL}_1(E); \mathbf{Z}/n\mathbf{Z}) \to \pi_2(\mathrm{BGL}(E); \mathbf{Z}/n\mathbf{Z}) \to \pi_2(\mathrm{BGL}(E)^+; \mathbf{Z}/n\mathbf{Z}) = K_2(E; \mathbf{Z}/p\mathbf{Z})\,.$$

The Bott element induces an isomorphism:

$$\beta : K_1(E; \mathbf{Z}/n\mathbf{Z}) \to K_3(E; \mathbf{Z}/n\mathbf{Z})\,.$$

Hence there is, given our choice of $\zeta \in E$, a canonically defined map:

$$\begin{tikzcd}
K_3(E) \arrow[d] & E^\times/E^{\times n} \arrow[dd, equal] \\
K_3(E; \mathbf{Z}/n\mathbf{Z}) \arrow[ur, "c_\zeta"] \arrow[d, equal, "\beta^{-1}"'] & \\
K_1(E, \mathbf{Z}/n\mathbf{Z}) \arrow[r, "c"] & E^\times/E^{\times n}
\end{tikzcd}$$

Here c_ζ is the composition of the Chern class map to $H^1(E, \mathbf{Z}/n\mathbf{Z}(2))$ which can be identified with $E^\times/E^{\times n}$ after a choice of $\zeta \in E$. Note that the definition of β also requires a similar choice. Thus it makes sense to make the following:

Assumption 5.6. The diagram above commutes.

We believe that it should be possible to prove this assumption, at least up to a choice of sign and a power of 2.

Using Assumption 5.6, we would like to show that $c_\zeta(\eta_E) = \zeta$, and hence that $c_\zeta(\eta_{E^+})$ and thus $c_\zeta(\eta_\zeta)$ are also both equal to ζ. This will follow if, under the Bott element, the class η_E corresponds to $\zeta \in K_1(E; \mathbf{Z}/n\mathbf{Z})$. To prove this, one roughly has to show that the following square commutes:

$$\begin{array}{ccc} \mu_E = H_1(\mu_E, \mathbf{Z}/n\mathbf{Z}) & \xrightarrow{*\beta} & H_3(\mu_E, \mathbf{Z}/n\mathbf{Z}) \\ \downarrow & & \downarrow \\ E^\times/E^{\times n} = K_1(E, \mathbf{Z}/n\mathbf{Z}) & \xrightarrow{\beta} & K_3(E; \mathbf{Z}/n\mathbf{Z}). \end{array}$$

The top line comes from the Pontryagin product structure of $H_1(\mu_E, \mathbf{Z}/n\mathbf{Z}) = \mu_E$ with

$$H_2(\mu_E, \mathbf{Z}/n\mathbf{Z}) = \ker(\mu_E \xrightarrow{[n]} \mu_E)\,,$$

and the bottom line comes from Pontryagin product with the Bott element β coming via the Bockstein map from

$$\ker(E^\times \xrightarrow{[n]} E^\times)\,.$$

We conveniently denote both maps by essentially the same letter in order to be more suggestive. One caveat is that the maps from $E^\times \to \mathrm{GL}_2(E)$ and $\mu_E \to \mathrm{SL}_2(E)$ considered above differ slightly in that x is sent to $\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}$ respectively; since n is odd such maps can be compared by comparing the cohomologies of GL, PGL, SL, and PSL respectively; it is quite possible that such comparisons might require that the maps above include a factor of 2 or -1 at some point.

The above discussion above makes the conjectured equation (13) plausible.

6. The connecting homomorphism to K-theory

In this section, we give a proof of Theorem 1.7. Assume that F is a field of characteristic prime to p which does not contain a pth root of unity. Recall that $Z(F)$ is the free abelian group on $F \setminus \{0, 1\}$ and $C(F)$ the subgroup generated by the 5-term relation.

Definition 6.1. Let $A(F; \mathbf{Z}/n\mathbf{Z})$ be the kernel of the map

$$d : Z(F) \longrightarrow \textstyle\bigwedge^2 F^\times \otimes \mathbf{Z}/n\mathbf{Z}\,, \qquad [X] \mapsto X \wedge (1 - X)\,.$$

The étale Bloch group $B(F; \mathbf{Z}/n\mathbf{Z})$ is the quotient

$$B(F; \mathbf{Z}/n\mathbf{Z}) = A(F; \mathbf{Z}/n\mathbf{Z})/(nZ(F) + C(F))\,.$$

It is annihilated by n.

There is a tautological exact sequence

$$0 \to B(F) \to Z(F)/C(F) \to \bigwedge^2 F^\times \to K_2(F) \to 0\,.$$

For appropriately defined R, we may break this into the two short exact sequences as follows:

$$\begin{tikzcd} 0 \arrow[r] & A(F) \arrow[r] \arrow[d] & Z(F) \arrow[r] \arrow[d] & R \arrow[r] \arrow[d, equal] & 0, \\ 0 \arrow[r] & B(F) \arrow[r] & Z(F)/C(F) \arrow[r] & R \arrow[r] & 0, \end{tikzcd}$$

$$0 \to R \to \bigwedge^2 F^\times \to K_2(F) \to 0. \tag{24}$$

Similarly, for some Q, we have corresponding short exact sequences:

$$\begin{tikzcd} 0 \arrow[r] & A(F) \arrow[r] \arrow[d] & A(F;\mathbf{Z}/n\mathbf{Z}) \arrow[r] \arrow[d] & Q \arrow[r] \arrow[d] & 0, \\ 0 \arrow[r] & A(F)/(nZ(F)+C(F)) \arrow[r] \arrow[d, equal] & A(F;\mathbf{Z}/n\mathbf{Z})/(nZ(F)+C(F)) \arrow[r] \arrow[d, equal] & Q/nR \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & B(F)/nB(F) \arrow[r] & B(F;\mathbf{Z}/n\mathbf{Z}) \arrow[r] & Q/nR \arrow[r] & 0, \end{tikzcd}$$

We have inclusions $Q \subseteq R$ and $nR \subseteq Q \subseteq n\bigwedge^2 F^\times$. From now on, we make the assumption that the number field F does not contain a pth root of unity for any p dividing n. This implies from the previous inclusions that Q and R are all p-torsion free for $p|n$. Tensor the exact sequence (24) with $\mathbf{Z}/n\mathbf{Z}$. The group $\mathrm{Tor}^1(\mathbf{Z}/n\mathbf{Z}, \wedge^2 F^\times)$ vanishes by our assumption. Hence we have an exact sequence:

$$0 \to K_2(F)[n] \to R/nR \to \bigwedge^2 F^\times \otimes \mathbf{Z}/n\mathbf{Z} \to K_2(F)/nK_2(F) \to 0. \tag{25}$$

Recall that R is the image of $Z(F)$ in $\bigwedge^2 F^\times$ and Q is the image of $A(F;\mathbf{Z}/n\mathbf{Z})$, which is precisely the kernel of the map from R to $\bigwedge^2 F^\times \otimes \mathbf{Z}/n\mathbf{Z}$. It follows that the image of Q in R/nR is the kernel of the map from R/nR to $\bigwedge^2 F^\times \otimes \mathbf{Z}/n\mathbf{Z}$. From the short exact sequence (25), this may be identified with $K_2(F)[n]$. Since the image of Q in R/nR is precisely Q/nR, however, this shows that $Q/nR \simeq K_2(F)$, we obtain the exact sequence:

$$0 \;\longrightarrow\; B(F)/nB(F) \;\longrightarrow\; B(F;\mathbf{Z}/n\mathbf{Z}) \;\longrightarrow\; K_2(F)[n] \;\longrightarrow\; 0\,,$$

completing the proof of Theorem 1.7.

The previous result was a diagram chase. The map $\delta : B(F;\mathbf{Z}/n\mathbf{Z}) \to K_2(F)$ can be given explicitly as follows: Lift $[x] \in B(F;\mathbf{Z}/n\mathbf{Z})$ to an element x of $A(F;\mathbf{Z}/n\mathbf{Z})/C(F)$, which is unique up to an element of $nZ(F)$. The image of x in $\bigwedge^2 F^\times \otimes \mathbf{Z}/n\mathbf{Z}$ is zero by definition. Hence, because $\bigwedge^2 F^\times$ is p-torsion free for $p|n$, there exists an element $y \in \bigwedge^2 F^\times$ such that the image of z in $\bigwedge^2 F^\times$ is ny, and now y is unique up to an element in the image of $C(F)$. Yet the projection z of $y \in \bigwedge^2 F^\times$ to $K_2(F)$ sends this ambiguity $C(F)$ to zero, and so $\delta([x]) := z \in K_2(F)$ is well defined.

If we assume that n is not divisible by any prime p which divides $w_2(F)$, we have constructed a map

$$R_\zeta : B(F;\mathbf{Z}/n\mathbf{Z}) \to (F_n^\times/F_n^{\times n})^{\chi^{-1}} \simeq H^1(F,\mathbf{Z}/n\mathbf{Z}(2))\,. \tag{26}$$

Taking $n = p^m$ for various m, and using the fact that $B(F)$ is finitely generated and so $\operatorname{proj\,lim} B(F)/p^m B(F) = B(F)\otimes \mathbf{Z}_p$, we obtain a commutative diagram as follows:

$$\begin{tikzcd}
0 \arrow[r] & B(F)/p^m B(F) \arrow[r] \arrow[d] & B(F;\mathbf{Z}/n\mathbf{Z}) \arrow[r] \arrow[d] & K_2(F)[n] \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H^1(F,\mathbf{Z}_p(2))/p^m \arrow[r] & H^1(F,\mathbf{Z}/n\mathbf{Z}(2)) \arrow[r] & H^2(F,\mathbf{Z}_p(2))[n] \arrow[r] & 0,
\end{tikzcd} \tag{27}$$

The first vertical map is an isomorphism by Theorem 3.2, and the last vertical map is also an isomorphism by a theorem of Tate [32]. It follows that the map R_ζ in equation 26 is an isomorphism for n prime to $w_2(F)$. This gives a link between our explicit construction of Chern class maps for $K_3(F)$ and the explicit construction of $K_2(F)$ in Galois cohomology by Tate [32].

We end this section with a remark on circular units. Let $F = \mathbf{Q}(\zeta_D)$. Associated to a primitive Dth root of unity ζ_D, Beilinson (see §9 of [15]) constructed special generating elements of $K_3(F)$, which correspond, on the Bloch group side, to the classes $D\,[\zeta_D] \in B(F)$. Soulé [29] proved that the images of these classes under the Chern class map consist exactly of the circular units. On the other hand, for p not dividing D, we see that the images of $D[\zeta_D]$ under the maps R_ζ are unit multiples of the elements

$$\prod_{k=0}^{p^{m-1}} (1-\zeta^k\,\zeta_D)^k\,;$$

these are exactly the compatible sequences of circular units which yield a finite index subgroup of $H^1(F,\mathbf{Z}_p(2))$ — the index being directly related to $K_2(\mathcal{O}_F)$ via the Quillen–Lichtenbaum conjectures.

7. Relation to quantum knot theory

As was mentioned in the introduction, the initial motivation for expecting a map as in (5) was the Quantum Modularity Conjecture, which concerns the asymptotics of a twisted version of the Kashaev invariant of a knot at roots of unity and a subtle transformation property (verified numerically for many knots and proved in the case of the figure 8 knot) of certain associated formal power series under the group $\mathrm{SL}(2,\mathbf{Z})$. In this section, we give a summary of this conjecture (a much more detailed discussion is given in [13]) and compare the near units appearing there with the ones studied in this paper.

Let K be a hyperbolic knot, i.e., an embedded circle in S^3 for which the 3-manifold $M_K = S^3 \smallsetminus K$ has a hyperbolic structure. This structure is then unique and gives several invariants: the *volume* $\mathrm{V}(K) \in \mathbf{R}_{>0}$ and *Chern-Simons-invariant* $\mathrm{CS}(K) \in \mathbf{R}/4\pi^2\mathbf{Z}$ of the 3-manifold M_K, the *trace field* $F_K = \mathbf{Q}\big[\{\mathrm{tr}(\gamma)\}_{\gamma\in\Gamma}\big]$ where $M_K = \mathbb{H}^3/\Gamma$ with $\Gamma \subset \mathrm{SL}(2,\mathbf{C})$ (the finitely generated group Γ is only unique up to conjugacy, but the set of traces of its elements is well-defined), and a *fundamental class* ξ_K in the Bloch group $B(F_K)$, defined as the class of $\sum[z_j]$ in $B(F_K)$, where $\sqcup\Delta_j = M_K$ is any ideal triangulation of M_K and z_j the cross-ratio of the four vertices of Δ_j. We also have two quantum invariants, the (normalized) *colored Jones polynomial* $J_N^K(q) \in \mathbf{Z}[q,q^{-1}]$ and the *Kashaev invariant* $\langle K\rangle_N \in \overline{\mathbf{Q}}$, which are computable expressions defined for any $N \in \mathbf{N}$ whose precise definitions, not needed here, we omit. The **Volume Conjecture**, due to Kashaev, says that the limit of $\frac{1}{N}\log|\langle K\rangle_N|$ as $N\to\infty$ equals $\frac{1}{2\pi}\mathrm{V}(K)$, the **Complexified Volume Conjecture** is the more precise statement $\langle K\rangle_N = e^{v(K)N+\mathrm{o}(N)}$ as $N\to\infty$, where $v(K) = \frac{1}{2\pi}(\mathrm{V}(K) - i\mathrm{CS}(K))$ (this makes sense because $v(K)$ is well-defined modulo $2\pi i$), and the yet stronger **Arithmeticity Conjecture**, stated in [6] and [9], says that there is a full asymptotic expansion

$$\langle K\rangle_N \;\sim\; \mu_8\delta(K)^{-1/2}\,N^{3/2}\,e^{v(K)N}\Big(1 \,+\, \kappa_1(K)\,\frac{2\pi i}{N} \,+\, \kappa_2(K)\Big(\frac{2\pi i}{N}\Big)^2 \,+\,\cdots\Big) \tag{28}$$

as $N\to\infty$, where $\delta(K)$ is a non-zero number related to the Ray-Singer torsion of K and where $\delta(K)$ and $\kappa_j(K)$ $(j\ge 1)$ belongs to the trace field F_K. An example, one of the few that are known rigorously (some other cases have now been proved by Ohtsuki et al; see [26]), is the expansion

$$\langle 4_1\rangle_N \;\sim\; \frac{N^{3/2}}{\sqrt[4]{3}}\,e^{v(4_1)N}\Big(1 \,+\, \frac{11\pi}{36\sqrt{3}\,N} \,+\, \frac{697\pi^2}{7776N^2} \,+\, \frac{724351\pi^3}{4199040\sqrt{3}\,N^3} \,+\,\cdots\Big)$$

for the knot $K = 4_1$ (figure 8), for which $F_K = \mathbf{Q}(\sqrt{-3})$.

Equation (28) is already a strong refinement of the Volume Conjecture. An even stronger is the **Modularity Conjecture** given in [39] and discussed further in [8] and [13]. The starting point is the famous theorem of H. Murakami and J. Murakami [23] saying that the Kashaev invariant $\langle K\rangle_N$, originally defined by Kashaev as a certain state sum, coincides with the value of the colored Jones polynomial $J_N^K(q)$ at $q = \zeta_N = e^{2\pi i/N}$. We define a function $\mathbf{J}^K : \mathbf{Q}\to\overline{\mathbf{Q}}$ by setting $\mathbf{J}^K(x) = J_N^K(e^{2\pi i x})$ for any $N\in\mathbf{N}$ with $Nx\in\mathbf{Z}$. This is independent of the choice of N since the values of the colored Jones polynomial $J_N^K(q)$ at an nth root of unity q is periodic in N of period n. From the Murakami-Murakami theorem and since the function $\mathbf{J}^K$ is Galois-invariant by its very definition, we have

$$\mathbf{J}^K\big(\frac{a}{N}\big) \;=\; \sigma_a\big(\langle K\rangle_N\big) \qquad \text{for any } a\in(\mathbf{Z}/N\mathbf{Z})^\times,$$

where σ_a is the automorphism of $\mathbf{Q}(\zeta_N)$ sending ζ_N to ζ_N^a, so the function $\mathbf{J}^K$ can be thought of as a Galois-twisted version of the Kashaev invariant. The Modularity Conjecture describes its asymptotic behavior of $\mathbf{J}^K(x)$ as the argument $x\in\mathbf{Q}$ approaches any fixed rational number α, not just 0. Specifically, it asserts that there exist formal power series $\Phi_\alpha^K(h)\in\overline{\mathbf{Q}}[[h]]$ ($\alpha\in\mathbf{Q}$, periodic in α with period 1), such that for any $\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in\mathrm{SL}(2,\mathbf{Z})$, we have

$$\mathbf{J}^{K^*}\Big(\frac{aX+b}{cX+d}\Big) \;\sim\; (cX+d)^{3/2}\,\mathbf{J}^{K^*}(X)\,e^{v(K)(X+d/c)}\,\Phi_{a/c}^K\Big(\frac{2\pi i}{cX+d}\Big) \tag{29}$$

to all orders in $1/X$ as $X \to \infty$ in $\mathbf{Q}$ with bounded denominator. Here, K^* is the mirror of K. If we take $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $X = N \to \infty$, then (29) reduces to $\langle K \rangle_N = \mathbf{J}^K(1/N) = \mathbf{J}^{K^*}(-1/N) \sim N^{3/2} e^{v(K)N} \Phi_0^K\big(\frac{2\pi i}{N}\big)$, so (29) is a generalization of (28) with $\Phi_0^K(h) = \mu_8\, \delta(K)^{-1/2}\, (1 + \kappa_1(K)h + \cdots)$.

The Quantum Modularity Conjecture, as already mentioned, was first given in [39], with detailed numerical evidence for the case of the figure 8 knot, and is further discussed in [8] and then in much more detail in [13], where this case is proved completely and numerical examples for several more knots are given. It is perhaps worth mentioning that regarding numerical evidence, one cannot use the standard `SnapPy` or `Mathematica` programs to compute the colored Jones polynomial (hence the Kashaev invariant) of a knot, since these work for small values of N (say, $N \sim 20$). Instead, we used a finite recursion for the colored Jones polynomial, whose existence was proven in [9], and concretely computed in several examples summarized in [8] and [13]. We could then compute the Kashaev invariant numerically to high precision up to N of the order of 5000, which with suitable numerical extrapolation techniques made it possible to compute and recognize several terms of the series $\Phi_{a/c}^K(h)$ with high confidence.

But already in the constant term of the series $\Phi_{a/c}^K(h)$, mysterious roots of algebraic units appear and those led to the main theorems of this paper. For instance, for the 4_1 knot, when a is an integer prime to 5, we have

$$\Phi_{a/5}^{4_1}(h) \;=\; 3^{\frac14}\big(\varepsilon^{(a)}\big)^{\frac{1}{10}}\left(\big(2-\varepsilon_1^{(a)}+\varepsilon_2^{(a)}+2\varepsilon_3^{(a)}\big) + \frac{2678 - 943\varepsilon_1^{(a)} + 1831\varepsilon_2^{(a)} + 2990\varepsilon_3^{(a)}}{2^3 3^2 5^2\sqrt{-3}}\, h + \cdots\right),$$

where $\varepsilon^{(a)} = \varepsilon_2^{(a)}/(\varepsilon_1^{(a)})^3\varepsilon_3^{(a)}$ and $\varepsilon_k^{(a)} = 2\cos\frac{2\pi(6a-5)k}{15}$. (See [39], p. 670 except that the formula is given there in terms of $\log\Phi$, which makes its coefficients much more complicated.) The number $\varepsilon_k^{(a)}$ is an algebraic unit in $F_5 = \mathbf{Q}(\zeta_{15})$, and it is the appearance of the 10th root of $\varepsilon^{(a)}$ that was the origin of the present investigation. In fact, although the unit $\varepsilon^{(a)}$ is not a square in the field $\mathbf{Q}(\zeta_{15})^+$ which it generates, its negative is a square in the larger field F_5: $\sqrt{-\varepsilon^{(a)}} = 2i\sin\frac{2\pi(6a-5)}{15}\varepsilon_2^{(a)}/\varepsilon_1^{(a)}$.

More generally, when we do numerical calculations for arbitrary knots K, we find:

- the power series $\Phi_\alpha^K(h)$ ($\alpha \in \mathbf{Q}$) belongs to $\overline{\mathbf{Q}}[[h]]$ and has a factorization of the form
$$\Phi_\alpha^K(h) \;=\; C_\alpha(K)\,\phi_\alpha^K(h) \tag{30}$$
with $C_\alpha(K) \in \overline{\mathbf{Q}}$ and $\phi_\alpha^K(h) \in F_c[[h]]$, where the number field F is independent of α (and is in fact is conjecturally the trace field F_K of K) and $F_c = F(\zeta_c)$;
- the constant $C_\alpha(K)$ factors as
$$C_\alpha(K) \;=\; \mu_{8c}(K)\delta(K)^{-1/2}\,\varepsilon_\alpha(K)^{1/c}$$
where μ_{8c} is a $8c$ root of unity and $\varepsilon_\alpha(K)$ is a unit of F_c;
- the units $\varepsilon_\alpha(K)$ for different rational numbers α and $\beta = k\alpha$ with the same denominator (assumed prime to some fixed integer depending on K) are related by both $\varepsilon_\beta(K) = \sigma_k(\varepsilon_\alpha(K))$ and $\varepsilon_\beta(K)^k = \varepsilon_\alpha(K)$ (the latter equality holding modulo cth powers), where $\sigma_k \in \mathrm{Gal}(F_c/F)$ is the map sending ζ_α to ζ_β. Compare this double Galois invariance with (6).

Notice that the factorization (30) is not canonical, since we can change both the constant C_α and the power series by a unit of F_c and its inverse, so that the formula involves a slight abuse of notation. Notice also that v depends only on the element of the Bloch group associated to the knot [25, 14] and so does ε_α, but not Φ_α. Specifically, as seen in [13], sister (or partner) knots do *not* have the same power series, but do (experimentally) have the same unit. It is this observation that led us to search for the map (5).

8. Nahm's conjecture and the asymptotics of Nahm sums at roots of unity

In the previous section, we saw that the near units constructed in this paper from elements of the Bloch group appear naturally (although in general only conjecturally) in connection with the asymptotic properties of the Kashaev invariant of knots and its Galois twists. A second place where these units appear is in the radial asymptotics of so-called Nahm sums, as was shown in [11] and is quoted (in a simplified form) in Theorem 8.1 below. In this section, we explain this and give two applications, the proof of Theorem 8.5 and the proof of one direction of Nahm's conjecture relating the modularity of Nahm sums to the vanishing of certain elements in Bloch groups.

Nahm sums are special q-hypergeometric series whose summand involves a quadratic form, a linear form and a constant. They were introduced by Nahm [24] in connection with characters of rational conformal field theories, and led to his above-mentioned conjecture concerning their modularity. They have also appeared recently in quantum topology in relation to the stabilization of the coefficients of the colored Jones polynomial (see Garoufalidis-Le [10]), and they are building blocks of the 3D-index of an ideally triangulated manifold due to Dimofte-Gaiotto-Gukov [5, 4]. Further connections between quantum topological invariants and Nahm sums are given in [12], where one sees once again the appearance of the units $R_\zeta(\xi)^{1/n}$.

In the first subsection of this section, we review Nahm sums and the Nahm conjecture and state Theorem 8.1 relating the asymptotics of Nahm sums at roots of unity to the near units of Theorem 1.2. This is then applied in §8.2 to a particular Nahm sum (namely, the famous Andrews-Gordon generalization of the Rogers-Ramanujan identities) to prove equation (12) of the introduction (Theorem 8.5). In the final subsection, we use Theorem 8.1 together with Theorem 1.2 to give a proof of one direction of Nahm's conjecture.

8.1. **Nahm's conjecture and Nahm sums.**

Nahm's conjecture gives a very surprising connection between modularity and algebraic K-theory. More precisely, it predicts that the modularity of certain q-hypergeometric series ("Nahm sums") is controlled by the vanishing of certain associated elements in the Bloch group $B(\overline{\mathbf{Q}}) = K_3(\overline{\mathbf{Q}})$.

The definition of Nahm sums and the question of determining when they are modular were motivated by the famous Rogers-Ramanujan identities, which say that

$$G(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q)_n} = \prod_{\substack{n>0 \\ (\frac{n}{5})=1}} \frac{1}{1-q^n}, \qquad H(q) := \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q)_n} = \prod_{\substack{n>0 \\ (\frac{n}{5})=-1}} \frac{1}{1-q^n},$$

where $(q)_n = (1-q)\cdots(1-q^n)$ is the q-Pochhammer symbol or quantum n-factorial. These identities imply via the Jacobi triple product formula that the two functions $q^{-1/60}G(q)$ and

$q^{11/60}H(q)$ are quotients of unary theta-series by the Dedekind eta-function and hence are modular functions. (Here and from now on we will allow ourselves the abuse of terminology of saying that a function $f(q)$ is modular if the function $\widetilde{f}(\tau) = f(e^{2\pi i\tau})$ is invariant under the action of some subgroup of finite index of $\mathrm{SL}(2,\mathbf{Z})$.) To see how general this phenomenon might be, Nahm [24] considered the three-parameter family

$$f_{A,B,C}(q) = \sum_{m\geq 0} \frac{q^{\frac{A}{2}m^2+Bm+C}}{(q)_m} \qquad (A \in \mathbf{Q}_{>0},\ B,\ C \in \mathbf{Q}) \tag{31}$$

These are formal power series with integer coefficients in some rational power of q, and are analytic in the unit disk $|q| < 1$, but they are very seldom modular: apart from the two Rogers-Ramanujan cases $(A,B,C) = (2,0,-\frac{1}{60})$ or $(2,1,\frac{11}{60})$, only five further cases $(1,0,-\frac{1}{48})$, $(1,\pm\frac{1}{2},\frac{1}{24})$, $(\frac{1}{2},0,-\frac{1}{40})$ and $(\frac{1}{2},\frac{1}{2},\frac{1}{40})$ were known for which $f_{A,B,C}$ is modular, and it was later proved ([33], [38]) that these are in fact the only ones. Since this list of seven examples is not very enlightening, Nahm introduced also a higher-order version

$$f_{A,B,C}(q) = \sum_{m\in\mathbf{Z}^r_{\geq 0}} \frac{q^{\frac{1}{2}m^tAm+Bm+C}}{(q)_{m_1}\cdots(q)_{m_r}}\,, \tag{32}$$

where $A = (a_{ij})$ is a symmetric positive definite $r \times r$ matrix with rational entries, $B \in \mathbf{Q}^r$ a column vector, and $C \in \mathbf{Q}$ a scalar, and asked for which triples (A,B,C) the function $\widetilde{f}_{A,B,C}(\tau) = f_{A,B,C}(e^{2\pi i\tau})$ is modular. His conjecture gives a partial answer to this question.

To formulate this conjecture, Nahm made two preliminary observations.

(i) Let $X = (X_1,\dots,X_r) \in \mathbf{C}^r$ be a solution of *Nahm's equations*

$$1 - X_i = \prod_{j=1}^r X_j^{a_{ij}} \qquad (1 \leq j \leq r) \tag{33}$$

(or symbolically $1-X = X^A$), and let F be the field they generate over $\mathbf{Q}$, which will typically be a number field since (33) is a system of r equations in r unknowns and generically defines a 0-dimensional variety. Then the element $[X] = [X_1]+\cdots[X_r]$ of $\mathbf{Z}[F]$ belongs to the kernel of the map (2), because

$$d\big([X]\big) = \sum_i (X_i)\wedge(1-X_i) = \sum_{i,j} a_{ij}\,(X_i)\wedge(X_j) = 0$$

by virtue of the symmetry of A. (This calculation makes sense as it stands if A has integer entries; if the entries are only rational, we have to tensor everything with $\mathbf{Q}$.) Therefore $[X]$ determines an element of the Bloch group $B(F)\otimes\mathbf{Q}$ and it makes sense to ask whether this element vanishes. This is equivalent to the vanishing of the numbers $D(\sigma X) = \sum D(\sigma X_i)$ for all embeddings $\sigma : F \hookrightarrow \mathbf{C}$, where $D(x)$ is the Bloch-Wigner dilogarithm function, and this condition can be either tested numerically to any precision or else verified rigorously by writing a multiple of $[X]$ as a linear combination of 5-term relations.

(ii) The first remark applies to any symmetric matrix A. If A is positive definite, then there is a distinguished solution of the Nahm equations, namely the unique solution $X^A = (X_1^A,\dots,X_r^A)$ with $0 < X_i^A < 1$ for all i. We denote by ξ_A the corresponding element $[X^A]$

of the Bloch group. Then since X^A is real, we obtain a further characteristic property when this element is torsion, namely that the real number $\mathrm{L}(\xi_A) = \sum \mathrm{L}(X_i)$, where $\mathrm{L}(x)$ is the Rogers dilogarithm function as defined below, is a rational multiple of π^2. But it can be shown fairly easily that $f_{A,B,C}(e^{-h})$ has an asymptotic expansion as $e^{\mathrm{L}(\xi_A)/h+\mathrm{O}(1)}$ as $h \to 0^+$ for any B and C (in fact, a full asymptotic expansion of the form $e^{\mathrm{L}(\xi_A)/h+c_0+c_1h+\cdots}$ is given in [38]). Since a modular function must have an expansion $e^{c/h+\mathrm{O}(1)}$ with $c \in \mathbf{Q}\pi^2$, this already gives a strong indication of a relation between the modularity of Nahm sums and the vanishing (up to torsion) of the associated elements of Bloch groups.

Based on these observations, one can consider the following three properties of a matrix A as above:

(a) The class $[X] \in B(\mathbf{C})$ vanishes for all solutions X of the Nahm equations (33).

(b) The special class $\xi_A \in B(\mathbf{C})$ associated to the solution X^A of (33) vanishes.

(c) The function $f_{A,B,C}(q)$ is modular for some $B \in \mathbf{Q}^r$ and $C \in \mathbf{Q}$.

Trivially (a) $\Rightarrow$ (b). Nahm's conjecture (see [24] and [38]) says that (a) $\Rightarrow$ (c) and (c) $\Rightarrow$ (b). (The possible stronger hypothesis that (b) alone might already imply (c) was eliminated in [38] using the 2×2 matrix $A = \left(\begin{smallmatrix} 8 & 5 \\ 5 & 4 \end{smallmatrix}\right)$, and the other possible stronger assertion that (c) might require (a) was shown to be false by Vlasenko and Zwegers [34] with the counterexample $A = \left(\begin{smallmatrix} 3/2 & 1/2 \\ 1/2 & 3/2 \end{smallmatrix}\right)$.) This conjecture had a dual motivation: on the one hand, the above-mentioned fact that both (b) and (c) force the rationality of $L(\xi_A)/\pi^2$, which is most unlikely to happen "at random," and on the other hand, a large number of supporting examples coming from the characters of rational conformal field theories, which are always modular functions and where the condition in the Bloch group can also be verified in many cases. Here we are concerned with an extension of the first of these two aspects, namely the asymptotics of the Nahm sum $f_{A,B,C}(q)$ as q tends radially to *any* root of unity, not just to 1.

In order to state the asymptotic formula, we need to define the Rogers dilogarithm. In our normalization (which is $\pi^2/6$ minus the standard one as given, for instance, in [38], §II.1A), this is the function defined on $\mathbf{R} \smallsetminus \{0, 1\}$ by

$$\mathrm{L}(x) \;=\; \begin{cases} \frac{\pi^2}{6} - \mathrm{Li}_2(x) - \frac{1}{2}\log(x)\log(1-x) & \text{if } 0 < x < 1, \\ -\mathrm{L}(1/x) & \text{if } x > 1, \\ \frac{\pi^2}{6} - \mathrm{L}(1-x) & \text{if } x < 0 \end{cases}$$

(here $\mathrm{Li}_2(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}$ is the standard dilogarithm) and extended by continuuity to a function $\mathbf{P}^1(\mathbf{R}) \to \mathbf{R}/\frac{\pi^2}{2}\mathbf{Z}$ by sending the three points 0, 1 and ∞ to $\frac{\pi^2}{6}$, 0, and $-\frac{\pi^2}{6}$. Its linear extension to $Z(\mathbf{R})$ vanishes on the group $C(\mathbf{R})$ as defined at the beginning of §1.1. (We comment here that there are several definitions of the Bloch group in the literature, all the same up to 6-torsion, and that the specific choice made in Definition 1.1, which forces $3[0] = 0$, $[X]+[1/X] = 0$ and $[X]+[1-X] = [0]$ for any field F and any element X of $\mathbf{P}^1(F)$, was chosen precisely so that L is well-defined on $B(\mathbf{R})$ and takes values in the full circle group $\mathbf{R}/\frac{\pi^2}{2}\mathbf{Z}$ rather than just its quotient $\mathbf{R}/\frac{\pi^2}{6}\mathbf{Z}$.)

Specifically, let A, B and C be as above let $X = X^A$ be the distinguished solution of (33) as in (ii) and F the corresponding number field, and for each integer n choose a primitive

nth root of unity ζ, set $F_n = F(\zeta)$ and denote by $H = H_n$ the Kummer extension of F_n obtained by adjoining the positive nth roots x_i of the X_i. We are interested in the asymptotic expansion of $f_{A,B,C}(\zeta e^{-h/n})$ as $h \to 0^+$. Strictly speaking, this only makes sense if A has integral coefficients, B is congruent to $\frac{1}{2}\mathrm{diag}(A)$ modulo $\mathbf{Z}^r$, and $C \in \mathbf{Z}$, since otherwise the quadratic function $q^{\frac{1}{2}nAn^t+nB+C}$ occurring in the definition of $f_{A,B,C}$ is not uniquely defined. We get around this by picking a representation of ζ as $\mathbf{e}(a/n)$ for some $a \in \mathbf{Z}$ and interpreting $f_{A,B,C}(\zeta e^{-h/n})$ as $\widetilde{f}_{A,B,C}\big(\frac{a+i\hbar}{n}\big)$, where $\hbar = \frac{h}{2\pi}$. The full asymptotic expansion of $f_{A,B,C}(\zeta e^{-h/n})$ as $h \to 0^+$ was calculated in [12] using the Euler-Maclaurin formula, generalizing an earlier result in [38] for the case $n = 1$. We do not give the complete formula here, but only the simplified form as needed for the applications we will give. In the statement of the theorem we have abbreviated by Δ_X the diagonal matrix whose diagonal is a vector X.

Theorem 8.1. [12] *Let (A, B, C) be as above. Then for every positive integer n (coprime to a finite set of primes that depend on A and B) and for every primitive nth root of unity ζ, we have*

$$f_{A,B,C}\big(\zeta\, e^{-h/n}\big) \;=\; \mu\, \omega\, e^{\mathrm{L}(\xi_A)/nh}\,\big(\Phi_\zeta(h) \;+\; O(h^K)\big) \tag{34}$$

for all $K > 0$ as $h \to 0^+$, where $\omega^2 \in F^\times$, $\mu^{24n} = 1$ and $\Phi_\zeta(h) = \Phi_{A,B,C,\zeta}(h)$ is an explicit power series satisfying the two properties $\Phi_\zeta(h)^n \in F_n[[h]]$ and $P_\zeta(\xi_A)^{1/n}\,\Phi_\zeta(h) \in H_n[[h]]$. Moreover, if $\Phi_\zeta(0)^n \neq 0$, then its image in $F_n^\times/F_n^{\times n}$ belongs to the χ^{-1} eigenspace.

Remark 8.2. If n is prime to 6, then we can choose μ to be a 24th root of unity, since the nth roots of unity are contained in F_n and can be absorbed into the power series Φ.

Corollary 8.3. If $\Phi_\zeta(0) \neq 0$, then the product of the power series $\Phi_\zeta(h)$ with $\varepsilon^{1/n}$ for any unit ε representing $R_\zeta(\xi_A)$ belongs to $F_n[[h]]$.

Proof. Let $\varepsilon \in F_n^\times$ denote a representative of $R_\zeta(\xi_A)$. On the one hand, Theorem 8.1 and Remark 2.5 imply that $\Phi_\zeta(0)\varepsilon^{1/n} \in F_n^\times$. On the other hand, Theorem 8.1 and our assumption implies that $(\Phi_\zeta(h)/\Phi_\zeta(0))^n \in F_n[[h]]$. Since $\Phi_\zeta(h)/\Phi_\zeta(0)$ is a power series with constant term 1, it follows that $\Phi_\zeta(h)/\Phi_\zeta(0) \in F_n[[h]]$. Combining both conclusions, it follows that $\varepsilon^{1/n}\Phi_\zeta(h) \in F_n[[h]]$. ☐

Remark 8.4. In the theorem, we do *not* assert that the power series Φ cannot vanish identically (which is why we wrote an equality sign and $\Phi(h) + \mathrm{O}(h^K)$ in (34) rather than writing an asymptotic equality sign and putting simply $\Phi(h)$ on the right), and indeed this often happens, for instance, when $f_{A,B,C}$ is modular and we are expanding at a cusp not equivalent to 0. Of course, the corollary is vacuous if Φ vanishes.

8.2. **Application to the calculation of $R_\zeta(\eta_\zeta)$.** In this subsection, we apply Theorem 8.1 and its corollary to a specific Nahm sum to prove equation (12) in the introduction.

Theorem 8.5. *Let n be positive and prime to 6 and η_ζ be the n-torsion element in $B(\mathbf{Q}(\zeta)^+)$ defined by (23), where ζ is a primitive nth root of unity. Then $R_\zeta(\eta_\zeta)^4 \;=\; \zeta$.*

Proof. Set $A_n = \big(2\min(i,j)\big)_{1\le i,j\le r}$, where $r = \frac{n-3}{2}$, and let f_n be the Nahm sum $f_{A_n,0,0}$ of order r. By a famous identity of Andrews and Gordon [1], which reduces to the first

Ramanujan-Rogers identity when $n = 5$, we have the product expansion

$$f_n(q) \;=\; \prod_{\substack{k>0 \\ 2k\not\equiv 0,\pm1 \,(\bmod n)}} \frac{1}{1-q^k}. \tag{35}$$

and this is modular up to a power of q for the same reason as for $G(q) = f_5(q)$ (quotient of a theta series by the Dedekind eta-function). This modularity allows us to compute its asymptotics as $q \to \zeta_n$, and by comparing the result with the general asymptotics of Nahm sums as given in 8.1, we will obtain the desired evaluation of η_n. We now give details.

It is easy to check that all solutions X of the Nahm equation $1 - X = X^{A_n}$ have the form

$$X = (X_1, \dots, X_r), \qquad X_k \;=\; \frac{(1-\zeta^{2k})(1-\zeta^{2k+4})}{(1-\zeta^{2k+2})^2}$$

with ζ a primitive n root of unity, and hence form a single Galois orbit. The distinguished solution $X^{A_n} \in (0,1)^r$ corresponds to $\zeta = \mathbf{e}(1/n) = \zeta_n$. From $1 - X_k = (\frac{\zeta-\zeta^{-1}}{\zeta^{k+1}-\zeta^{-k-1}})^2$ and the functional equation $L(1-X) = \frac{\pi^2}{6} - L(X)$ we find

$$\mathrm{L}(X^{A_n}) \;=\; \frac{1}{2}\sum_{0<\ell<n}\left(\frac{\pi^2}{6} - \mathrm{L}\Big(\frac{\sin^2(\pi/n)}{\sin^2(\ell\pi/n)}\Big)\right) \;=\; \frac{(n-3)\pi^2}{6n},$$

the final equality being a well-known identity for the Rogers dilogarithm of which a proof can be found at the end of [38], §II.2C. Denote the right-hand side of this by $-4\pi^2 C_n$ and set $\widetilde{f}_n(\tau) = q^{C_n} f_n(q)$. Using the Jacobi theta function and Jacobi triple product formula

$$\theta(\tau, z) \;=\; \sum_{n\in\mathbf{Z}+1/2} (-1)^{[n]}\, q^{n^2/2}\, y^n \;=\; q^{1/8}\, y^{1/2} \prod_{n=1}^{\infty}\big(1-q^n\big)\big(1-q^n y\big)\big(1-q^{n-1}y^{-1}\big)$$

(where $\Im(\tau) > 0$, $z \in \mathbf{C}$, $q = \mathbf{e}(\tau)$, and $y = \mathbf{e}(z)$), together with the Dedekind eta-function $\eta(\tau) = q^{1/24}\prod_{n>0}(1-q^n)$, we can rewrite (35) as

$$f_n(\tau) \;=\; -q^{(r+1)^2/2n}\,\frac{\theta(n\tau,\,(r+1)\tau)}{\eta(\tau)},$$

which in conjunction with the standard transformation properties of θ and η implies that $f_n(\tau)$ is a modular function (with multiplier system) on the congruence subgroup $\Gamma_0(n)$ of $\mathrm{SL}(2,\mathbf{Z})$. We need only the special case $\tau \mapsto \frac{\tau}{n\tau+1}$, where the transformation law is given by

$$f_n\Big(\frac{\tau}{n\tau+1}\Big) \;=\; \mathbf{e}\Big(\frac{n-3}{24}\Big)\, f_n(\tau)\,, \tag{36}$$

whose proof we sketch for completeness. The well-known modular transformation properties of θ and η under the generators $T = \left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$ and $S = \left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)$ of $\mathrm{SL}(2,\mathbf{Z})$ are given by

$$\theta(\tau+1, z) \;=\; \mathbf{e}(1/8)\,\theta(\tau, z)\,, \quad \theta(-1/\tau,\, z/\tau) \;=\; \sqrt{\tau/i}\;\mathbf{e}(z^2/2\tau)\,\theta(\tau, z)$$
$$\eta(\tau+1) \;=\; \mathbf{e}(1/24)\,\eta(\tau)\,, \qquad \eta(-1/\tau) \;=\; \sqrt{\tau/i}\;\eta(\tau)\,.$$

Hence, using $\overset{T}{\sim}$ and $\overset{S}{\sim}$ to denote an equality up to an elementary factor (the product of a power of τ with the exponential of a linear combination of 1, τ and z^2/τ) that can be deduced from the T- or S-transformation behavior of the function in question, we have

$$\theta\Big(\frac{n\tau}{n\tau+1},\frac{(r+1)\tau}{n\tau+1}\Big) \overset{T}{\sim} \theta\Big(\frac{-1}{n\tau+1},\frac{(r+1)\tau}{n\tau+1}\Big) \overset{S}{\sim} \theta\left(n\tau+1,(r+1)\tau\right) \overset{T}{\sim} \theta\left(n\tau,(r+1)\tau\right),$$

$$\eta\Big(\frac{\tau}{n\tau+1}\Big) \overset{S}{\sim} \eta\Big(-n-\frac{1}{\tau}\Big) \overset{T}{\sim} \eta\Big(-\frac{1}{\tau}\Big) \overset{S}{\sim} \eta(\tau)\,.$$

Inserting all omitted factors and dividing the first equations by the second, we obtain (36).

Now applying (36) to $\tau=\frac{1+i\hbar}{n}$, with $\hbar=\frac{h}{2\pi}$, where h positive and small, we find

$$\begin{aligned} f_{A_n,0,C_n}\big(\zeta_n e^{-h/n}\big) \;&=\; f_n\Big(\frac{1+i\hbar}{n}\Big) \;=\; \mathbf{e}\Big(\frac{n-3}{24}\Big)\, f_n\Big(\frac{-1+i/\hbar}{n}\Big) \\ &=\; \mathbf{e}\Big(\frac{n}{24}-\frac{1}{8}+\frac{1}{24n}-\frac{1}{8n^2}\Big)\, e^{\mathrm{L}(X^{A_n})/nh}\,\big(1\,+\,O\big(e^{-4\pi^2/nh}\big)\big)\,. \end{aligned} \tag{37}$$

Taking the $8n$-th power of this and combining with Theorem 8.1 and its Corollary 8.3, we find that $R_\zeta(\xi_{A_n})^8=\mathbf{e}(1/n)\in(F_n^\times/F_n^{\times n})^{\chi^{-1}}$. On the other hand, using the same identity $1-X_k=(\frac{\zeta-\zeta^{-1}}{\zeta^{k+1}-\zeta^{-k-1}})^2$ as before, we find that the Bloch element ξ_{A_n} associated to the distinguished real solutions X^{A_n} of the Nahm equation is *equal* to twice the Bloch element η_ζ defined in (23). This completes the proof of Theorem 8.5. ∎

8.3. **Application to Nahm's conjecture.** In this final subsection, we give an application of the asymptotic Theorem 8.1 and Theorem 1.2 to proving one direction of Nahm's conjecture about the modularity of Nahm sums. The notations and assumptions are as before, but for convenience we repeat them here.

Let $A\in M_r(\mathbf{Q})$ be a positive definite symmetric matrix, $B\in\mathbf{Q}^r$, and $C\in\mathbf{Q}$. We denote $X^A=(X_1,\dots,X_r)$ denote the unique solution in $(0,1)^r$ to the Nahm equation, by $F=F_A$ the real number field generated by the X_i and by $\xi_A=\sum_i[X_i]\in B(F_A)$ the corresponding element of the Bloch group. Finally, when we say that $F_{A,B,C}$ is modular, we mean that the function $\widetilde{f}(\tau)=f_{A,B,C}(\mathbf{e}(\tau))$ is invariant with respect to a subgroup of finite index of $\mathrm{SL}(2,\mathbf{Z})$.

Theorem 8.6. *If $f_{A,B,C}(\tau)$ is a modular function, then $\xi_A\in B(F_A)$ is a torsion element.*

Proof. On p. 56 of [38] it is shown that any Nahm sum has an expansion near $q=1$ of the form

$$f_{A,B,C}(e^{-\epsilon}) \;=\; e^{\mathrm{L}(\xi_A)/\epsilon}\,\big(K\,+\,\mathrm{O}(\epsilon)\big) \qquad (\epsilon\to 0), \tag{38}$$

where K (given explicitly in eq. (29) of [38]) is a non-zero algebraic number some power of which belongs to $F=F_A$ and where the error term $\mathrm{O}(\epsilon)$ can be replaced by $\mathrm{O}(e^{-c/\epsilon})$ with some $c>0$ if $f_{A,B,C}$ is assumed to be modular ([38], eq. (28)). Notice that in this case the number $\lambda=\mathrm{L}(\xi_A)/4\pi^2$ must be rational, since the modularity of $\widetilde{f}(\tau)=f_{A,B,C}(\mathbf{e}(\tau))$ implies that the function $\widetilde{f}(-1/\tau)$ is invariant under some power of $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$.

Now assume that $\widetilde{f}$ is modular with respect to a finite index subgroup Γ of $\mathrm{SL}(2,\mathbf{Z})$. Then for $h \to 0^+$, $\hbar = \frac{h}{2\pi}$, and any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$, taking $\epsilon = \frac{dh}{1-ic\hbar}$, we find

$$f_{A,B,C}(e^{-\epsilon}) \;=\; \widetilde{f}\Big(\frac{i\epsilon}{2\pi}\Big) \;=\; \widetilde{f}\Big(\frac{ai\epsilon/2\pi + b}{ci\epsilon/2\pi + d}\Big) \;=\; \widetilde{f}\Big(\frac{b + i\hbar}{d}\Big) \;=\; f_{A,B,C}(\zeta e^{-h/d}),$$

where $\zeta = \mathbf{e}(b/d)$, and now comparing the asymptotic formulas (38) and (34) (with $n = d$), we find

$$\mu\, e^{\mathrm{L}(\xi_A)/hd}\, \Phi(h) \;=\; e^{L(\xi_A)/dh}\left(K\mathbf{e}(\lambda c/d) \,+\, \mathrm{O}(h)\right)$$

or $\Phi_\zeta(0) = \mu^{-1}K\mathbf{e}(\lambda c/d)$, with $\lambda \in \mathbf{Q}$ as above. This implies in particular that $\Phi_\zeta(0) \neq 0$, and now, using that some bounded power of both μ and K belong to F_n, we deduce that $\Phi(0)^r$ belongs to F_n for some fixed integer $r > 0$ independent of $n = d$. We can also assume that d is prime to M for any fixed integer M, since by intersecting Γ with the full congruence subgroup $\Gamma(M)$, we may assume that Γ is contained in $\Gamma(M)$. This shows that there are infinitely many integers n and primitive nth roots of unity ζ for which $\Phi_\zeta(0)^r$ in Theorem 8.1 is a non-zero element of F_n. Now Corollary 8.3 implies that the rth power of $R_\zeta(\xi_A)$ has trivial image in $F_n^\times/F_n^{\times n}$ for infinitely many n, and in view of the injectivity statement in Theorem 1.2 this proves that ξ_A is a torsion element in the finitely generated group $B(F)$. $\square$

Remark 8.7. The proof of the theorem would have been marginally shorter if we had assumed that $f_{A,B,C}$ was a modular function on a congruence subgroup, rather than just a subgroup of finite index of $\mathrm{SL}(2,\mathbf{Z})$. We did not make this assumption since it was not needed, but should mention that $f_{A,B,C}$, if modular at all, is expected automatically to be modular for a congruence subgroup, because it has a Fourier expansion with integral coefficients in some rational power of q and a standard conjecture says that the Fourier expansion of a modular function on a non-congruence subgroup of $\mathrm{SL}(2,\mathbf{Z})$ always has unbounded denominators.

Remark 8.8. Conversely, we could have stated Theorem 8.6 in an apparently more general form by writing "modular form" instead of "modular function." We did not do this since it is easy to see that if a Nahm sum is modular at all, it is actually a modular function, because if it were a modular form of non-zero rational weight k, there would be an extra factor h^{-k} in the right-hand side of (38).

Acknowledgements. F.C. and S.G. were supported in part by grants DMS-1404620 and DMS-1406419 of the US National Science Foundation.

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