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# Finite type 3-manifold invariants and the structure of the Torelli group. I

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**Abstract.** Using the recently developed theory of finite type invariants of integral homology 3-spheres we study the structure of the Torelli group of a closed surface. Explicitly, we construct (a) natural cocycles of the Torelli group (with coefficients in a space of trivalent graphs) and cohomology classes of the abelianized Torelli group; (b) group homomorphisms that detect (rationally) the nontriviality of the lower central series of the Torelli group. Our results are motivated by the appearance of trivalent graphs in topology and in representation theory and the dual role played by the Casson invariant in the theory of finite type invariants of integral homology 3-spheres and in Morita's study [Mo2, Mo3] of the structure of the Torelli group. Our results generalize those of S. Morita [Mo2, Mo3] and complement the recent calculation, due to R. Hain [Ha2], of the $I$-adic completion of the rational group ring of the Torelli group. We also give analogous results for two other subgroups of the mapping class group.

## Contents



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## 1. Introduction

**1.1. Background.** The notion of finite type invariants for oriented integral homology 3-spheres was introduced not long ago by Ohtsuki [Oh]. More recently Le, Murakami and Ohtsuki [L, LMO] used the Kontsevich integral to give a complete classification of these invariants in terms of a certain space of trivalent graphs.

In another very recent paper [GL3] the present authors gave several different formulations of the notion of finite type invariants. In particular we showed that one could use the lower central series of the Torelli group (or certain other subgroups of the mapping class group), in conjunction with Heegaard decompositions, to define finite type invariants in terms of higher genus surgery formulas.

It is the purpose of the present paper to exploit this connection, using the classification theorem of [LMO], to investigate the structure of the Torelli group. Explicitly, we:

• Construct canonical cocycles of the Torelli group (with coefficients in a space of trivalent graphs), and cohomology classes in the abelianized Torelli group.
• Show, by very explicit and geometric construction, that the (rational) lower central series quotients of the Torelli group (and certain other subgroups of the mapping class group) map *onto* a space of trivalent graphs.

In a recent paper [Ha2] Hain has given a presentation of (the Lie algebra associated to) the lower central series of the Torelli group using mixed Hodge structures. We do not yet understand the relationship between our results and his. However, it would be interesting to compare them.

Finally we point out that the relation between trivalent graphs (in the theory of finite type invariants) and the Torelli group has been foreshadowed by the work of Morita in:

• The appearance of trivalent graphs in invariant theory applied to the Torelli group, see [Mo6, KM].
• The study of the *Casson invariant* in terms of the Torelli group and other subgroups of the mapping class group, see [Mo2, Mo3].

**1.2. Trivalent graphs in topology and in representation theory.** We begin by recalling the appearance of trivalent graphs in topology (in the theory of finite type invariant s of integral homology 3-spheres) and in representation theory (related to invariant tensors of the abelianization of the Torelli group). *Finite type invariants* of integral homology 3-spheres were introduced by Ohtsuki [Oh], in terms of a decreasing filtration $\mathscr{F}^{as}_{*}\mathscr{M}$ on the vector space $\mathscr{M}$ (over $\mathbb{Q}$) of isomorphism classes of oriented, connected integral homology 3-spheres. A linear map $v : \mathscr{M} \to \mathbb{Q}$ is called a *type m invariant* of integral homology 3-spheres if $v(\mathscr{F}^{as}_{m+1}\mathscr{M}) = 0$. The associated graded quotients $\mathscr{G}^{as}_{*}\mathscr{M}$ of the filtration $\mathscr{F}^{as}_{*}\mathscr{M}$ has recently been related to trivalent graphs in the following way. Let $\mathscr{A}(\phi)$ denote the vector space over $\mathbb{Q}$ on the set of trivalent, vertex-oriented graphs, modulo the *AS* and the *IHX* relations; see Fig. 1 and [GO1, LMO]. $\mathscr{A}(\phi)$ has a natural grading $\mathscr{G}_*\mathscr{A}(\phi)$; the degree of a trivalent graph is half the number of its vertices. Thus a degree $m$ trivalent graph has $3m$ edges and $2m$ vertices.

One can define a map (for details see Sect. 2.2):

**Fig. 1.** The *IHX* and the *AS* relations on $\mathscr{A}(\phi)$

(1) $$\mathscr{G}_m\mathscr{A}(\phi) \to \mathscr{G}^{as}_{3m}\mathscr{M}$$

which was shown in [GO1] to be well defined and onto. According to the *fundamental theorem* of finite type invariants of integral homology 3-spheres [LMO, L] the map (1) is one-to-one and thus a vector space isomorphism. We wish to think of the above isomorphism as a relation between finite type invariants of integral homology 3-spheres and *trivalent graphs* (decorated by a choice of a vertex orientation, and considered modulo the *AS* and *IHX* relations).

As it turns out, one can reformulate [GL3] the notion of finite type invariants of integral homology 3-spheres in such a way that makes explicit the dependence of the values of finite type invariants on manifolds obtained by cutting, twisting and gluing of higher genus surfaces. This reformulation, given in [GL3] in terms of six filtrations on $\mathscr{M}$, will be used crucially on the present paper. Three filtrations on $\mathscr{M}$ were defined in [GL3] using surgery on special classes of links, and three more filtrations by using cutting, twisting and gluing along embedded surfaces. Even though the results of the present paper can be stated using only a filtration denoted by $\mathscr{F}^T_*\mathscr{M}$ in [GL3], the proofs of our results will require the use of a few more filtrations from [GL3]. Following the notation as in [GL3], we briefly recall the definition of $\mathscr{F}^T_*\mathscr{M}$. Let $M$ be an integral homology 3-sphere and $f : \Sigma \hookrightarrow M$ an embedded, oriented, connected, closed genus $g$ surface in $M$. Such a surface will be called *admissible* in $M$. Note that an admissible surface has no boundary. Since $M$ is an integral homology 3-sphere it follows that an admissible surface is *separating*, i.e., $M - f(\Sigma)$ is the union of two connected components $M^{\mathrm{o}}_+$ and $M^{\mathrm{o}}_-$, where the positive normal vector to $f(\Sigma)$ points into $M^{\mathrm{o}}_+$. Let $M_\epsilon$ (for $\epsilon = \pm$) denote their closures; they are compact 3-manifolds with boundary $f(\Sigma)$. There is a natural decomposition $H_1(f(\Sigma)) = L_+ \oplus L_-$, where $L_\epsilon = \mathrm{Ker}\{(i_\epsilon)_* : H_1(f(\Sigma)) \to H_1(M_\epsilon)\}$ and $i_\epsilon : f(\Sigma) \to M_\epsilon$ is the inclusion. Here the homology is taken with integer coefficients. We refer to $(L_+, L_-)$ as the *Lagrangian pair* of the symplectic module $H = H_1(f(\Sigma))$ associated to the admissible surface $f : \Sigma \hookrightarrow M$. If $h \in \Gamma(f(\Sigma))$ (the mapping class group of $f(\Sigma)$, i.e., the group of isotopy classes of orientation preserving diffeomorphisms of $f(\Sigma)$) let $M_h$ denote $M_+ \amalg M_-$ with the identifications: $i_+(x) \leftrightarrow i_-h(x)$ for every $x \in f(\Sigma)$. The notation $M_h$ does not explicitly indicate the dependence of $M_h$ on the admissible surface which we keep fixed; we hope that this will not confuse the reader. Note that if $h \in \mathcal{T}(f(\Sigma))$, the *Torelli group* of $f(\Sigma)$ (i.e., all elements of the mapping class group that act trivially on the homology of the surface) and $M$ is an integral homology 3-sphere then the resulting manifold $M_h$ will also be an integral homology 3-sphere. For a closed surface $\Sigma$ of genus $g$, let $\mathcal{T}_g = \mathcal{T}(\Sigma)$. The assignment $h \to M_{fhf^{-1}}$ defines a map

$$\Phi_f^T : \mathbb{Q}\mathcal{T}_g \to \mathcal{M} \tag{2}$$

where $\mathbb{Q}\mathcal{T}_g$ is the rational group ring of $\mathcal{T}_g$. Let $I\mathcal{T}_g$ denote the augmentation ideal (i.e., the two sided ideal of $\mathbb{Q}\mathcal{T}_g$ generated by elements of the form $1-f$, for $f \in \mathcal{T}_g$). We *define* $\mathcal{F}_m^T\mathcal{M}$ to be the union in $\mathcal{M}$ of the image $\Phi_f^T((I\mathcal{T}_g)^m)$ for all admissible surfaces $f : \Sigma \hookrightarrow M$ in all integral homology 3-spheres $M$. Alternatively we can choose a single *Heegaard* embedding (i.e., $M_+$ and $M_-$ are handlebodies) into $S^3$ for each genus $g$ and let $\mathcal{F}_m^T\mathcal{M}$ be the union in $\mathcal{M}$ of the images $\Phi_f^T((I\mathcal{T}_g)^m)$ for these embeddings. It is shown in [GL3] that this gives the same filtration, and that $\mathcal{F}_{3m}^{as}\mathcal{M} = \mathcal{F}_{2m}^T\mathcal{M} = \mathcal{F}_{2m-1}^T\mathcal{M}$.

Let us now recall one more ingredient, the Johnson homomorphism [Jo1], related to the *abelianization* of the Torelli group. For a more detailed description, as well as a summary of properties of the Johnson homomorphism, see Sect. 2.3. If $\Sigma$ is a closed surface, D. Johnson defined a homomorphism $\tau : \mathcal{T}_g \to U = \Lambda^3 H/H$ where $H = H_1(\Sigma, \mathbb{Z})$. There are several versions of Johnson's homomorphism, depending on the surface being closed, or punctured, or with boundary components. The following are three important properties of Johnson's homomorphism (and its various versions):

- It coincides (modulo torsion) with the abelianization of the Torelli group.
- It is equivariant with respect to the action of the mapping class group of the surface.
- It is stable with respect to an inclusion of a surface with one boundary component into another.

These properties have been used extensively by Hain [Ha2] and Morita [Mo2], [Mo3] to study questions relating to the lower central series of the Torelli group and the mapping class group. From its very definition, the image of $\tau$ is a quotient $U$ of the third exterior power of $H$. Furthermore, it turns out that the invariant space of $\otimes^{2m}U$ under the symplectic or the general linear group can be described in terms of suitably "decorated" *trivalent graphs*, modulo an *AS* relation. For a precise statement, see Sect. 2.4 and especially Definition 2.17.

It is a natural question to ask whether the above two appearances of trivalent graphs in the theory of finite type invariants of integral homology

3-spheres and in the abelianization of the Torelli group are related to each other. For a positive answer (in terms of a stably onto map $\Psi_{L^{\pm},m}$) see Theorem 3.

**1.3. The role of the Casson invariant.** As was stated above, a main motivation for the present work was the role of the Casson invariant in the theory of finite type invariants and in the work of S. Morita [Mo3]. Explicitly, the Casson invariant $\lambda$ [AM] has the following properties:

• $\lambda$ is a type 3 invariant of integral homology 3-spheres, [Oh].
• Given an admissible genus $g$ surface $f : \Sigma \hookrightarrow M$, Morita [Mo3] used the Casson invariant to construct a map:

$$2\delta_f : U \otimes U \to \mathbb{Q} \tag{3}$$

where the notation is as in [Mo3].
• Furthermore, given an admissible surface Morita [Mo2] used the Casson invariant to construct a group homomorphism:

$$\mathscr{K}_g \to \mathbb{Q} \tag{4}$$

where $\mathscr{K}_g$ is the kernel of the Johnson homomorphism $\mathscr{T}_g \to U$.

It is a natural question to ask whether one can use finite type invariants of integral homology 3-spheres generalize the two maps constructed above. For a positive answer, see Theorems 1, 2, 5 and especially Corollary 1.1 and Theorem 4.

**1.4. Statement of the results.** In this section we state the main results of the paper. For a group $G$, and a positive integer $n$, let us inductively define the *lower central series* subgroups of $G$ by $G_{n+1} = [G, G_n]$, with $G_1 = G$. Let us also define $G(n)$ to consist of all elements of $G$ for which a nonzero power lies in $G_n$. We call $G(n)$ the $n^{th}$ term in the *rational lower central series* of $G$. With the notation as in Sect. 1.1, we have the following:

**Theorem 1.** *Let $f : \Sigma \hookrightarrow M$ be an admissible surface. For every non-negative integer $m$, there is a map*

$$C_{f,m} : \otimes^{2m} U \to \mathscr{G}_m \mathscr{A}(\phi) \tag{5}$$

*with the following properties*:

• *The map $C_{f,m}$ is multilinear and $Sp(H)$ equivariant, i.e., satisfies the following property* (*for $\alpha_i \in U, h \in \Gamma_g$*):

$$C_{f,m}(h_*\alpha_1, \ldots h_*\alpha_{2m}) = C_{hf,m}(\alpha_1, \ldots \alpha_{2m}) \tag{6}$$

• *$C_{f,m}$ is a $2m$ cocycle of the abelian group $U$ with coefficients in the trivial $U$ module $\mathcal{G}_m\mathcal{A}(\phi)$. In particular, it represents a cohomology class $[C_{f,m}] \in H^{2m}(U, \mathcal{G}_m\mathcal{A}(\phi))$*

• *The pullback of the cocycle $C_{f,m}$* to *$\mathcal{T}_g$ and, in fact, to $\mathcal{T}_g/\mathcal{T}_g(3)$, under the projection maps $\mathcal{T}_g \to \mathcal{T}_g/\mathcal{T}_g(3) \to U$, is a coboundary.*

**Addendum 1.** *With the above notation, given an admissible surface $f : \Sigma \hookrightarrow M$, the following diagram commutes*:

$$\begin{array}{ccc} \mathcal{T}_g^{2m} & \longrightarrow & \mathcal{F}_{2m}^T\mathcal{M} \\ {\scriptstyle \tau^{\otimes 2m}}\big\downarrow & & \big\downarrow \\ \otimes^{2m} U & \xrightarrow{C_t,m} & \mathcal{G}_m\mathcal{A}(\phi) \end{array}$$

*where the right vertical map is the composition of maps* (*defined in Sect.* 1.1):

$$\mathcal{F}_{2m}^T\mathcal{M} = \mathcal{F}_{3m}^{as}\mathcal{M} \to \mathcal{G}_{3m}^{as}\mathcal{M} \simeq \mathcal{G}_m\mathcal{A}(\phi)$$

*and the top horizontal map is the map*: $(h_1, \dots, h_{2m}) \to \Phi_f^T((1-h_1)\dots(1-h_{2m}))$.

**Corollary 1.1.** *If $v$ is a type $3m$ invariant of integral homology* 3-*spheres*, *then its associated weight system is an element $W_v \in \mathcal{G}_m\mathcal{A}^*(\phi)$*, [GO1]. *Given an admissible surface $f : \Sigma \hookrightarrow M$, we thus get a $2m$ cocycle $W_v \circ C_{f,m}$ of $U$ with rational coefficients.*

**Theorem 2.** *Let $f : \Sigma \hookrightarrow M$ be an admissible* Heegaard *surface. Then*, *for every non*-*negative integer $m$*,

• *The cocycle $C_{f,m}$ depends only on the associated Lagrangian pair $(L^+, L^-)$ of the admissible surface*, *and will thus be denoted by $C_{L^\pm,m}$.*

• *Using the natural onto maps $U \to \Lambda^3(H/L^\mp) \simeq \Lambda^3 L^\pm$, $C_{L^\pm,m} : \otimes^{2m}U \to \mathcal{G}_m\mathcal{A}(\phi)$ factors though a $GL(L^+_{\mathbb{Q}})$-invariant map*:

$$\Lambda^3 L^+_{\mathbb{Q}} \otimes (\otimes^{2m-2} U_{\mathbb{Q}}) \otimes \Lambda^3 L^-_{\mathbb{Q}} \to \mathcal{G}_m\mathcal{A}(\phi) \tag{7}$$

• *If we change the orientation of the integral homology* 3-*sphere $M$*, *this results in a permutation of the Lagrangian pair $L^\pm$ and the associated cocycle satisfies*:

$$C_{L^\pm,m}(g_1, g_2, \dots, g_{2m}) = (-1)^m C_{L^\mp,m}(g_{2m}, g_{2m-1}, \dots, g_1) \tag{8}$$

*Passing to cohomology classes though*, *we have*

$$[C_{L^\pm,m}] = [C_{L^\mp,m}] \tag{9}$$

**Addendum 2.** *In the case of an admissible Heegaard surface of genus* $g$, *the map* $C_{L^\pm,m}$ *is stable with respect to an inclusion of one (punctured) Heegaard surface into another. Furthermore, the map* $C_{L^\pm,m}$ *is stably (i.e.,* for $g \gg m$) *onto. In particular, the cocycles* $C_{L^\pm,m}$ *of* $U$ *are stably nontrivial.*

See Sect. 3.1 for a more precise assertion of Addendum 2.

Before we state the next theorem we need to define $\mathscr{G}_m\mathscr{A}^{rp,nl,cl}$, a vector space over $\mathbb{Q}$ on the set of "decorated" trivalent graphs with $3m$ edges, modulo a colored antisymmetry relation (see Fig. 7). These decorations involve a choice of *ordering for the vertices*, as well as a choice of *vertex orientation* and a choice of 2-*coloring* for the edges. For a precise definition, as well as motivation coming from representation theory of classical Lie groups, see Definition 2.17 and Sect. 2.4.

**Theorem 3.** *Let* $f : \Sigma \hookrightarrow M$ *be an admissible* Heegaard *surface. Then, for every non-negative integer m there is an onto map:* $\mathscr{G}_m\mathscr{A}^{rp,nl,cl} \to \otimes^{2m} U$, *which combined with the map* $C_{L^\pm,m} : \otimes^{2m} U \to \mathscr{G}_m\mathscr{A}(\phi)$, *induce a (stably onto) map:*

$$\Psi_{L^\pm,m} : \mathscr{G}_m\mathscr{A}^{rp,nl,cl} \to \mathscr{G}_m\mathscr{A}(\phi) \tag{10}$$

The above map compares trivalent graphs related to the abelianization of the Torelli group (on the left) to trivalent graphs related to finite type invariants of integral homology 3-spheres (on the right), and fulfills one of the goals of the present paper.

In the case of $m = 1$, we have an explicit description of the cocycle $C_{L^\pm,1}$ of theorem 2 and corollary 1.1 and of the map $\Psi_{L^\pm,1}$ of Theorem 3. We need to recall first that the Casson invariant $\lambda$ [AM] is a type 3 invariant, [Oh]. Let $W_\lambda \in \mathscr{G}_1\mathscr{A}(\phi)^*$ denote its associated manifold weight system as in [GO1]. Let $\Theta_w \in \mathscr{G}_1\mathscr{A}(\phi)$ denote the trivalent graph $\Theta$ with a fixed choice of vertex orientation. Let $f : \Sigma \hookrightarrow M$ be an admissible Heegaard surface, and let $C_\Theta : \otimes^6 H \to \mathbb{Q}$ be given by:

$$C_\Theta(a_1 \otimes a_2 \otimes a_3, b_1 \otimes b_2 \otimes b_3) = \omega(a_1, b_1)\omega(a_2, b_2)\omega(a_3, b_3) \tag{11}$$

for $a_i, b_i \in H$, where $\omega$ is the intersection pairing on $H$. Recall the onto maps $U_\mathbb{Q} \to \Lambda^3 L^\pm_\mathbb{Q}$ from theorem 2, their tensor product $\otimes^2 U_\mathbb{Q} \to \Lambda^3 L^+_\mathbb{Q} \otimes \Lambda^3 L^-_\mathbb{Q}$ and the natural inclusion $\Lambda^3 L^+_\mathbb{Q} \otimes \Lambda^3 L^-_\mathbb{Q} \hookrightarrow \otimes^2 \Lambda^3 H_\mathbb{Q} \hookrightarrow \otimes^6 H_\mathbb{Q}$. Let us denote by $C^U_\Theta$ the pullback of $C_\Theta$ to $\otimes^2 U_\mathbb{Q}$ under the composition of the above maps. Then, we have the following theorem:

**Theorem 4.** *Given an admissible Heegaard surface* $f : \Sigma \hookrightarrow M$,

- *The map* $(W_\lambda) \circ C_{L^\pm,1} : \otimes^2 U_\mathbb{Q} \to \mathbb{Q}$ *is given by:*

$$(W_\lambda) \circ C_{L^\pm,1} = 2C^U_\Theta \tag{12}$$

• *the map* $C_{L^\pm,1} : \otimes^2 U \to \mathcal{G}_1\mathcal{A}(\phi)$ *is given as follows. For* $\alpha_1, \alpha_2 \in U$ *we have*:

$$C_{L^\pm,1}(\alpha_1, \alpha_2) = -C^U_\Theta(\alpha_1, \alpha_2) \cdot \Theta_w \tag{13}$$

• *The vector space* $\mathcal{G}_1\mathcal{A}^{rp,nl,cl}$ *is four dimensional, with a basis given in the southeast part of Fig.* 8. *The map* (10) *of Theorem* 3 *is given as follows*:
• *The cocycle* $C_{L^\pm,1}$ *defines a nonzero cohomology class* $[C_{L^\pm,1}] \in H^2(U;\mathbb{Q})$, *if* $\dim H \geq 6$. *Moreover* $[C_{L^\pm,1}]$ *depends on the Lagrangian pair* $L^\pm$ *in the following strong sense. If* $K^\pm$ *is another Lagrangian pair then* $[C_{L^\pm,1}] = [C_{K^\pm,1}]$ *if and only if one of the following holds*:
•: $\dim H < 6$
•: $L^+ = K^+, L^- = K^-$, *or*
•: $L^+ = K^-, L^- = K^+$.

*Remark 1.2*. In coordinates, the map $C^U_\Theta$ is given as follows. Let $\{x_i\}_{i=1}^g$ (respectively, $\{y_i\}_{i=1}^g$) be basis for $L^+$ (respectively, $L^-$) such that $\omega(x_i, y_j) = \delta_{i,j}$. Using the natural projection $\Lambda^3 H_\mathbb{Q} \to U_\mathbb{Q}$, consider $\alpha_1, \alpha_2 \in U_\mathbb{Q}$ and let $\bar\alpha_1, \bar\alpha_2 \in \Lambda^3 H_\mathbb{Q}$ be their lifts written as:

$$\bar\alpha_1 = \sum_{i<j<k} \alpha^1_{ijk} x_i \wedge x_j \wedge x_k \ + \text{other terms}$$
$$\bar\alpha_2 = \sum_{i<j<k} \alpha^2_{ijk} y_i \wedge y_j \wedge y_k \ + \text{other terms}$$

Then, we have:

$$C^U_\Theta(\bar\alpha_1, \bar\alpha_2) = \sum_{i<j<k} \alpha^1_{ijk}\alpha^2_{ijk} \tag{14}$$

*Remark 1.3*. Note that the above map $(W_\lambda) \circ C_{L^\pm,1}$ coincides with the map $2\delta_f$ of [Mo3, Definition 4.1, Theorem 4.3], and that the first part of the above theorem was originally proven by Morita [Mo3, Theorem 4.3]. Morita's result was a starting point for the results of the present paper. It is interesting to note that the factor of 2 in $2\delta_f$ in the above mentioned paper of Morita was derived from a representation theory calculation (counting irreducible components of $Sp(H)$ representations), whereas in our context it comes from the identity $\Theta_w = 2Y_w$.

The maps $C_{f,m}$ assemble well to define a map: $C_f : T_{ev}(U) \to \mathcal{A}(\phi)$, where $T_{ev}(U) = \oplus_{m=0}^\infty(\otimes^{2m}U)$. Recall that $T_{ev}(U)$ and $\mathcal{A}(\phi)$ are graded Hopf algebras, where the comultiplication in $T_{ev}(U)$ is given by declaring $U$ to be the set of primitive elements. $C_f$ respects the multiplication in the following sense. For $i = 1, 2$ let $f_i : \Sigma_{g_i} \to M$ be two admissible genus $g_i$ surfaces disjointly embedded in an integral homology 3-sphere $M$. Without loss of generality, let us assume that $f_i$ are inclusions. Assume that there is an embedded 2-sphere $S \hookrightarrow M$ with the following properties:

• The intersections $S \cap \Sigma_{g_i} = D_i$ for $i = 1, 2$ are disjoint discs. Let $\Sigma_i = \Sigma_{g_i} - \mathrm{Int} D_i$.
• Recall that $S$ separates $M - S$ in two components. Assume that $\Sigma_i$ for $i = 1, 2$ lie in different components of $M - S$.

Then we can form the composite admissible surface $f_1 \cup f_2 : \Sigma = \Sigma_1 \cup_{S-(D_1 \cup D_2)} \Sigma_2 \to M$. Considering the homology of $\Sigma_{g_1}, \Sigma_{g_2}$ and $\Sigma$ we get natural onto maps $U_{g_1+g_2} \to U_{g_i}$ for $i = 1, 2$ which in turn induce onto maps: $T_{ev}(U_{g_1+g_2}) \to T_{ev}(U_{g_i})$. Let $C^i_{f_i}$ denote the pullbacks of the maps $C_{f_i}$ to $T_{ev}(U_{g_1+g_2})$ for $i = 1, 2$.

**Proposition 1.4.** *With the above assumptions we have the following*:

$$C^1_{f_1} \cdot C^2_{f_2} = C_{f_1 \cup f_2} \tag{15}$$

*Remark 1.5*. The above proposition makes necessary the existence of an operadic formalism of the above cocycles. Such a formalism, which may

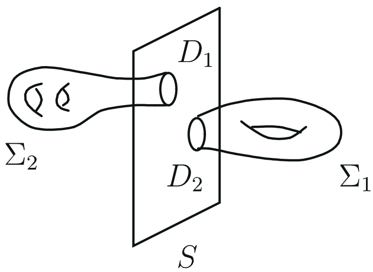


**Fig. 2.** Gluing two admissible surfaces to form a third one. Note that only part of the surface $S$ is drawn in the figure

make more transparent the relation with the ideas from 2*D* gravity [Ko1], [Ko2], [Wi1], [Wi2] will be the subject of a future study.

Before we state the next theorem, we need some notation: for a group $G$ and a positive integer $n$ let us *denote* by $\mathcal{G}_n G$ the (abelian) quotient $G(n)/G(n+1)$. Let $\mathcal{A}^{conn}(\phi)$ denote the subspace of $\mathcal{A}(\phi)$ consisting of linear combinations of *connected* admissible graphs. We also define two binary operations: $[^0 x, y] = x \otimes y$ and $[^1 x, y] = -y \otimes x$. Then, we have the following theorem:

**Theorem 5.** *Given an admissible surface* $f : \Sigma \hookrightarrow M$, *and a nonnegative integer* $m$, *there is a linear map*:

$$D_{f,m} : \mathcal{G}_{2m} \mathcal{T}_g \otimes \mathbb{Q} \to \mathcal{G}_m \mathcal{A}^{conn}(\phi) \tag{16}$$

*with the following properties*:

• *$D_{f,m}$ is determined by the cocycle $C_{f,m}$ as follows:*

$$\begin{aligned} &D_{f,m}([x_1,[x_2,\ldots,[x_{2m-1},x_{2m}]]]) \\ (17)\qquad &= -\sum_a C_{f,m}([^{a(1)}y_1,[^{a(2)}y_2,\ldots,[^{a(2m-1)}y_{2m-1},y_{2m}]]]) \end{aligned}$$

*where $x_i \in \mathscr{T}_g$, $y_i = \tau(x_i) \in U$, the summation runs over all functions $a : \{1,2,\ldots,2m-1\} \to \{0,1\}$, and we set $[^0x,y] = x \otimes y$* and *$[^1x,y] = -y \otimes x$.*
*Assume in addition that $f$ is an admissible Heegaard surface. Then,*
• *$D_{f,m}$ depends only on the associated Lagrangian pair $(L^+,L^-)$, and will be denoted by $D_{L^\pm,m}$.*
• *$D_{L^\pm,m}$ satisfies the following symmetry property:*

$$(18)\qquad D_{L^\pm,m}(x^{-1}) = (-1)^m D_{L^\mp,m}(x)$$

*for every $x \in \mathscr{G}_{2m}\mathscr{T}_g \otimes \mathbb{Q}$.*
• *Assume that $f$ is the standard Heegaard splitting of $S^3$. If $g \geq 5m+1$, then $D_{L^\pm,m}$ is onto.*
• *For an arbitrary admissible Heegaard surface $f$, let $D_m : (\mathscr{G}_{2m}\mathscr{T}_g \otimes \mathbb{Q})^{Sp_g} \to \mathscr{G}_m\mathscr{A}^{conn}(\phi)$ denote the restriction of $D_{L^\pm,m}$ on the symplectic invariant part of its domain. For $m=1$, the composition of $D_1$ with the weight system $W_\lambda : \mathscr{G}_1\mathscr{A}^{conn}(\phi) \to \mathbb{Q}$ coincides with the restriction of $-\frac{1}{24}d_1 : \Gamma_g \to \mathbb{Q}$ of* [Mo2, section 5] *to $(\mathscr{G}_2\mathscr{T}_g \otimes \mathbb{Q})^{Sp(H_\mathbb{Q})}$*

*Remark 1.6*. The proof of Theorem 5 (and Theorem 7 below) exhibits an explicit construction of enough *stably* non-trivial elements of the lower central series quotients $\mathscr{G}_{2m}\mathscr{T}_g$ of the Torelli group when $g \geq 5m+1$ to prove that the map $D_{L^\pm,m}$ is onto for a standard Heegaard splitting of genus at least $5m+1$. This construction may prove useful in further study of the Torelli group.

As an application of the proof of Theorem 5 (and Theorem 7 below) we define and determine an analogue of the (rational) Gusarov group of knots [Gu] for integral homology 3-spheres. This result has been obtained independently by Le [L, Theorem 10]. Following the ideas of Gusarov, [Gu], we define a sequence of equivalence relations on the set of (orientation preserving diffeomorphism classes of ) integral homology 3-spheres as follows. Given a nonnegative integer $n$, and two integral homology 3-spheres $M$ and $N$, we define $M$ to be *n-equivalent* to $N$, (and write $M \sim_n N$) if $M - N \in \mathscr{F}^{as}_{3n}\mathscr{M}$. Let $\mathscr{E}_n$ denote the set of $\sim_{n+1}$-equivalence classes. Connected sum induces the structure of an abelian semigroup on $\mathscr{E}_n$. We have natural projections $\mathscr{E}_n \to \mathscr{E}_{n-1}$ whose kernel $\mathscr{O}_n$ is an abelian *semigroup*. We can define a map

$$(19)\qquad \tau_n : \mathscr{O}_n \to \mathscr{G}^{as}_{3n}\mathscr{M}$$

by $\tau_n(M) = S^3 - M$. This map is additive. Indeed, we have

$$S^3 - M\sharp N = -(S^3 - M) \cdot (S^3 - N) + (S^3 - M) + (S^3 - N)$$

and, if $M, N \in \mathcal{O}_n$, then $(S^3 - M) \cdot (S^3 - N) \in \mathscr{F}^{as}_{6n}\mathscr{M} \subseteq \mathscr{F}^{as}_{3(n+1)}\mathscr{M}$, for $n \geq 1$. We now have:

**Theorem 6.** [L] *$\mathcal{O}_n$ is a group and $\tau_n$ induces an isomorphism of $\mathcal{O}_n \otimes \mathbb{Q}$ with the subspace of primitive elements in $\mathscr{G}^{as}_{3n}\mathscr{M}$.*

**Corollary 1.7.** *$\mathscr{E}_n$ is an abelian group.*

**1.5. Plan of the proof.** In Sect. 2 we review the definition and a few essential properties of the Johnson homomorphism and discuss invariant theory for the symplectic and general linear group. In Sect. 3 we prove our main results. In Sect. 4 we discuss analogous constructions for some other subgroups of the mapping class group. In Sect. 5 we discuss related results by Hain [Ha2] and Morita [Mo6]. Finally in Sect. 6 we formulate a question which will be studied in a subsequent publication.

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## 2. Preliminaries

**2.1. Generalities in group theory.** In this section we review some general facts about group cohomology of discrete groups. Let $G$ be a discrete group and $\mathbb{Q}G$ the rational group ring of $G$. Let $IG$ (or simply $I$, in case we fix the group $G$) denote the augmentation ideal of $\mathbb{Q}G$ and $I^n$ the $n$-th power of $I$. We first recall the definition of the chain complex defined by the bar construction. Since we will only be dealing with coefficients with trivial $G$-action we can define, for a *trivial* $G$ module $M$, $C_n(G, M) \stackrel{\text{def}}{=} \mathrm{Hom}(C_n(G), M)$. Here $C_n(G)$ is the free $\mathbb{Z}$-module generated by $n$-tuples $[g_1|\ldots|g_n]$, where $g_i \in G$, and the boundary operator is defined by the formula:

$$\partial[g_1|\ldots|g_n] = [g_2|\ldots|g_n] + \sum_{i=1}^{n-1}(-1)^i[g_1|\ldots|g_ig_{i+1}|\ldots|g_n] + (-1)^n[g_1|\ldots|g_{n-1}]$$

As usual, we let $Z^n(G,M)$ (resp., $B^n(G,M)$) denote the *n-cocycles* (resp., *n-coboundaries*). For a cocyle $c_n \in Z^n(G,M)$, let $[c_n] \in H^n(G,M)$ denote the associated cohomology class.

Before proceeding to the main results of this section, we will prove a lemma which will be needed below in Sect. 3.2. Define an involution $\gamma$ of $C_*(G)$ by the formula

$$\gamma[g_1|\dots|g_n] = (-1)^{\binom{n}{2}}[g_n^{-1}|\dots|g_1^{-1}]$$

We leave it to the reader to check that this, indeed, is a chain map. We have the following lemma:

**Lemma 2.1.** *If $G$ is a free abelian group, then $\gamma_* : H_n(G) \to H_n(G)$ is multiplication by $(-1)^n$.*

*Proof*. It is sufficient to prove this is true for $\gamma^*$ on the exterior algebra $H^*(G;\mathbb{Z})$. Notice that for $n=1$ this is clearly true. Since $H^1(G)$ generates $H^*(G)$, as an algebra, we will be done if we prove that $\gamma^*$ preserves cup-products. Recall (see [Mac]) that the formula for cup-product, in the context of the bar construction, is:

$$(\xi \cup \eta)[g_1|\dots|g_n|h_1|\dots|h_m] = \xi[g_1|\dots|g_n] \cdot \eta[h_1|\dots|h_m]$$

where $\xi \in C^n(G), \eta \in C^m(G)$. Now we compute:

$$\begin{aligned}
\gamma^\sharp(\xi \cup \eta)[g_1|\dots|g_m|h_1|\dots|h_n] &= (-1)^{\binom{m+n}{2}}(\xi \cup \eta)[h_n^{-1}|\dots|h_1^{-1}|g_m^{-1}|\dots|g_1^{-1}] \\
&= (-1)^{\binom{m+n}{2}}\xi[h_n^{-1}|\dots|h_1^{-1}] \cdot \eta[g_m^{-1}|\dots|g_1^{-1}] \\
&= (-1)^{\binom{m+n}{2}+\binom{m}{2}+\binom{n}{2}}\gamma^\sharp(\xi)[h_1\dots|h_n] \cdot \gamma^\sharp(\eta)[g_1|\dots|g_m] \\
&= (-1)^{mn}(\gamma^\sharp(\eta) \cup \gamma^\sharp(\xi))[g_1|\dots|g_m|h_1|\dots|h_n]
\end{aligned}$$

So we see that, for any cohomology classes $\alpha \in H^n(G), \beta \in H^m(G)$, $\gamma^*(\alpha \cup \beta) = (-1)^{mn}\gamma^*(\beta) \cup \gamma^*(\alpha)$. But now we just invoke the commutativity of cup-product on the cohomology level. $\square$

Turning to our main results, we will now define for every nonnegative integer $n$ a *cochain* $\phi_n \in C^n(G;I^n)$, where $I^n$ is given the trivial $G$-module structure, as follows:

$$\phi_n[g_1|\dots|g_n] = (1-g_1)\dots(1-g_n)$$

Let $i_n : I^{n+1} \to I^n$ be the inclusion and $(i_n)_\sharp$ be the corresponding coefficient homomorphism of cochains.

**Lemma 2.2.**

$$\delta(\phi_n) = \begin{cases} 0 & n \quad \text{even} \\ (i_n)_\sharp(\phi_{n+1}) & n \quad \text{odd} \end{cases}$$

*Proof*. We have the following formula for the coboundary:

$$\begin{aligned}
\delta\phi_n([g_1|\dots|g_{n+1}]) &= \phi_n([g_2|\dots|g_{n+1}]) + \sum_{i=1}^{n}(-1)^i\phi_n([g_1|\dots|g_ig_{i+1}|\dots|g_{n+1}]) \\
&\quad + (-1)^{n+1}\phi_n([g_1|\dots|g_n]) \\
&= \prod_{i=2}^{n+1}(1-g_i) \\
&\quad + \sum_{i=1}^{n}(-1)^i(1-g_1)\dots(1-g_ig_{i+1})\dots(1-g_{n+1}) \\
&\quad + (-1)^{n+1}\prod_{i=1}^{n}(1-g_i)
\end{aligned}$$

Now making the substitution $1-g_ig_{i+1} = -(1-g_i)(1-g_{i+1}) + (1-g_i) + (1-g_{i+1})$, the summation term in the above equation becomes:

$$\begin{aligned}
&\sum_{i=1}^{n}(-1)^i(1-g_1)\dots(1-g_ig_{i+1})\dots(1-g_{n+1}) \\
&= \sum_{i=1}^{n}(-1)^i(1-g_1)\dots\{-(1-g_i)(1-g_{i+1}) \\
&\quad + (1-g_i) + (1-g_{i+1})\}\dots(1-g_{n+1}) \\
&= \sum_{i=1}^{n}(-1)^{i+1}(1-g_1)\dots(1-g_i)(1-g_{i+1})\dots(1-g_{n+1}) \\
&\quad + \sum_{i=1}^{n}(-1)^i(1-g_1)\dots\{\widehat{(1-g_i)} + \widehat{(1-g_{i+1})}\}\dots(1-g_{n+1}) \\
&= \left(\sum_{i=1}^{n}(-1)^{i+1}\right)(1-g_1)\dots(1-g_{n+1}) - \prod_{i=2}^{n+1}(1-g_i) + (-1)^n\prod_{i=1}^{n}(1-g_i)
\end{aligned}$$

Inserting this into the previous equation we obtain:

$$\begin{aligned}
\delta\phi_n([g_1|\dots|g_{n+1}]) &= \left(\sum_{i=1}^{n}(-1)^{i+1}\right)(1-g_1)\dots(1-g_{n+1}) \\
&= \left(\sum_{i=1}^{n}(-1)^{i+1}\right)\phi_{n+1}([g_1|\dots|g_{n+1}])
\end{aligned}$$

and the result follows. □

**Corollary 2.3.** *For every even nonnegative integer $n$, there is a well-defined cohomology class $[\phi_n]$ in $H^n(G; I^n)$.*

*Proof*. Immediate by Lemma 2.2 above. □

We now point out another useful fact. Recall from Sect. 1.4 that for any positive integer $q$, $G_q$ is defined inductively by $G_{q+1} = [G, G_q]$ with the understanding that $G_1 = G$. Recall also that $G(q)$ is the (normal) subgroup of $G$ that contains all elements of $G$ for which a nontrivial power belongs to $G_q$. It is easy to see that $\{G(n)\}_{n\geq 1}$ is a decreasing sequence of normal subgroups of $G$ with the property: $[G(n), G(m)] \subseteq G(n+m)$.

**Lemma 2.4.** *If $g_i \in G(q_i)$ then $\phi_n([g_1|\dots|g_n]) \subseteq I^{q_1+\dots+q_n}$.*

*Proof*. With the notation $[g,h] = ghg^{-1}h^{-1}$, the following formula

$$1 - [g,h] = (-(1-g)(1-h) + (1-h)(1-g))g^{-1}h^{-1} \in I^2 \tag{20}$$

shows that if $g \in G_q$, then $1 - g \in I^q$. The following formula

$$1 - g^m = \sum_{i=1}^{m} (-1)^i \binom{m}{i} (1-g)^i$$

and the above shows by induction on $q$ that if $g^m \in G_q$ for some nonnegative integer $m$, then

$$m(1-g) \equiv 1 - g^m \bmod I^{q+1} \tag{21}$$

thus deducing that $m(1-g) \in I^q$, and since we are using rational coefficients, this proves the lemma. □

**Corollary 2.5.** *Given nonnegative integers $n, q$, $\phi_n$ induces cochains*

$$\phi_{n,q} \in C^n(G/G(q); I^n/I^{n+q-1}) \tag{22}$$

*Furthermore, for even $n$, $\phi_{n,q}$ is a cocycle and, for odd $n$, $\phi_{n,2}$ is a cocycle. Moreover, $\phi_{n,2}$ is multilinear.*

Now suppose that $\mathscr{B}$ is a vector space (over $\mathbb{Q}$), carrying a decreasing filtration $\mathscr{F}_*\mathscr{B}$ and $\rho : \mathbb{Q}G \to \mathscr{B}$ is a linear map preserving the filtration, i.e., $\rho(I^n) \subseteq \mathscr{F}_n\mathscr{B}$. Suppose also that the filtration of $\mathscr{B}$ is *p-step*, for some positive integer $p$, i.e., $\mathscr{F}_i\mathscr{B} = \mathscr{F}_{i+1}\mathscr{B}$ unless $p$ divides $i$. Now $\phi_{n,q}$ induces via $\rho$, a cochain $\phi^\rho_{n,q} \in C^n(G/G(q); \mathscr{F}_n\mathscr{B}/\mathscr{F}_{n+q-1}\mathscr{B})$. Let $\mathscr{G}_n\mathscr{A} = \mathscr{F}_{pn}\mathscr{B}/\mathscr{F}_{pn+1}\mathscr{B}$.

We will consider the cochains $\phi^\rho_{pn-q,q+2} \in C^{pn-q}(G/G(q+2); \mathscr{G}_n\mathscr{A})$, for $0 \leq q < p$, (since $\mathscr{F}_{pn-q}\mathscr{B}/\mathscr{F}_{pn+1}\mathscr{B} = \mathscr{G}_n\mathscr{A}$). These cochains are cocycles if either $pn-q$ is odd and $q = 0$, or $pn-q$ is even and $0 \leq q < p$.

**Proposition 2.6.** *If $pn-q$ is even and $0 \leq q < p-1$, then $[s^\sharp\phi^\rho_{pn-q,q+2}] = 0 \in H^{pn-q}(G/G(q+3); \mathscr{G}_n\mathscr{A})$, where $s : G/G(q+3) \to G/G\ (q+2)$ is the obvious projection.*

*Proof*. Consider the cochain $\phi^{\rho}_{pn-q-1,q+3} \in C^{pn-q-1}(G/G(q+3); \mathcal{G}_n\mathcal{A})$. It follows from Lemma 2.2 that, when $pn-q$ is even

$$\delta\phi^{\rho}_{pn-q-1,q+3} = s^{\sharp}\phi^{\rho}_{pn-q,q+2} \tag{23}$$

□

**Corollary 2.7.** *With the above notation, if* $pn$ *is even, we have that* $[s^{\sharp}\phi^{\rho}_{pn,2}] = 0 \in H^{pn}(G/G(3); \mathcal{G}_n\mathcal{A})$.

We can also identify a family of *secondary* cohomology classes, although we will not, at this time, explore the application of these to our considerations. For $pn-q$ odd, we define $\mu_{q,n} \overset{\text{def}}{=} j^{\sharp}\phi^{\rho}_{pn-q,q+2} \in C^{pn-q}(G(q+1)/G(q+2); \mathcal{G}_n\mathcal{A})$, where $j : G(q+1)/G(q+2) \to G/G(q+2)$ is the obvious inclusion. Since $s \circ j$ is trivial, it follows from equation (23) that $\mu_{q,n}$ is a cocycle. Clearly for $q > 0$,

$$\begin{aligned} &[\mu_{q,n}] \in H^{pn-q}(G(q+1)/G(q+2); \mathcal{G}_n\mathcal{A}) \\ &\text{and } [\phi^{\rho}_{pn-q+1,q+1}] \in H^{pn-q+1}(G/G(q+1); \mathcal{G}_n\mathcal{A}) \end{aligned}$$

are related by transgression in the fibration $G(q+1)/G(q+2) \to G/G(q+2) \to G/G(q+1)$, but neither one is determined by the other. The extra information is encoded in the particular cocycle representatives. If $q = 0$, then this gives nothing new since $[\mu_{0,n}] = [\phi^{\rho}_{pn,2}]$.

We end this section with a lemma that will be used in the proof of Theorem 5. Recall first the map $G \to \mathbb{Q}G$ given by $g \to 1-g$. According to Lemma 2.4 for every positive integer $n$, we get an induced map $G(n) \to I^n$, and thus a linear map:

$$\mathcal{G}_nG \otimes \mathbb{Q} \to I^n/I^{n+1} \tag{24}$$

Note that addition in $\mathcal{G}_nG \otimes \mathbb{Q}$ is given by group multiplication in $G$.

These maps can be assembled together in the following way. Recall first that $\mathcal{G}G \otimes \mathbb{Q}$ can be given the structure of a graded Lie algebra (over $\mathbb{Q}$). Let $U(\mathcal{G}G \otimes \mathbb{Q})$ denote the universal enveloping algebra. Note that $U(\mathcal{G}G \otimes \mathbb{Q})$ is a Hopf algebra. Note also that $\mathbb{Q}G$ is a filtered algebra with respect to powers of the augmentation ideal. Let $\mathcal{G}\mathbb{Q}G$ be the associated graded algebra, i.e., $\mathcal{G}_n\mathbb{Q}G = I^n/I^{n+1}$. Note that $\mathbb{Q}G$ is a Hopf algebra with comultiplication defined by $\Delta(g) = g \otimes g$ for $g \in G$. Then the maps of equation (24) induce a map:

$$U(\mathcal{G}G \otimes \mathbb{Q}) \to \mathcal{G}\mathbb{Q}G \tag{25}$$

This map was shown by Jennings (see Quillen [Qu]) to be a Hopf algebra isomorphism. In particular, the primitive elements of $\mathcal{G}\mathbb{Q}G$ are isomorphic to the Lie algebra $\mathcal{G}G \otimes \mathbb{Q}$.

We end the section with the following lemma:

**Lemma 2.8.** *For $x_i \in G$ we have the following identity in the graded quotient $I^n/I^{n+1}$:*

$$1 - [x_1, \ldots, [x_{n-1}, x_n]] = (-1)^{n-1} \sum_a \uparrow^{a(1)} (z_1, \ldots, \uparrow^{a(n-1)} (z_{n-1}, z_n)) \tag{26}$$

*where $z_i = 1 - x_i \in I$ and the summation is over all functions $a : \{1, 2, \ldots, n-1\} \to \{0, 1\}$, and we set $\uparrow^0 (a, b) = ab$ and $\uparrow^1 (a, b) = -ba$. Furthermore, the map* (24) *is a linear map.*

*Proof.* Using the identity (20) the first part follows by induction on $n$. Indeed, (20) implies that

$$\begin{aligned} 1 - [x_1, x_2] &\equiv -(1 - x_1)(1 - x_2) + (1 - x_2)(1 - x_1) \bmod I^3 \\ &= -(\uparrow^0 (1 - x_1, 1 - x_2) + \uparrow^1 (1 - x_1, 1 - x_2)) \bmod I^3 \end{aligned}$$

which concludes the proof of the first part for $n = 1$. The induction step follows the same way using identity (20).

The second part follows immediately using the following identity:

$$1 - ab = (1 - a)b + (1 - b)$$

□

**2.2. A review of finite type invariants of integral homology 3-spheres.** In this section we review some essential properties of finite type invariants of integral homology 3-sphere s that will be used in the present paper.

We begin by recalling the definition of the map (1) from [Oh], [GO1]: for an admissible (i.e., trivalent, vertex-oriented) graph $G$ with $3m$ edges and $2m$ vertices, let $L_w(G)$ denote the (linear combination of $2^{2m}$) algebraically split links in $S^3$ with framing $f = +1$ on each component obtained by choosing some of the vertices of $G$, replacing each non-chosen vertex of $G$ by a Borromean ring, each chosen vertex by a trivial 3-component link and each edge of $G$ by a band as in Fig. 3. The coefficient of that term is $(-1)^k$, where $k$ is the number of chosen vertices. Due to the fact that $G$ is an abstract graph (i.e., non-embedded in $S^3$), the links whose sum with signs is $L_w(G)$ are not well defined (modulo isotopy). Nevertheless, (with the notation of [GL1], [GO1]) one can associate a well defined element $[S^3, L_w(G), f] \in \mathcal{G}^{as}_{3m}\mathcal{M}$ in the associated graded space. This map was shown in [Oh] (see also [ GL1]) to be onto. Furthermore, in [GO1] it was shown that it actually descends to a map $\mathcal{G}_m\mathcal{A}(\phi) \to \mathcal{G}^{as}_{3m}\mathcal{M}$ which, therefore, is also onto. This defines the map (1). According to the *fundamental theorem* of finite type invariants of integral homology 3-spheres [LMO], [L], the map (1) is one-to-one, and therefore a vector space isomorphism. The isomorphisms of (1) can be assembled together for various $m$. Indeed, $\mathcal{A}(\phi)$ is equipped with a multiplication (induced by the disjoint union of graphs) and a comultiplication (induced by all ways of splitting a graph into its connected components), compatible with the grading, thus giving $\mathscr{A}(\phi)$ the structure of a commutative cocommutative Hopf algebra. Let $\hat{\mathscr{A}}(\phi)$ denote the completion. Furthermore, $\mathscr{M}$ is equipped with a multiplication (induced by connected sums of integral homology 3-spheres) and a comultiplication defined by $\Delta(M) = M \otimes M$ for an integral homology 3-sphere $M$, thus giving $\mathscr{F}^{as}_{\star}\mathscr{M}$ and $\mathscr{G}^{as}_{\star}\mathscr{M}$ the structure of a commutative cocommutative Hopf algebra. The above mentioned results of [LMO], [L] additionally imply that the maps (1) combine to give an isomorphism of Hopf algebras $\hat{\mathscr{A}}(\phi) \to \mathscr{G}^{as}_{*}\mathscr{M}$.

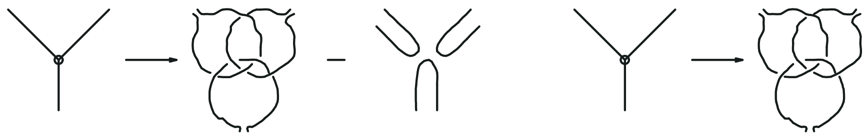


**Fig. 3.** Two maps from admissible graphs to (linear combinations of) algebraically split links in $S^3$. The map on the left is denoted by $G \to L_w(G)$ and the one on the right is denoted by $G \to L_b(G)$

In the rest of this section we recall several facts about the combinatorics of finite type invariants that will be used *exclusively* in the proof of Theorem 7. The reader may choose to postpone them until needed.

We begin with an equivalent description of the Hopf algebra $\mathscr{A}(\phi)$ taken from [GO1]. It turns out (see [GO1]) that there is a vector space (over $\mathbb{Q}$) $\mathscr{A}_b(\phi)$ on the set of vertex oriented graphs with univalent and trivalent vertices only, modulo an appropriate set of relations, described in detail in [GO1], together with a *deframing* map:

$$F : \mathscr{A}(\phi) \to \mathscr{A}_b(\phi) \tag{27}$$

defined as follows: for a vertex oriented trivalent graph $\Gamma$

$$F(\Gamma) = \sum_{s:v(\Gamma)\to\{0,1\}} (-1)^{|s^{-1}(1)|}\Gamma_s$$

where $\Gamma_s$ is obtained by splitting $\Gamma$ along every vertex $v$ such that $s(v) = 1$. We will not use explicitly the set of relations in $\mathscr{A}_b(\phi)$; note however [GO1] that $F$ is a vector space isomorphism, thus giving $\mathscr{A}_b(\phi)$ the structure of a graded Hopf algebra. For a trivalent vertex oriented graph $\Gamma$, let $\Gamma_w$ (resp., $\Gamma_b$) denote the associated element in $\mathscr{A}(\phi)$ (resp., $\mathscr{A}_b(\phi)$). The subscripts $w, b$ denote *white* and *black* vertices resp., the terminology is taken from [GO1]. Given a trivalent vertex oriented graph $\Gamma$, the associated elements under the maps $\mathscr{G}\mathscr{A}(\phi) \to \mathscr{G}\mathscr{M}$, $\mathscr{G}\mathscr{A}_b(\phi) \to \mathscr{G}\mathscr{M}$ are shown in the left and right hand of Fig. 3 and are denoted by $\Gamma \to [S^3, L_w(\Gamma), +1]$ and $\Gamma \to [S^3, L_b(\Gamma), +1]$ respectively. Putting together the isomorphism of (1) with the isomorphism (27) we get an isomorphism:

$$F' : \mathcal{G}_n\mathcal{A}_b(\phi) \simeq \mathcal{G}^{as}_{3n}\mathcal{M} \tag{28}$$

$\mathcal{A}(\phi)$ (resp., $\mathcal{A}_b(\phi)$) has a naturally defined ideal: $\mathcal{Y}_w = \Theta_w \cdot \mathcal{A}(\phi)$ (resp., $\mathcal{Y}_b = Y_b \cdot \mathcal{A}_b(\phi)$) where $\Theta_w$ (resp., $Y_b$) are the obvious generators of the degree 1 parts $\mathcal{G}_1\mathcal{A}(\phi)$ (resp., $\mathcal{G}_1\mathcal{A}_b(\phi)$). Note that the ideals $\mathcal{Y}_w, \mathcal{Y}_b$ correspond under the isomorphism $F$ of (27), since $F(\Theta_w) = 2Y_b$ (see [GO1]). Let $\tilde{\mathcal{A}}(\phi) = \mathcal{A}(\phi)/\mathcal{Y}_w, \tilde{\mathcal{A}_b}(\phi) = \mathcal{A}_b(\phi)/\mathcal{Y}_b$ denote the quotient spaces.

We now have the following very useful lemma:

**Lemma 2.9.** *Let* $\Gamma$ *denote a* connected *vertex oriented trivalent graph of degree* $n \neq 1$, *and let* $a \in \mathcal{G}_n\mathcal{A}^{conn}(\phi)$ *be such that*:

$$\Gamma_b \operatorname{mod} \mathcal{Y}_b = F(a \operatorname{mod} \mathcal{Y}_w) \in \mathcal{G}_n\tilde{\mathcal{A}_b}(\phi) \tag{29}$$

*Then, we have that* $\Gamma_w = a$.

*Proof*. Recall first from [GO1] that $F(\Gamma_w) \equiv \Gamma_b \operatorname{mod} \mathcal{Y}_b$. In fact more is true: namely $F(\Gamma_w) = \Gamma_b + k(\Gamma)(\coprod_n Y_b)$ for some integer $k(\Gamma)$. This follows by definition of the map $F$ of (27) and the relations in $\mathcal{G}_n\mathcal{A}_b(\phi)$. Therefore, we have that $F(\Gamma_w) \equiv F(a) \operatorname{mod} \mathcal{Y}_b$, thus $\Gamma_w \equiv a \operatorname{mod} \mathcal{Y}_w$, thus $\Gamma - a \equiv 0 \in \mathcal{G}_n\tilde{\mathcal{A}}(\phi)$. Recall that $\mathcal{A}(\phi)$ is graded by $\mathcal{G}$, and thus filtered, where $\mathcal{F}_n\mathcal{A}(\phi) = \oplus_{k \geq n}\mathcal{G}_k\mathcal{A}(\phi)$. Using the fact that $\mathcal{A}^{conn}(\phi) = \mathcal{A}(\phi)/(\mathcal{F}_1\mathcal{A}(\phi) \cdot \mathcal{F}_1\mathcal{A}(\phi))$, we can see that, for any $k \neq 1$, there is an *onto* map $\mathcal{G}_k\tilde{\mathcal{A}}(\phi) \to \mathcal{G}_k\mathcal{A}^{conn}(\phi)$. Moreover, the composite map $\mathcal{G}_k\mathcal{A}^{conn}(\phi) \to \mathcal{G}_k\tilde{\mathcal{A}}(\phi) \to \mathcal{G}_k\mathcal{A}^{conn}(\phi)$ is the identity map on $\mathcal{G}_k\mathcal{A}^{conn}(\phi)$. This finishes the proof of the lemma. $\square$

*Remark 2.10*. For $n = 1$ the above lemma is obviously not true, since we can take $\Gamma = \Theta$ and $a = 0$.

We now recall a few essential facts from [GL3] relating the various filtrations on $\mathcal{M}$ that will only be used in the proof of Theorem 7. Following the notation of [GL3], consider $f : \Sigma_g \hookrightarrow M$ an admissible genus $g$ surface. We need to recall from [GL3, Sect. 1.3] an important subgroup $\mathcal{L}^L_g$ of the mapping class group. We call a Lagrangian $L \subseteq H$ $f$*-compatible* if $L = (L \cap L^+) + (L \cap L^-)$, where $(L^+, L^-)$ is the associated Lagrangian pair of the admissible surface $f$. (For example, $L^+$ and $L^-$ themselves are $f$-compatible). For any $f$-compatible Lagrangian $L$, let $\mathcal{L}^L_g$ denote the subgroup of the mapping class group generated by *Dehn twists* along simple closed curves that homologically represent elements of $L$. Let $\mathcal{J}$ denote any of the subgroups $\mathcal{T}_g, \mathcal{K}_g, \mathcal{L}^L_g$ of the mapping class group. Consider the maps $\Phi^J_f : \mathbb{Q}\mathcal{J} \to \mathcal{M}$ defined the same way as the map $\Phi^T_f$ of (2). Let $\mathcal{G}\Phi^J_f : \mathcal{G}\mathbb{Q}\mathcal{J} \to \mathcal{G}^J\mathcal{M}$ denote the associated graded maps. Recall from Sect. 2.1 that $\mathcal{G}\mathbb{Q}\mathcal{J}$ is a coalgebra, and so is $\mathcal{G}^J\mathcal{M}$ (with the comultiplication of $\mathscr{G}^J\mathscr{M}$ induced by the one on $\mathscr{M}$). Then, with the above conventions we have the following lemma:

**Lemma 2.11.** *For $\mathscr{J}$ as above, the maps $\mathscr{G}\Phi_f^J : \mathscr{G}\mathbf{Q}\mathscr{J} \to \mathscr{G}^J\mathscr{M}$ are maps of coalgebras.*

*Proof*. Recalling that the coproduct on $\mathbf{Q}\mathscr{J}$ is defined by $\Delta(g) = g \otimes g$, and the coproduct on $\mathscr{M}$ defined by $\Delta(M) = M \otimes M$, it follows that the map $\Phi_f^J : \mathbf{Q}\mathscr{J} \to \mathscr{M}$ preserves the coalgebra structure. Therefore, the associated graded map preserves the coalgebra structure as well. □

**Corollary 2.12.** *For $\mathscr{J}$ as above, we get an induced map*:

$$\phi_f^J : \mathscr{G}\mathscr{J} \otimes \mathbf{Q} \to \mathscr{G}\mathscr{A}^{conn}(\phi) \tag{30}$$

*Proof*. We need to recall from [GL3] the following filtrations on $\mathscr{M}$: for a nonnegative integer $m$, $\mathscr{F}_m^T\mathscr{M}$ (resp., $\mathscr{F}_m^K\mathscr{M}, \mathscr{F}_m^L\mathscr{M}$) are defined to be the span of the images (over all admissible surfaces $f$) of $\Phi_f^T(I\mathscr{T}_g)^m$ (resp., $\Phi_f^K(I\mathscr{K}_g)^m$, $\Phi_f^L(I\mathscr{L}_g^{L^+})^m$). Let $\mathscr{F}_m^{H\mathscr{L}}\mathscr{M}$ denote the span of the images, for Heegaard surfaces $f$ and $f$-compatible Lagrangians $L$ of $\Phi_f^L((I\mathscr{L}_g^L)^m)$. In [GL3] we showed that $\mathscr{F}_m^{H\mathscr{L}}\mathscr{M} = \mathscr{F}_m^L\mathscr{M}$, and from now on we will identify these two filtrations. The filtrations considered above can be compared to the $\mathscr{F}^{as}$ filtration on $\mathscr{M}$ as follows [GL3, Corollary 1.20]:

$$\mathscr{F}_m^K\mathscr{M} \subseteq \mathscr{F}_{2m}^T\mathscr{M} = \mathscr{F}_{2m-1}^T\mathscr{M} = \mathscr{F}_{3m}^L\mathscr{M} = \mathscr{F}_{3m-2}^L\mathscr{M} = \mathscr{F}_{3m}^{as}\mathscr{M}$$

inducing associated graded maps

$$\mathscr{G}_m^K\mathscr{M} \to \mathscr{G}_{2m}^T\mathscr{M} = \mathscr{G}_{2m-1}^T\mathscr{M} = \mathscr{G}_{3m}^L\mathscr{M} = \mathscr{G}_{3m-2}^L\mathscr{M} = \mathscr{G}_{3m}^{as}\mathscr{M}$$

Using the isomorphism of Hopf algebras $\mathscr{G}_\star\mathscr{A}(\phi) \simeq \mathscr{G}_\star^{as}\mathscr{M}$ the above graded maps and Lemma 2.11 show that there are coalgebra maps $\mathscr{G}\Phi_f^J : \mathscr{G}\mathbf{Q}\mathscr{J} \to \mathscr{G}\mathscr{A}(\phi)$, which induce maps $\phi_f^J$ on the primitive elements. Recall finally from section 2.1 that the subspace of primitive elements of $\mathscr{G}\mathbf{Q}\mathscr{J}$ is the Lie algebra $\mathscr{G}\mathscr{J} \otimes \mathbf{Q}$, and that the subspace of primitive elements of $\mathscr{G}\mathscr{A}(\phi)$ is $\mathscr{G}\mathscr{A}^{conn}(\phi)$. □

*Remark 2.13*. $\mathscr{G}^J\mathscr{M}$ is a Hopf algebra and the map $\mathscr{G}^J\mathscr{M} \to \mathscr{G}\mathscr{A}(\phi)$ is a Hopf algebra isomorphism for $\mathscr{J} = \mathscr{T}_g$ and $\mathscr{L}_g^L$ and a Hopf algebra epimorphism for $\mathscr{J} = \mathscr{K}_g$, see [GL2]. Furthermore, $\mathscr{G}\mathbf{Q}\mathscr{J}$ is also a Hopf algebra, see Sect. 2.1. The map $\mathscr{G}\Phi_f^J$ however is *not* an algebra map.

We close the section with the following lemma which will be used in the proof of Theorem 5.

**Lemma 2.14.** *Given an admissible surface and compatible Lagrangian L, we have inclusions* $\mathscr{L}_g^L \supseteq \mathscr{K}_g \subseteq \mathscr{T}_g$. *Then*:

$$\Phi_f^L|_{\mathbb{Q}\mathscr{K}_g} = \Phi_f^T|_{\mathbb{Q}\mathscr{K}_g} \tag{31}$$

*Proof*. This is a straightforward consequence of the definitions of the maps. □

**2.3. Johnson's homomorphism and representation theory.** In this section we review well known properties of the Johnson homomorphism, [Jo1], and some essential facts about representation theory of the symplectic group.

We begin with the following:

*Remark 2.15*. Even though in the present paper we are interested mainly in closed surfaces embedded in closed 3-manifolds, we will for a variety of reasons, also consider surfaces *with boundary*. These reasons include $(a)$ historical traditions [Jo1], [Mo2], [Mo3], $(b)$ technical reasons (the fact that the fundamental group of a closed surface is not free, whereas that of one with boundary is. Also, one can glue surfaces along boundary to increase the genus and consider stability problems, whereas there is no canonical way of increasing the genus of closed surfaces), and $(c)$ modern interpretation in terms of open string field theory [Ko1, Ko2, Wi1, Wi2]. For all of the above reasons, we usually first decorate surfaces by boundary components or punctures, and only afterwards do we discuss closed surfaces. At any rate, the reader should keep in mind that there are exact sequences that relate invariants of decorated surfaces to invariants of closed surfaces.

Let $\Sigma_g$ denote a closed, oriented surface of genus $g$, and let $D \subseteq \Sigma_g$ be a fixed embedded disk. Let $\Sigma_{g,1}$ denote the associated surface $\Sigma - \mathrm{Int}D$ with one boundary component. Let $\Gamma_g$ (resp. $\Gamma_{g,1}$) denote the mapping class group, i.e., the group of isotopy classes of orientation preserving surface diffeomorphisms (resp. that are identity on the boundary). Let $\mathscr{T}_g$ (resp. $\mathscr{T}_{g,1}$) (the Torelli group) denote the subgroup of $\Gamma_g$ (resp. $\Gamma_{g,1}$) of elements that act trivially on the homology of the surface. Let $H = H_1(\Sigma_g, \mathbb{Z})$, and $\omega$ be the intersection form. Note that the inclusion $\Sigma_{g,1} \hookrightarrow \Sigma_g$ induces a canonical isomorphism $H_1(\Sigma_{g,1}, \mathbb{Z}) \simeq H$, and in this section we will identify $H_1(\Sigma_{g,1}, \mathbb{Z})$ with $H$. The groups $\Gamma_g, \Gamma_{g,1}, \mathscr{T}_g, \mathscr{T}_{g,1}$ are related in the following (exact) commutative diagram [Jo1]:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \pi_1(T_\Sigma) & \longrightarrow & \mathscr{T}_{g,1} & \longrightarrow & \mathscr{T}_g & \longrightarrow & 1 \\
 & & \| & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \pi_1(T_\Sigma) & \longrightarrow & \Gamma_{g,1} & \longrightarrow & \Gamma_g & \longrightarrow & 1 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & Sp(H) & \longrightarrow & Sp(H) & &
\end{array}$$

where $T_\Sigma$ denotes the unit tangent bundle of the surface $\Sigma_g$. Note that the two rightmost vertical sequences are also short exact.

With the above notation, we can recall a few essential facts from representation theory. For proofs, we refer the reader to [FH]. Recall that for an abelian group $A$, we let $A_{\mathbb{Q}}$ denote the rational vector space $A \otimes_{\mathbb{Z}} \mathbb{Q}$. All the linear maps to be described in this section will be $Sp(H_{\mathbb{Q}})$ equivariant. Recall first that the intersection form $\omega \in \Lambda^2 H^*$ induces an isomorphism $H \simeq H^*$. Let $H \to \Lambda^3 H$ denote the map defined by $x \to x \wedge \omega$ (where we think of $\omega \in \Lambda^2 H$, via the isomorphism $H \simeq H^*$). In terms of a symplectic basis $\{x_i, y_i\}$ of $H$, the above map is given by $x \to \sum_i x \wedge x_i \wedge y_i$. Let $U = \Lambda^3 H/H$ denote the quotient. Note that our notation differs from [ Mo6] (Morita denotes $\frac{1}{2}\Lambda^3 H/H$ by $U$). Since both Morita and we are only dealing with rational results, this is not really a problem.

We can think of $\Lambda^3 H_{\mathbb{Q}}$ as a quotient module of $\otimes^3 H_{\mathbb{Q}}$ (in the natural way), or as a submodule of $\otimes^3 H_{\mathbb{Q}}$ as follows: we let $\Lambda^3 H_{\mathbb{Q}} \hookrightarrow \otimes^3 H_{\mathbb{Q}}$ be the (one-to-one) map: $x_1 \wedge x_2 \wedge x_3 \to \sum_{s \in \mathrm{Sym}_3} \mathrm{sgn}(s) x_{s(1)} \wedge x_{s(2)} \wedge x_{s(3)}$, where $\mathrm{Sym}_3$ denotes the symmetric group and *sgn* denotes the sign homomorphism. Note that we do not divide out the above map by $1/6$. As a result, the composite map $\Lambda^3 H_{\mathbb{Q}} \to \otimes^3 H_{\mathbb{Q}} \to \Lambda^3 H_{\mathbb{Q}}$ is multiplication by 6. Let $\otimes^3 H \to H$ denote the map $x_1 \otimes x_2 \otimes x_3 \to \omega(x_1, x_2) x_3$, and let $\Lambda^3 H \to H$ denote the composite map $\Lambda^3 H \hookrightarrow \otimes^3 H \to H$. More explicitly, the above composite map is the following:

$$x_1 \wedge x_2 \wedge x_3 \to 2(\omega(x_1, x_2) x_3 - \omega(x_1, x_3) x_2 + \omega(x_2, x_3) x_1) \tag{32}$$

It is easy to see that there is a rational isomorphism $U_{\mathbb{Q}} \simeq \mathrm{Ker}(\Lambda^3 H_{\mathbb{Q}} \to H_{\mathbb{Q}})$.

In his pioneering work [Jo1], [Jo2] D. Johnson described a homomorphism $\tau : \mathscr{T}_{g,1} \to \Lambda^3 H$. We briefly summarize its properties here:

- $\tau$ is onto.
- $\tau$ is equivariant with respect to the conjugation action of the mapping class group $\Gamma_{g,1}$ on $\mathscr{T}_{g,1}$ and the natural action of the symplectic group $Sp(H)$ on $\Lambda^3 H$.
- $\tau$ coincides, modulo 2 torsion, with the abelianization of the Torelli group. In fact, $[\mathscr{T}_{g,1}, \mathscr{T}_{g,1}] \subset \mathrm{Ker}(\mathscr{T}_{g,1} \to \Lambda^3 H)$ is a normal subgroup with quotient a 2-group.
- $\tau$ factors through a map $\tau : \mathscr{T}_g \to U$ (denoted by the same name) making the following diagram commute:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \pi_1(T_\Sigma) & \longrightarrow & \mathscr{T}_{g,1} & \longrightarrow & \mathscr{T}_g & \longrightarrow & 1 \\
 & & \downarrow & & \tau\downarrow & & \tau\downarrow & & \\
0 & \longrightarrow & H & \longrightarrow & \Lambda^3 H & \longrightarrow & U & \longrightarrow & 1
\end{array}$$

Furthermore, $\mathscr{T}_g \to U$ is equivariant, onto, and coincides (modulo 2 torsion) with the abelianization of $\mathscr{T}_g$. From this and the preceding property we have that $\mathscr{K}_g = \mathscr{T}_g(2)$ and that $\tau : \mathscr{T}_g/\mathscr{T}_g(2) \otimes \mathbb{Q} \simeq U_{\mathbb{Q}}$.

- $\tau$ is stable with respect to an inclusion $\Sigma_{g,1} \hookrightarrow \Sigma_{g+h,1}$.

**2.4. Representations of the symplectic and the general linear group.** In this section we review a few essential facts about representations of the symplectic and the general linear group. The main result is Proposition 2.18 which will be used in the proof of Theorem 3.

Let $(H, \omega)$ denote a symplectic space, and assume a given splitting $H = L^+ \oplus L^-$ into two Lagrangian subspaces. An example is given by the Lagrangian pair of an admissible surface in an integral homology 3-sphere. Let us denote $L^+$ by $V$; then, the symplectic form induces isomorphisms $L^- \simeq V^*$ and $H \simeq V \oplus V^*$. In this section we will identify $L^+$, $L^-$ and $H$ with $V, V^*$ and $V \oplus V^*$, respectively.

Consider the subgroup of the symplectic group $Sp(H)$ formed by all matrices of the form $\begin{pmatrix} A & 0 \\ 0 & (A^t)^{-1} \end{pmatrix}$, where $A \in GL(V)$ and $A^t$ stands for the transpose of a matrix $A$. This subgroup of $Sp(H)$ is obviously isomorphic to the group $GL(V)$. Note that the action of $GL(V)$ on $H$ preserves the decomposition $H = V \oplus V^*$ and as a subgroup of $Sp(H)$ extends to an action on $\Lambda^3 H$ and $U$.

We will be mainly concerned with describing a generating set for the vector space of invariants $(\otimes^{2m} U_{\mathbb{Q}})^{GL(V_{\mathbb{Q}})}$. First, however, we need to recall several ideas about irreducible representations of $Sp(H_{\mathbb{Q}})$ and $GL(V_{\mathbb{Q}})$. For more details see [FH] and [KK].

We begin by recalling some results about the invariant theory of the symplectic group as formulated by Morita [Mo6]. There is a one-to-one correspondence between irreducible $Sp(H_{\mathbb{C}})$ representations and *dominant integral weights* (with respect to a standard choice of a Weyl chamber). Let us denote by $V(\lambda)$ the rational representation associated to weight $\lambda$. Dominant integral weights are parametrized as follows: if $dim(H_{\mathbb{Q}}) = 2n$ and $\{\epsilon_i\}_{i=1}^n$ is the set of dominant weights, then every dominant integral weight can be written uniquely in the form $\sum_i f_i \epsilon_i$ for integers $f_i$ such that $f_1 \geq f_2 \ldots \geq f_n \geq 0$. It is often customary to denote the representation $V(\lambda)$ by a *Young diagram* of $n$ rows with $f_i$ boxes on each row. Due to typographical limitations though, we will not denote them by Young diagrams. In this language we have the following identifications (as $Sp(H_{\mathbb{Q}})$ representations):

$$H_{\mathbb{Q}} = V(\epsilon_1) \qquad \Lambda^3 H_{\mathbb{Q}} = V(\epsilon_3) + V(\epsilon_1) \qquad U_{\mathbb{Q}} = V(\epsilon_3)$$

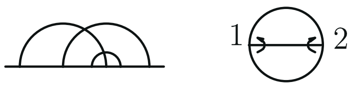
Before we state the next result, we need a few definitions: a degree $m$ *linear chord diagram* is an involution on the set $\{1, \ldots, 2m\}$ without fixed points [B–N]. Let $\mathcal{G}_m \mathcal{D}^l$ denote the vector space over $\mathbb{Q}$ on the set of linear chord diagrams with $m$ chords. It is easy to see that $\mathcal{G}_m \mathcal{D}^l$ is a vector space of dimension $(2m-1)!! = 1.3.5. \ldots (2m-1)$.

The symplectic form gives a $Sp(H_{\mathbb{Q}})$ invariant map $\omega : H_{\mathbb{Q}} \otimes H_{\mathbb{Q}} \to \mathbb{Q}$, thus given a degree $m$ linear chord diagram, we get an induced $Sp(H_{\mathbb{Q}})$

invariant map $\otimes^{2m} H_{\mathbb{Q}} \to \mathbb{Q}$, and dually (using the isomorphism $H \simeq H^*$ induced by the symplectic form) a $Sp(H_{\mathbb{Q}})$ invariant element in $\otimes^{2m} H_{\mathbb{Q}}$. The definition is clear from the example shown in figure 4. Thus we have a map:

$$\mathscr{G}_m \mathscr{D}^l \to (\otimes^{2m} H_{\mathbb{Q}})^{Sp(H_{\mathbb{Q}})} \tag{33}$$

According to the *first fundamental theorem* of representation theory (see [W, p. 167]), the above map (33) is onto, and according to the *second fundamental theorem* of representation theory [W, p. 168]), provided $\dim(H_{\mathbb{Q}}) = 2n \geq 2m$, (33) is one-to-one and therefore a vector space isomorphism.




$$v_1 \otimes v_2 \otimes v_3 \otimes v_4 \otimes v_5 \otimes v_6 \to \omega(v_1, v_4)\omega(v_2, v_6)\omega(v_3, v_5)$$

**Fig. 4.** A degree 3 linear chord diagram and its associated trivalent graph. The trivalent graph $G$ comes equipped with an ordering $od_{V(G)}$ of its vertices, as well as with a vertex orientation $or_{V(G)}$, indicated by a choice of cyclic order of the edges around each vertex. The linear chord diagram corresponds (under the map (33)) to the contraction shown $\otimes^6 H \to \mathbb{Q}$ shown in the figure

Our next goal is to describe the invariant spaces $(\otimes^{2m} \Lambda^3 H_{\mathbb{Q}})^{Sp(H_{\mathbb{Q}})}$. In order to do so, we need one more definition. Given a degree $3m$ linear chord diagram, its associated trivalent graph is defined as follows: the set of vertices is the quotient set $\{1, 2, \ldots, 6m\}/\sim$, modulo the relation $3j-2 \sim 3j-1 \sim 3j$ (for $1 \leq j \leq m$), and the set of edges is given by the quotient map of the chord diagram. There is an orientation at every vertex, induced by the ordering $3j-2 < 3j-1 < 3j$ of the edges around it. For an example, see Fig. 4. The trivalent graphs constructed above have extra data: they come equipped with an ordering $od_{V(G)}$ of the vertices. Let $\mathscr{G}_m \mathscr{A}^{rp}$ denote the vector space over $\mathbb{Q}$ on the set of isomorphism classes of tuples $(G, od_{V(G)}, or_{V(G)})$ divided out by the usual *antisymmetry* relation (denoted by $AS$ in Fig. 1). Here $G$ is a trivalent graph with $3m$ edges and $2m$ vertices, $od_{V(G)}$ is an ordering of its vertices and $or_{V(G)}$ is a vertex orientation of $G$, i.e., a choice of cyclic order for the 3 edges that emanate through each vertex of $G$. The above discussion defines a map:

$$\mathscr{G}_{3m} \mathscr{D}^l \to \mathscr{G}_m \mathscr{A}^{rp} \tag{34}$$

Due to the projection $\otimes^3 H \to \Lambda^3 H$ of Sect. 2.3, and the choice of cyclic order for the above mentioned trivalent graphs, the map of equation (33) factors through the map of (34), thus inducing a map:

$$\mathscr{G}_m \mathscr{A}^{rp} \to (\otimes^{2m} \Lambda^3 H_{\mathbb{Q}})^{Sp(H_{\mathbb{Q}})} \tag{35}$$

Furthermore, there are natural inclusion maps $\mathcal{G}_m\mathcal{A}^{rp} \to \mathcal{G}_{3m}\mathcal{D}^l$ and $(\otimes^{2m}\Lambda^3 H_{\mathbb{Q}})^{Sp(H_{\mathbb{Q}})} \to \mathcal{G}_m\mathcal{A}(\phi)$, such that the composite maps $\mathcal{G}_m\mathcal{A}^{rp} \to \mathcal{G}_{3m}\mathcal{D}^l \to \mathcal{G}_m\mathcal{A}^{rp}$ and $(\otimes^{2m}\Lambda^3 H_{\mathbb{Q}})^{Sp(H_{\mathbb{Q}})} \to (\otimes^{6m} H_{\mathbb{Q}})^{Sp(H_{\mathbb{Q}})} \to (\otimes^{2m}\Lambda^3 H_{\mathbb{Q}})^{Sp(H_{\mathbb{Q}})}$ are multiplication by a nonzero scalar. Moreover, we have two commutative diagrams:

$$
\begin{array}{ccc}
\mathcal{G}_{3m}\mathcal{D}^l & \longrightarrow & (\otimes^{6m} H_{\mathbb{Q}})^{Sp(H_{\mathbb{Q}})} \\
\big\downarrow & & \big\downarrow \\
\mathcal{G}_m\mathcal{A}^{rp} & \longrightarrow & (\otimes^{2m}\Lambda^3 H_{\mathbb{Q}})^{Sp(H_{\mathbb{Q}})}
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
\mathcal{G}_m\mathcal{A}^{rp} & \longrightarrow & (\otimes^{2m}\Lambda^3 H_{\mathbb{Q}})^{Sp(H_{\mathbb{Q}})} \\
\big\downarrow & & \big\downarrow \\
\mathcal{G}_{3m}\mathcal{D}^l & \longrightarrow & (\otimes^{6m} H_{\mathbb{Q}})^{Sp(H_{\mathbb{Q}})}
\end{array}
$$

where the vertical maps on the left diagram are onto, and on the right diagram are one-to-one. Using the fact that the map of (33) is an isomorphism (provided that $n \geq m$) and the above commutative diagrams, it is easy to see that the map (35) is a vector space isomorphism, provided that $n \geq 3m$.

Finally, we describe a generating set for the invariant vector space $(\otimes^{2m} U_{\mathbb{Q}})^{Sp(H_{\mathbb{Q}})}$: let $\mathcal{G}_m\mathcal{A}^{rp,nl}$ denote the quotient space $\mathcal{G}_m\mathcal{A}^{rp}$ divided out by the subspace of tuples $(G, od_{V(G)}, or_{V(G)})$, where $G$ contains a loop. For an example of a loop, see Fig. 5.

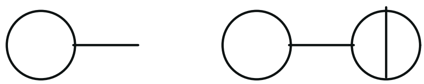


**Fig. 5.** An example of a loop on the left and of a trivalent graph containing a loop on the right

Due to the projection $\Lambda^3 H_{\mathbb{Q}} \to U_{\mathbb{Q}}$ it is easy to see that the map of (35) induces a map:

$$\mathcal{G}_m\mathcal{A}^{rp,nl} \to (\otimes^{2m} U_{\mathbb{Q}})^{Sp(H_{\mathbb{Q}})} \tag{36}$$

Using the fact of an inclusion $U_{\mathbb{Q}} \simeq \mathrm{Ker}(\Lambda^3 H_{\mathbb{Q}} \to H_{\mathbb{Q}}) \subseteq \Lambda^3 H_{\mathbb{Q}}$ and the same reasoning as that of $\Lambda^3 H_{\mathbb{Q}}$ above, it follows that equation (36) is an isomorphism, provided $n \geq 3m$. The above isomorphism has already been discussed in previous work of Morita [Mo6, p. 11] which has been a source of inspiration for us. The reason that we recall it here in detail is to show the similarities and differences between the invariant theory of $Sp(H_{\mathbb{Q}})$ and the invariant theory of the general linear group $GL(V_{\mathbb{Q}}) \hookrightarrow Sp(H_{\mathbb{Q}})$, to which we now turn.

We begin by recalling a few standard facts about representations of $GL(V_{\mathbb{Q}})$. In this case too, it turns out that there is a one-to-one correspondence between irreducible $GL(V_{\mathbb{C}})$ representations and *dominant integral weights* (with respect to a standard choice of a Weyl chamber). Let us denote by $V(\lambda)$ the representation (over $\mathbb{Q}$) associated to weight $\lambda$. Dominant integral weights are parametrized as follows: if $\dim(V_{\mathbb{Q}}) = n$ and $\{\epsilon_i\}_{i=1}^n$ is the set of dominant weights, then every dominant integral weight can be written uniquely in the form $\sum_i f_i \epsilon_i$ for integers $f_i$ such that $f_1 \geq f_2 \ldots \geq f_n$. (In case $f_n \geq 0$, such representations are called *polynomial*, and a usual graphical way of representing them is by a Young diagram of $n$ rows with $f_i$ boxes on each row. However, if $V(\lambda)$ is not polynomial, there is *no* graphical way to represent it. Since we will be dealing with non-polynomial representations of $GL(V_{\mathbb{Q}})$, we will not use the Young diagram method.) In this language we have the following lemma:

**Lemma 2.16.** *As representations of* $GL(V_{\mathbb{Q}})$, *we have the following decomposition*:

$$\begin{aligned} H_{\mathbb{Q}} &= V(\epsilon_1) + V(-\epsilon_n) \\ \Lambda^3 H_{\mathbb{Q}} &= V(\epsilon_1 + \epsilon_2 + \epsilon_3) + V(\epsilon_1) + V(\epsilon_1 + \epsilon_2 - \epsilon_n) \\ &\quad + V(-\epsilon_n) + V(\epsilon_1 - \epsilon_{n-1} - \epsilon_n) + V(-\epsilon_{n-2} - \epsilon_{n-1} - \epsilon_n) \\ U_{\mathbb{Q}} &= V(\epsilon_1 + \epsilon_2 + \epsilon_3) + V(\epsilon_1 + \epsilon_2 - \epsilon_n) \\ &\quad + V(\epsilon_1 - \epsilon_{n-1} - \epsilon_n) + V(-\epsilon_{n-2} - \epsilon_{n-1} - \epsilon_n) \end{aligned}$$

*Proof.* The first part is obvious. The second one follows using the identity

$$\Lambda^3(V + V^*) = \Lambda^3 V + \Lambda^2 V \otimes V^* + V \otimes \Lambda^2 V^* + \Lambda^3 V^*$$

together with the facts that

$$\Lambda^3 V = V(\epsilon_1 + \epsilon_2 + \epsilon_3) \qquad \Lambda^2 V \otimes V^* = V(\epsilon_1) + V(\epsilon_1 + \epsilon_2 - \epsilon_n)$$

The third part follows from first and the second. We thank D. Vogan for a crash course on representation theory of $GL(V_{\mathbb{Q}})$. $\square$

Before we state the main proposition of this section, we need a few more definitions:

**Definition 2.17.** A **2-coloring** *of a linear chord diagram* (*resp., of a graph*) *is an orientation for each of the edges of the chord diagram* (*resp., graph*). *For examples, see Fig.* 6. *Let* $\mathcal{G}_m\mathcal{D}^{l,cl}$ *denote the vector space over* $\mathbb{Q}$ *on the set of* 2*-colored linear chord diagrams with m chords. It is easy to see that* $\mathcal{G}_m\mathcal{D}^{l,cl}$ *is a vector space of dimension* $(2m-1)!! \ldots 2^m = 1.3.5.\ldots(2m-1)2^m$. *Let* $\mathcal{G}_m\mathcal{A}^{rp,cl}$ *denote the vector space over* $\mathbb{Q}$ *on the set of isomorphism classes of tuples* $(G, od_{V(G)}, or_{V(G)}, cl_{E(G)})$ *divided out by the* **colored antisymmetry** *relation* (*denoted by* $AS^{cl}$) *of figure* 7. Here G is a trivalent graph with 3m edges and 2m vertices, $od_{V(G)}$ *is an ordering of its vertices*, $or_{V(G)}$ *is a vertex orientation of G*, (*i.e., a choice of cyclic order for the* 3 *edges that emanate through each vertex of G*) *and* $cl_{E(G)}$ *is a* 2*-coloring of the edges* $E(G)$ *of G. Let*

*$\mathcal{G}_m\mathcal{A}^{rp,nl,cl}$ denote the quotient space of $\mathcal{G}_m\mathcal{A}^{rp,cl}$ divided out by all graphs which contain a loop. As is maybe apparent from the notation, the superscripts on $\mathcal{A}$ are explained as follows: rp stands for representation theory, nl stands for no loops and cl stands for 2-coloring (of the edges).*

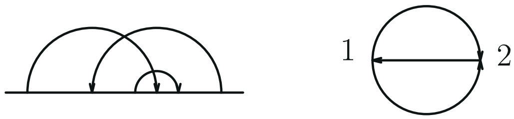



**Fig. 6.** An example of a 2-coloring of a linear chord diagram and of its associated trivalent graph


**Fig. 7.** The colored antisymmetry relation on 2-colored, vertex oriented trivalent graphs. All edges are 2-colored by the choice of an arrow. The left hand side corresponds to 8 identities (for all possible 2-colorings of the edges of the $Y$ graph). Similarly, the right hand side corresponds to 2 identities

With the notation as in the above definition, we are ready to state the main result of this section:

**Proposition 2.18.** *For every nonnegative integer m, there are onto maps:*

$$\mathcal{G}_m\mathcal{D}^{l,cl} \to (\otimes^{2m} H_{\mathbb{Q}})^{GL(V_{\mathbb{Q}})}$$
$$\mathcal{G}_m\mathcal{A}^{rp,cl} \to (\otimes^{2m} \Lambda^3 H_{\mathbb{Q}})^{GL(V_{\mathbb{Q}})}$$
$$\mathcal{G}_m\mathcal{A}^{rp,nl,cl} \to (\otimes^{2m} U_{\mathbb{Q}})^{GL(V_{\mathbb{Q}})}$$

*which are isomorphisms provided that $n \geq m$, $n \geq 3m$ and $n \geq 3m$ respectively.*

*Proof.* Recall first that $H = V \oplus V^*$, thus we get two $GL(V_{\mathbb{Q}})$ invariant maps: $H_{\mathbb{Q}} \otimes H_{\mathbb{Q}} \to \mathbb{Q}$, obtained as follows:

$$H_{\mathbb{Q}} \otimes H_{\mathbb{Q}} \to V_{\mathbb{Q}} \otimes V_{\mathbb{Q}}^* \to \mathbb{Q}$$
$$H_{\mathbb{Q}} \otimes H_{\mathbb{Q}} \to V_{\mathbb{Q}}^* \otimes V_{\mathbb{Q}} \to \mathbb{Q}$$

by projecting $H_{\mathbb{Q}} \to V_{\mathbb{Q}}$ or $H_{\mathbb{Q}} \to V_{\mathbb{Q}}^*$ appropriately. A choice between each of the above mentioned invariant maps will be associated with a 2-coloring. Therefore, given a degree $m$ 2-colored linear chord diagram, we get an invariant map $\otimes^{2m} H_{\mathbb{Q}} \to \mathbb{Q}$, and dually (using the isomorphism $H \simeq H^*$) a $GL(V_{\mathbb{Q}})$ invariant element in $\otimes^{2m} H_{\mathbb{Q}}$. This defines a map: $\mathcal{G}_m\mathcal{D}^{l,cl} \to$

$(\otimes^{2m} H_{\mathbb{Q}})^{GL(V_{\mathbb{Q}})}$, which according to the *first fundamental theorem* of invariant theory is onto, and according to the *second fundamental theorem* (provided that $n \geq m$) is one-to-one and thus a vector space isomorphism. This shows the first part of Proposition 2.18. Next, recall the map of (34) that assigns to each degree $3m$ linear chord diagram its associated degree $m$ trivalent graph. This map by definition respects the 2-coloring of linear chord diagrams and trivalent graphs and therefore descends to onto maps:

$$\mathscr{G}_{3m}\mathscr{D}^{l,cl} \to \mathscr{G}_m\mathscr{A}^{rp,cl} \text{ and } \mathscr{G}_{3m}\mathscr{D}^{l,cl} \to \mathscr{G}_m\mathscr{A}^{rp,cl,nl} \tag{37}$$

Arguing as in (34) and (36), due to the projections $\otimes^3 H_{\mathbb{Q}} \to \Lambda^3 H_{\mathbb{Q}}$ and $\otimes^3 H_{\mathbb{Q}} \to \Lambda^3 H_{\mathbb{Q}}$ we obtain that the map $\mathscr{G}_m\mathscr{D}^{l,cl} \to (\otimes^{2m} H_{\mathbb{Q}})^{GL(V_{\mathbb{Q}})}$ induces quotient maps:

$$\mathscr{G}_m\mathscr{A}^{rp,cl} \to (\otimes^{2m}\Lambda^3 H_{\mathbb{Q}})^{GL(V_{\mathbb{Q}})} \text{ and } \mathscr{G}_m\mathscr{A}^{rp,nl,cl} \to (\otimes^{2m} U_{\mathbb{Q}})^{GL(V_{\mathbb{Q}})} \tag{38}$$

The same reasoning as that of (34) and (36) together with the isomorphism of the first part of the proposition implies the rest of Proposition 2.18. □

**Corollary 2.19.** *In particular, for* $m = 1$, *we have the following table of dimensions for the various invariant spaces*:

| A | dim $(A)^{Sp(H_{\mathbb{Q}})}$ | dim $(A)^{GL(H_{\mathbb{Q}})}$ |
|---|---|---|
| $\oplus^6 H_{\mathbb{Q}}$ | $5!! = 15$ | $2^3.5!! = 120$ |
| $\oplus^6 \Lambda^3 H_{\mathbb{Q}}$ | 2 | 6 |
| $\oplus^2 U_{\mathbb{Q}}$ | 1 | 4 |

*In terms of the isomorphisms of Proposition* 2.18, *the graphs in Fig*. 8 *form a basis for the invariant spaces* $(\Lambda^3 H_{\mathbb{Q}})^{Sp(H_{\mathbb{Q}})}$, $(U_{\mathbb{Q}})^{Sp(H_{\mathbb{Q}})}$, $(\Lambda^3\ H_{\mathbb{Q}})^{GL(V_{\mathbb{Q}})}$ *and* $(U_{\mathbb{Q}})^{GL(V_{\mathbb{Q}})}$.

*Proof*. For $m = 1$, there are only two trivalent graphs with 3 edges and no decorations. After including the possible decorations and taking into account the colored $AS$ relation of Fig. 7, we arrive at the table of Fig. 8. □

Two remarks are in order:

*Remark 2.20*. An alternative way of counting the dimensions of the invariant spaces $(\Lambda^3 H_{\mathbb{Q}})^{Sp(H_{\mathbb{Q}})}$, $(U_{\mathbb{Q}})^{Sp(H_{\mathbb{Q}})}$, $(\Lambda^3 H_{\mathbb{Q}})^{GL(V_{\mathbb{Q}})}$ and $(U_{\mathbb{Q}})^{GL(V_{\mathbb{Q}})}$ is by decomposing into irreducible representations and using Schur's lemma. Indeed, since $\Lambda^3 H_{\mathbb{Q}}^* \simeq \Lambda^3 H_{\mathbb{Q}}$, we obtain:

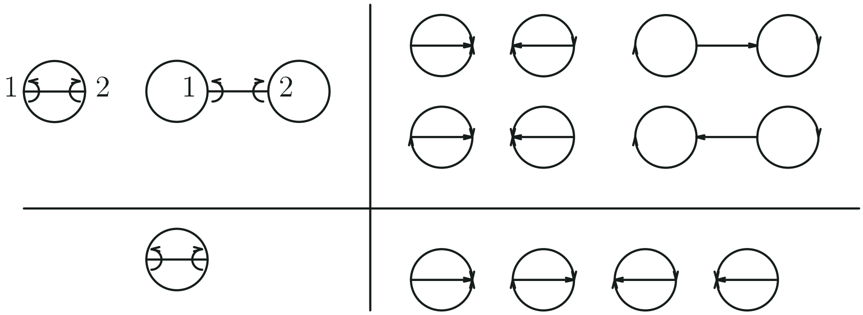



**Fig. 8.** Basis for the invariant spaces corresponding to the last two rows and columns of the table of Corollary 2.19. Note that all graphs are vertex ordered, vertex oriented and 2-colored. For simplicity, the ordering and the orientation of the vertices is indicated only in the northwest part of the figure

$$\begin{aligned}(\otimes^2 \Lambda^3 H_{\mathbb{Q}})^{GL(V_{\mathbb{Q}})} &= (\Lambda^3 H_{\mathbb{Q}}^* \otimes \Lambda^3 H_{\mathbb{Q}})^{GL(V_{\mathbb{Q}})} \\ &= \mathrm{Hom}_{GL(V_{\mathbb{Q}})}(\Lambda^3 H_{\mathbb{Q}}, \Lambda^3 H_{\mathbb{Q}})\end{aligned}$$

Using Lemma 2.16, we see that $\Lambda^3 H_{\mathbb{Q}}$ is a sum of six irreducible nonisomorphic representations of $GL(V_{\mathbb{Q}})$; therefore it follows by *Schur's lemma* that the dimension of the invariant space $\mathrm{Hom}_{GL(V_{\mathbb{Q}})}(\Lambda^3 H_{\mathbb{Q}}, \Lambda^3 H_{\mathbb{Q}})$ is six. We can deduce the rest of the dimensions of the invariant spaces the same way as above. Note however, that this alternative way can only provide us with a dimension count but not with a choice of basis for the above mentioned invariant spaces.

*Remark 2.21.* On the one hand, weights (or Young diagrams) are a classical and convenient way of parametrizing irreducible representations of classical Lie groups. On the other hand, trivalent graphs seem to be a very convenient way of parametrizing invariant tensors of representations of classical groups.

## 3. Proofs

### *3.1. Proof of Theorem 1 and Addenda*

Proof. [of Theorem 1] We fix an admissible surface $f : \Sigma \to M$ and consider the (discrete) group $G = \mathscr{T}_g$ as in Sect. 2.1. For every nonnegative integer $m$, consider the following cocycle as in Corollary 2.5

$$\phi_{2m,2} \in C^{2m}(\mathscr{T}_g/\mathscr{T}_g(2); I^{2m}/I^{2m+1})$$

where $I$ denotes the augmentation ideal of the group ring $\mathbb{Q}\mathscr{T}_g$. According to the properties of the Johnson homomorphism reviewed in Sect. 2.3 we have that the abelianization (modulo torsion) of the Torelli group is isomorphic, via the Johnson homomorphism, with $U$, i.e., that $\tau : \mathscr{T}_g/\mathscr{T}_g(2) \simeq U$. Recall from (2) the linear map $\Phi_f^T : \mathbb{Q}\mathscr{T}_g \to \mathscr{M}$ preserving (by definition) the filtration $\{I^n\}$ of $\mathbb{Q}\mathscr{T}_g$ and $\mathscr{F}_*^T\mathscr{M}$ of $\mathscr{M}$. In [ GL3] it was shown that the filtration $\mathscr{F}_*^T\mathscr{M}$ is 2-step (following the definition of section 2.1), i.e., we have that $\mathscr{F}_{2m-1}^T\mathscr{M} = \mathscr{F}_{2m}^T\mathscr{M}$. In [GL3] it was additionally shown that $\mathscr{F}_{2m}^T\mathscr{M} = \mathscr{F}_m^{\mathrm{as}}\mathscr{M}$, and that $\mathscr{F}_*^{as}\mathscr{M}$ is 3-step, from which follows the equality $\mathscr{G}_{2m}^T\mathscr{M} = \mathscr{G}_{3m}^{\mathrm{as}}\mathscr{M}$ of the associated graded spaces. According to the *fundamental theorem* of [LMO] and [ L] it follows that there is an isomorphism $\mathscr{G}_{3m}^{as}\mathscr{M} \simeq \mathscr{G}_m\mathscr{A}(\phi)$. Thus we get a linear map $I^{2m}/I^{2m+1} \to \mathscr{G}_m\mathscr{A}(\phi)$, and putting everything together, according to Corollary 2.5, we get a $2m$ cocycle

$$C_{f,m} \in C^{2m}(U, \mathscr{G}_m\mathscr{A}(\phi)) \tag{39}$$

Of course, this cocycle depends on the choice of an admissible surface $f : \Sigma \hookrightarrow M$, as the notation indicates. It follows from Corollary 2.7 that $C_{f,m}$ is multilinear. In order to show that $C_{f,m}$ is equivariant, consider a diffeomorphism $h : \Sigma \to \Sigma$. It follows by definition of the map $\Phi_f^T$ of (2) that for $a \in \mathscr{T}(\Sigma)$ we have: $\Phi_{hf}(a) = \Phi_f(h^{-1}ah)$. Thus $\Gamma_g$ acts by conjugation on the graded quotients $(I\mathscr{T}_g)^n/(I\mathscr{T}_g)^{n+1}$, and since the action of $\mathscr{T}_g \subseteq \Gamma_g$ is trivial, and the quotient $\Gamma_g/\mathscr{T}_g$ is the symplectic group $Sp(H)$, equivariance follows as stated in the first part of Theorem 1. Furthermore, it follows by Proposition 2.6 that the pullback of the cocycle $C_{f,m}$ to $\mathscr{T}_g/\mathscr{T}_g(3)$ (and therefore, to $\mathscr{T}_g$) represents a trivial cohomology class. The proof of Theorem 1 is complete. □

*Proof*. (of Addendum 1) The commutativity of the diagram follows by definition of the map $C_{f,m}$. □

*Proof*. (of Addendum 2) We explain the statement of this addendum more fully. Suppose, for each $g$, we choose a Heegaard embedding of a closed surface of genus $g$ into the 3-sphere. Fixing a disk, and considering surface diffeomorphisms on the disc complement that pointwise fix a neighborhood of the boundary, we obtain a map $\mathbb{Q}\mathscr{T}_{g,1} \to \mathscr{M}$ by $h \to S^3_{\hat{h}}$, where $\hat{h}$ is the obvious extension of $h$ to a diffeomorphism of the closed surface. Moreover, with respect to an inclusion of such a surface with boundary in another one, we obtain an inclusion $\mathscr{T}_{g,1} \subseteq \mathscr{T}_{g+1,1}$ and we can arrange that the following diagram is commutative:

$$\begin{tikzcd} \mathbb{Q}\mathscr{T}_{g,1} \arrow[r] \arrow[d] & \mathscr{M} \arrow[d, equal] \\ \mathbb{Q}\mathscr{T}_{g+1,1} \arrow[r] & \mathscr{M} \end{tikzcd}$$

This is, in fact, a special case of Proposition 1.4. We can define $\mathscr{T} = \lim_{g\to\infty} \mathscr{T}_{g,1}$ and thus combine these into a single map $\mathbb{Q}\mathscr{T} \to \mathscr{M}$.

Then it is proved in [GL3] that this map sends $(I\mathcal{T})^n$ onto $\mathcal{F}_n^T\mathcal{M}$, thus we have epimorphisms $(I\mathcal{T})^n/(I\mathcal{T})^{n+1} \to \mathcal{G}_n^T\mathcal{M}$. It follows that the stable cocycle $C_{2m} \in C^{2m}(\mathcal{T}/\mathcal{T}(2);\mathcal{G}_m\mathcal{A}(\phi))$ is onto. From this it is clear, since $\mathcal{G}_m\mathcal{A}(\phi)$ is finitely-generated, that $C_{f,m}$ is onto for large enough $g$. $\square$

### *3.2. Proof of Theorem 2*

*Proof*. For the convenience of the reader, we divide the proof of Theorem 2 into several lemmas. Recall first that for a diffeomorphism $\theta$ of a manifold, we denote by $\theta_*$ the action of it on the homology of the manifold. $\square$

**Lemma 3.1.** *Let $Q$ be a handlebody and $L = Ker\{i_* : H_1(\partial Q,\mathbb{Z}) \to H_1(Q,\mathbb{Z})\}$. Suppose $\alpha$ is a symplectic automorphism of $H_1(\partial Q,\mathbb{Z})$ such that $\alpha(L) = L$. Then there exists a diffeomorphism $h$ of $Q$ such that $(h|\partial Q)_* = \alpha$.*

*Proof*. Let $\alpha_Q$ be the automorphism of $H_1(Q,\mathbb{Z}) \cong H_1(\partial Q,\mathbb{Z})/L$ induced by $\alpha$. Then there exists an orientation preserving diffeomorphism $\bar{h}$ of $Q$ such that $\bar{h}_* = \alpha_Q$. Now consider the symplectic automorphism $\beta = (\bar{h}|\partial Q)\circ\alpha^{-1}$ of $H_1(\partial Q,\mathbb{Z})$. If we write $H_1(Q,\mathbb{Z}) = L \oplus L'$, where $L'$ is a complementary Lagrangian to $L$, then a matrix representative of $\beta$ has the form $\begin{pmatrix} I & C \\ 0 & X \end{pmatrix}$. Since $\beta$ is symplectic, it follows that $X = I$ and $C$ is symmetric. It suffices to see that any such matrix can be realized by a diffeomorphism of $Q$. But this is proved in [GL3]. $\square$

**Lemma 3.2.** *Suppose that $f_1, f_2 : \Sigma \hookrightarrow M$ are admissible Heegaard surfaces in an integral homology 3-sphere $M$ satisfying*:

- $f_1(\Sigma) = f_2(\Sigma)$
- $(f_{1,*})^{-1}(L_\epsilon) = (f_{2,*})^{-1}(L_\epsilon) \subseteq H_1(\Sigma,\mathbb{Z})$ *where $L_\epsilon$ is the Lagrangian pair in $H_1(f_1(\Sigma),\mathbb{Z})$ as in section* 1.1.

*Then $C_{m,f_1} = C_{m,f_2}$.*

*Proof*. Consider the diffeomorphism $g = f_2f_1^{-1}$ of $f_1(\Sigma)$. Since $g_*$ preserves $L_\epsilon$, for $\epsilon = \pm$, we can apply Lemma 3.1to deduce the existence of a diffeomorphism $\hat{h}_\epsilon$ of $M_\epsilon$ which induces the same automorphism of $H_1(f_1(\Sigma),\mathbb{Z})$ as $g$. In other words we can write $f_2 = h_\epsilon f_1 g_\epsilon$, where $g_\epsilon \in \mathcal{T}(\Sigma)$ and $h_\epsilon$ is the restriction of $\hat{h}_\epsilon$ to $\partial M_\epsilon$. Recalling the map $\Phi^T_{f_i}$ of equation (2), for $g_i \in \mathcal{T}(\Sigma)$ we have:

$$\begin{aligned} C_{f_2,k}(g_1,\ldots,g_k) &= \Phi^T_{f_2}((1-g_1)\ldots(1-g_k)) \\ &= M_{f_2(1-g_1)\ldots(1-g_k)(f_2)^{-1}} \\ &= M_{h_+f_1g_+(1-g_1)\ldots(1-g_k)g_-^{-1}f_1^{-1}h_-^{-1}} \\ &= M_{f_1g_+(1-g_1)\ldots(1-g_k)g_-^{-1}f_1^{-1}} \end{aligned}$$

$$= \Phi^T_{f_1}(g_+(1-g_1)\dots(1-g_k)g_-^{-1})$$

But $g(1-g_1)\dots(1-g_k) \equiv (1-g_1)\dots(1-g_k)g \equiv (1-g_1)\dots(1-g_k)$mod $I^{k+1}$ for any $g \in \mathscr{T}(\Sigma)$ and so, if $k = 2m$, this last term is the same as $\Phi^T_{f_1}((1-g_1)\dots(1-g_k)) \in \mathscr{G}_m\mathscr{A}(\phi)$. □

**Lemma 3.3.** *Suppose that $f_i : \Sigma \hookrightarrow M_i$ for $i = 1, 2$ are admissible Heegaard surfaces in integral homology 3-spheres $M_i$ satisfying*:

- $(f_*)^{-1}(L_{1,\epsilon}) = (f_{2,*})^{-1}(L_{2,\epsilon}) \subseteq H_1(\Sigma, \mathbb{Z})$, *where $L_{i,\epsilon}$ for $i = 1, 2, \epsilon = \pm$ are the Lagrangian pairs associated to $(M_i, f_i)$ respectively.*

*Then $C_{m,f_1} = C_{m,f_2}$.*

*Proof.* We reduce this to the preceding Lemma 3.2 by the following observation. Let $N$ be an integral homology 3-sphere , $f : \Sigma \hookrightarrow N, \Sigma' = f(\Sigma)$. Let $h \in \mathscr{T}(\Sigma')$ and $N' = N_h$. Define $f' : \Sigma \hookrightarrow N'$ from $f$ by identifying $N_+ \subseteq N$ with $N_+ \subseteq N_+ \cup_h N_- = N_h$. Then $\Phi^T_{f'}(g) = N'_{f'g(f')^{-1}} = N_{hfgf^{-1}} = N_{f(f^{-1}hf)}$ $gf^{-1} = \Phi^T_f(h'g)$, where $h' = f^{-1}hf \in \mathscr{T}(\Sigma)$. Therefore, when $k = 2m$:

$$\begin{aligned} C_{f',k}(g_1,\dots,g_k) &= \Phi^T_{f'}((1-g_1)\dots(1-g_k)) \\ &= \Phi^T_f(h'(1-g_1)\dots(1-g_k)) \\ &\equiv \Phi^T_f(1-g_1)\dots(1-g_k) \bmod \mathscr{F}^T_{k+1}\mathscr{M} \\ &= C_{f,k}(g_1|\dots|g_k) \in \mathscr{G}_m\mathscr{A}(\phi) \end{aligned}$$

since $h'(1-g_1)\dots(1-g_k) \equiv (1-g_1)\dots(1-g_k) \bmod I^{k+1}$. Thus $C_{f',k} = C_{f,k}$.

Now, since $M_1$ and $M_2$ are endowed with Heegaard decompositions of the same genus by $f_1, f_2$, we may identify $M_2$ as $(M_1)_h$, for some $h \in \Gamma(\Sigma')$, where $\Sigma' = f(\Sigma)$. We may even assume that $h \in \mathscr{T}(\Sigma')$ since we can increase the genus of the Heegaard decompositions without affecting the hypotheses, and every integral homology 3-sphere has some Heegaard decomposition where the gluing map is an element of the Torelli group. The above observation and Lemma 3.2 complete the proof of the present lemma. □

Lemma 3.3 implies that the cocycle $C_{f,m}$ in the case of a Heegaard admissible surface depends only on the Lagrangian pair $(L^+, L^-)$, and will thus be denoted by $C_{L^\pm,m}$. In addition the $Sp(H)$-equivariance of $C_{f,m}$ shows that $C_{L^\pm,m}$ is $GL(L^+_{\mathbb{Q}})$-invariant.

We next recall a useful lemma due to Morita [Mo3]. Let $\mathscr{N}^+_{g,1}$ (resp., $\mathscr{N}^-_{g,1}$) be the subgroup of the mapping class group $\Gamma_{g,1}$ of $\Sigma$ that extend to $M^+$ (resp., $M^-$). Let $W^+$ (resp., $W^-$) denote the quotient space: $\Lambda^3 H/\Lambda^3 L^+$ (resp., $\Lambda^3 H/\Lambda^3 L^-$). Note that we can identify $W^+$ (and similarly, $W^-$) with a subgroup of $\Lambda^3 H$ in the following way: Let

$$\eta_\pm : \Lambda^3 H \to \Lambda^3(H/L^\mp) \simeq \Lambda^3 L^\pm$$

be the projection followed by the natural isomorphism. Then the projection $\Lambda^3 H \to W^\pm$ induces an isomorphism $W^\pm \simeq \mathrm{Ker}\eta_\pm$. Alternatively, if we choose $\{x_i\}_{i=1}^g$ (resp., $\{y_i\}_{i=1}^g$) a basis for $L^+$ (resp., $L^-$) such that $\omega(x_i, y_j) = \delta_{i,j}$, (where $\omega$ is the symplectic form on $H$) then $W^+$ is isomorphic to the subgroup of $\Lambda^3 H$ generated by all elements of the form: $x_i \wedge x_j \wedge y_k$, $x_i \wedge y_j \wedge y_k$, $y_i \wedge y_j \wedge y_k$ for all $1 \leq i < j < k \leq g$.

Recall the Johnson homomorphism $\tau$ of Sect. 2.3. Then, we have the following lemma:

**Lemma 3.4.** [ Mo3, lemma 4.6] *With the above notation, identifying $W^\pm$ with a subgroup of $\Lambda^3 H$, we have the following:*

$$\tau(\mathscr{T}_{g,1} \cap \mathscr{N}^\pm_{g,1}) = W^\pm \tag{40}$$

We can now show that the cocycle $C_{L^\pm,m}$ factors through a map as in equation (7).

First notice that the map $H \to \Lambda^3 H$ from Sect. 2.3 followed by the projection $\Lambda^3 H \to \Lambda^3 L^\pm$ vanishes, and therefore induces onto maps $U = \Lambda^3 H/H \to \Lambda^3 L^\pm$.

Let us now consider elements $g_i, h_i \in \mathscr{T}_g$ (for $i = 1, 2, \dots, 2m$) such that $[\tau(g_1)] = [\tau(h_1)] \in \Lambda^3 L^+$, and $[\tau(g_{2m})] = [\tau(h_{2m})] \in \Lambda^3 L^-$, and $g_i = h_i$ for $2 \leq i \leq 2m-1$. The notation is as follows: recall that $\tau(g_i), \tau(h_i) \in U$, and temporarily denote both maps $U \to \Lambda^3 L^\pm$ by $x \to [x]$.

Now it is clear that $W^\pm \cap U = \mathrm{Ker}\{U \to \Lambda^3 L^\pm\}$. Since $h_1 g_1^{-1} \in W^+ \cap U$ and $h_{2m} g_{2m}^{-1} \in W^- \cap U$, we can choose liftings $\tilde{g}_1, \tilde{h}_1, \tilde{g}_{2m}, \tilde{h}_{2m} \in \mathscr{T}_{g,1}$ so that $\tau(\tilde{h}_{2m}\tilde{g}_{2m}^{-1}) \in W^-$. Using Lemma 3.4 above, there exist $b^+ \in \mathscr{T}_{g,1} \cap \mathscr{N}^+_{g,1}$ (resp., $b^- \in \mathscr{T}_{g,1} \cap \mathscr{N}^-_{g,1}$) such that $\tilde{h}_1 = b^+ \tilde{g}_1$ (resp., $\tilde{h}_{2m} = \tilde{g}_{2m} b^-$). Since $f : \Sigma \hookrightarrow M$ is a Heegaard embedding, it follows by definition of $\mathscr{N}^\pm_{g,1}$ that $M_{b^+\tilde{h}} = M_{b^+h} = M_h = M_{hb^-} = M_{\tilde{h}b^-}$, and therefore that $M_{(1-g_1)\dots(1-g_{2m})} = M_{(1-b^+g_1)(1-g_2)\dots(1-g_{2m-1})(1-g_{2m}b^-)}$, and therefore that

$$C_{L^\pm,m}(g_1, \dots, g_{2m}) = C_{L^\pm,m}(h_1, \dots, h_{2m}) \tag{41}$$

This completes the second part of Theorem 2.

In order to show the third part of Theorem 2, recall first that all integral homology 3-spheres are oriented. The change of orientation of an integral homology 3-sphere induces an involution on $\mathscr{M}$, and thus on $\mathscr{G}^{as}_m\mathscr{M}$ for every $m$. Recalling the isomorphism $\mathscr{G}^{as}_m\mathscr{M} \simeq \mathscr{G}_m\mathscr{A}(\phi)$, the above involution on $\mathscr{G}_m\mathscr{A}(\phi)$ is simply multiplication with $(-1)^m$, [LMO, Proposition 5.2]. On the other hand, given an admissible Heegaard surface $f : \Sigma \hookrightarrow M$ in an integral homology 3-sphere $M$, let $\overline{f} : \Sigma \hookrightarrow \overline{M}$ denote the same embedding but with different orientation on the ambient space $M$.

It is easy to see that the associated change to the set of Lagrangian pairs is given by $(L^+, L^-) \to (L^-, L^+)$.

Furthermore, note that for an element $g$ of the Torelli group of $\Sigma$, we have the following identity: $(\overline{M})_g = \overline{M_{g^{-1}}}$; see also Fig. 9. Thus, by the above discussion, we deduce that $C_{\overline{f},m}(g_1, \dots, g_{2m}) = (-1)^m C_{f,m}(g_{2m}^{-1}, \dots, g_1^{-1})$. Since passing from $f$ to $\overline{f}$ interchanges the Lagrangians, and since $C_{f,m}(g_{2m}^{-1}, \dots, g_1^{-1}) = C_{f,m}(g_{2m}, \dots, g_1)$ (due to the fact that $C_{f,m}$ is multilinear, and the fact that we use multiplicative notation here to denote addition) this proves the third part on the cocycle level. The assertion about the cohomology class follows from Lemma 2.1, using the fact that $[C_{\overline{f},m}] = (-1)^{\binom{2m}{2}} + 1\gamma^*[C_{f,m}]$ and $\binom{2m}{2} \equiv m \pmod 2$. This completes the proof of Theorem 2. □

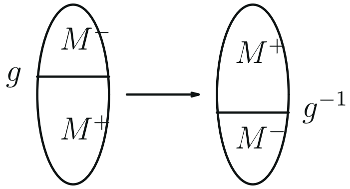



**Fig. 9.** An orientation reversing of $M$ corresponds to a reflection along the $x$-axis

### 3.3. *Proof of Theorem 3*

*Proof.* Let $f : \Sigma \hookrightarrow M$ be an admissible Heegaard surface, and $(L^+, L^-)$ the associated Lagrangian pair of the symplectic vector space $(H, \omega)$ as in Sect. 1.1. Consider the cocycle $C_{L^\pm,m} : \otimes^{2m} U \to \mathcal{G}_m\mathcal{A}(\phi)$. Recall from Sect. 2.4 the subgroup of the symplectic group $Sp(H)$ isomorphic to $GL(L^+)$. Since this group acts on $H$ preserving the Lagrangian pair $(L^+, L^-)$, it follows from Theorem 2 that $C_{L^\pm,m}$ factors through an invariant map: $(\otimes^{2m}U)^{GL(L^+)} \to \mathcal{G}_m\mathcal{A}(\phi)$. Composing with the onto map $\mathcal{G}_m\mathcal{A}^{rp,nl,cl} \to (\otimes^{2m}U)^{GL(L^+)}$ of Proposition 2.18, we get a composite map:

$$\Psi_{L^\pm,m} : \mathcal{G}_m\mathcal{A}^{rp,nl,cl} \to \mathcal{G}_m\mathcal{A}(\phi) \tag{42}$$

thus finishing the proof of Theorem 3. □

### 3.4. *Proof of theorem 4*

*Proof.* Let $f : \Sigma \hookrightarrow M$ be an admissible Heegaard surface, and let $(L^+, L^-)$ be the associated Lagrangian pair of the symplectic vector space $(H, \omega)$ as in Sect. 1.1.

Let $\lambda$ denote the Casson invariant, and $W_\lambda$ its associated manifold weight system. Consider the associated 2-cocycle $W_\lambda \circ C_{L^\pm,1}$ of $U$ with coefficients in $\mathbb{Q}$ as in corollary 1.1. According to theorem 2, $W_\lambda \circ C_{L^\pm,1}$ factors through a $GL(L^+_{\mathbb{Q}})$ invariant map: $\Lambda^3 L^+_{\mathbb{Q}} \otimes \Lambda^3 L^-_{\mathbb{Q}} \to \mathbb{Q}$. According to Corollary 2.19, the vector space of such invariant maps is 1 dimensional, and a nonzero such map is the restriction of the map $C_\Theta$ of (11) to $\Lambda^3 L^+_{\mathbb{Q}} \otimes \Lambda^3 L^-_{\mathbb{Q}} \hookrightarrow \otimes^2 \Lambda^3 H_{\mathbb{Q}} \hookrightarrow \otimes^6 H_{\mathbb{Q}}$, which we will denote by the same name before the $\otimes^2 U_{\mathbb{Q}}$. Using the definition of the map $C^U_\Theta$ (given before the statement of Theorem 4) and the above discussion, we deduce that $W_\lambda \circ C_{L^\pm,1} = c_g C^U_\Theta$ for some rational number $c_g$ depending on the genus $g$ of $\Sigma$. According to Addendum 2, $C_{L^\pm,1}$ is stable (with respect to an inclusion of a surface in another) and so is $C^U_\Theta$, therefore $c_g = c$ independent of the genus.

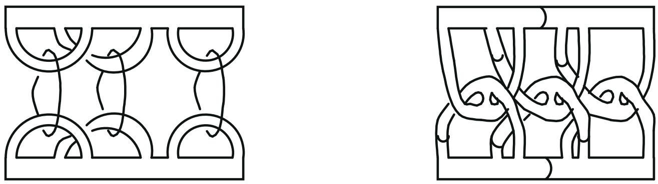


**Fig. 10.** On the left, a special case of a 2-pair blink $L'_{bl}$ (bounding the disjoint union of two genus 2 surfaces) union a 3-component algebraically split link $L'$. On the right the boundary surface $\Sigma_9$ of a genus 9 handlebody

To determine the value of $c$, we need to calculate a particular example. Consider a 2-pair blink $L'_{bl}$ and a 3-component algebraically split link $L'$ in $S^3$ shown in the left part of Fig. 10. Part of this figure appeared first in [GL3, Sect. 3, Fig. 39]. Note that $L'$ is a trivial 3-component link, bounding a disjoint union of obvious disks. Choose a unit Seifert-framing for the blink $L'_{bl}$, and for $L'$. Perform a Dehn twist on each of the three disks that $L'$ bounds, and let $L_{bl}$ denote the image of $L'_{bl}$. After performing the twists, thicken the surface that $L_{bl}$ bounds in order to get two disconnected genus 3 solid surfaces, and join them along three tubes to form a genus 9 surface $\Sigma_9$, see the right part of Fig. 10. We can assume that $L_{bl}$ lies in $\Sigma_9$. It is easy to see that $\Sigma_9$ is a genus 9 Heegaard splitting of $S^3$. Let $L^i_{bl}$ (for $i = 1, 2$) denote the two pairs of the blink $L_{bl}$; each gives rise to an element of the Torelli group of $\Sigma_9$, see [GL3]. Let $\alpha^i$ (for $i = 1, 2$) denote the image in $U$ of each of the above mentioned elements of the Torelli group under the Johnson homomorphism. Using the definition of $C^U_\Theta$ and the definition of the Johnson homomorphism, it is easy to show that $C^U_\Theta(\alpha^1, \alpha^2) = -1$.

The first part follows from the following lemma:

**Lemma 3.5.** *With the above normalizations, we have the following equalities*:

$$W_\lambda \circ C_{L^\pm,1}(\alpha^1, \alpha^2) = -2$$
$$W_\lambda(\Theta_w) = -2$$

*Proof.* For the first part, note first that for $f : \Sigma_9 \subseteq S^3$ the admissible Heegaard surface we have:

$$\begin{aligned}\Phi_f^T((1-\alpha^1)(1-\alpha^2)) &= S^3 - S^3_{\alpha^1} - S^3_{\alpha^2} + S^3_{\alpha^1,\alpha^2} \\ &= -S^3 + S^3_{\alpha^1,\alpha^2} \\ &= -[S^3, L_{bl}, f] \in \mathscr{F}_2^T\mathscr{M} = \mathscr{F}_3^{as}\mathscr{M}\end{aligned}$$

where the first equality follows from the fact that $S^3_{\alpha^1} = S^3_{\alpha^2} = S^3$. We also have that:

$$-[S^3, L_{bl}, f] = [S^3, L'_{bl} \cup L', f] = \Theta_w = 2Y_b = 2(S^3 - S^3_{\text{Trefoil},+1})$$

where the first equality follows from the fact that $L'$ is an unlink, and the second follows from the fact that surgery on each of the pairs of $L'_{bl}$ corresponds to the alternating sum of cutting or not each vertex of the $\Theta$ graph, see also [GL3, Sect. 3, Fig. 36]. The third equality follows from the main identities of [Oh], [GL2], and the last by the fact that $S^3_{\text{Trefoil},+1}$ equals the result of Dehn surgery on a Borromean ring of three components with framing $+1$. Note that Trefoil is a right (or left) handed trefoil in $S^3$ depending on the Borromean ring; either case does not affect the validity of our calculation.

This and the definition of $W_\lambda \circ C_{L^\pm,1}$ imply that

$$\begin{aligned}W_\lambda \circ C_{L^\pm,1}(\alpha^1, \alpha^2) &= \lambda(\Phi_f^T((1-\alpha^1)(1-\alpha^2))) \\ &= \lambda(\Theta_w) \\ &= 2\lambda(S^3) - 2\lambda(S^3_{\text{Trefoil},+1})\end{aligned}$$

Using the normalizations of the Casson invariant it follows that $\lambda(S^3) = 0$ and $\lambda(S^3_{\text{Trefoil},+1}) = 1$ which proves the lemma. $\square$

In order to show the second part, since $\mathscr{G}_1\mathscr{A}(\phi)$ is 1 dimensional, the first part implies that:

$$C_{L^\pm,1} = c'_g C^U_\Theta \cdot \Theta_w \tag{43}$$

for some rational number $c'_g$. The stability of $C_{L^\pm,1}$ implies that $c'_g = c'$ independent of the genus. Composing (43) with $W_\lambda : \mathscr{G}_1\mathscr{A}(\phi) \to \mathbb{Q}$ we obtain that $W_\lambda \circ C_{L^\pm,1} = c' W_\lambda(\Theta_w) C^U_\Theta$, which (due to the first part of Theorem 4) implies that $2 = c' W_\lambda(\Theta_w)$. Using lemma 3.5, the second part of Theorem 4 follows.

In order to show the third part of Theorem 4, recall first from Theorem 2 that $C_{L^\pm,1} : \otimes^2 U \to \mathscr{G}_1\mathscr{A}(\phi)$ factors through a $GL(L^+)$ invariant map $\Lambda^3 L^+_{\mathbb{Q}} \otimes \Lambda^3 L^-_{\mathbb{Q}} \to \mathscr{G}_1\mathscr{A}(\phi)$. Thus using the basis of the four dimensional vector space $\mathscr{G}_1\mathscr{A}^{rp,nl,cl}(\phi)$ shown in the southeast part of Fig. 8 and the definition of $\Psi_{L^\pm,1}$ and Corollary 2.19, the second part of Theorem 4 implies the third part.

In order to show the fourth part of Theorem 4, namely that $C_{L^\pm,1}$ represents a non-zero cohomology class, we interpret it as a cup-product.

Consider the elements $\xi^{\pm} \in H^1(\Lambda^3 H; \Lambda^3 L^{\pm})$ defined by the homomorphisms $\Lambda^3 H \to \Lambda^3(H/L^{\mp}) \cong \Lambda^3 L^{\pm}$. Then the intersection pairing on $H$ defines a non-singular pairing on $\Lambda^3 H$ which induces a non-singular pairing $\Lambda^3 L^+ \otimes \Lambda^3 L^- \to \mathbb{Q}$. Using this pairing we have a cup-product

$$H^1(\Lambda^3 H; \Lambda^3 L^+) \otimes H^1(\Lambda^3 H; \Lambda^3 L^-) \to H^2(\Lambda^3 H; \mathbb{Q})$$

Now it is straightforward to check, from (13), that $j^*[C_{L^{\pm},1}] = \xi^+ \cup \xi^-$, where $j : \Lambda^3 H \to U$ is the projection.

Recall that, for any finitely-generated abelian group $A$, there is an isomorphism of $\mathbb{Q}$-algebras:

$$H^*(A; \mathbb{Q}) \cong \Lambda^*(A^*) \otimes \mathbb{Q}$$

where $A^* = \mathrm{Hom}(A, \mathbb{Z})$. The product structure on $\Lambda^*(A^*)$ is defined as follows:

$$\phi^p \cdot \psi^q(v_1 \wedge \ldots \wedge v_{p+q}) = \sum_{\pi} \mathrm{sgn}(\pi) \phi^p(v_{\pi 1} \wedge \ldots \wedge v_{\pi p}) \psi^q(v_{\pi(p+1)} \wedge \ldots \wedge v_{\pi(p+q)})$$

where the sum ranges over all *shuffle* permutations $\pi$.

Thus $\xi^+ \cup \xi^- \in H^2(\Lambda^3 H; \mathbb{Q}) \cong \Lambda^2(\Lambda^3 H^*)$ is defined by

$$u \wedge v \mapsto \omega(\xi^+(u), \xi^-(v)) - \omega(\xi^+(v), \xi^-(u))$$

for $u, v \in \Lambda^3 H$. To see this is non-trivial we note, for example, that, if $u \in \Lambda^3 L^+$ and $v \in \Lambda^3 L^-$, then $\xi^+ \cup \xi^-(u \wedge v) = \omega(u, v)$.

Now suppose that $K^{\pm}$ is another Lagrangian pair with associated classes $\eta^{\pm} \in H^1(\Lambda^3 H; K^{\pm})$, and suppose that $\xi^+ \cup \xi^- = \eta^+ \cup \eta^-$. We first point out that $\Lambda^3 L^+ + \Lambda^3 L^- = \Lambda^3 K^+ + \Lambda^3 K^- \subseteq \Lambda^3 H$. This follows from the observation that $\xi^+ \cup \xi^-(u \wedge w) = 0$ for all $v \in \Lambda^3 H$ if and only if $\omega(u, \Lambda^3 L^+ + \Lambda^3 L^-) = 0$.

We can assume that $\dim H \geq 6$. Let $p : K^+ \to L^+$ be the restriction of the projection of $H$ onto $L^+$ with kernel $L^-$. Choose any basis $v_1, \ldots, v_n$ of $K^+$ such that $p(v_i) = 0$ for $i \leq r$, and $p(v_i)$ are linearly independent in $L^+$ for $i > r$. Write $v_i = v_i^+ + v_i^-$ for $i > r$, where $v_i^{\pm} \in L^{\pm}$. We consider several cases.

$\mathbf{1 < r < n}$. Then $v_1, v_2 \in L^-$ and $v_1 \wedge v_2 \wedge v_n = v_1 \wedge v_2 \wedge v_n^+ + v_1 \wedge v_2 \wedge v_n^-$. If we now consider the direct sum decomposition:

$$\Lambda^3 H = (\Lambda^3 L^+ \oplus \Lambda^3 L^-) \oplus (L^+ \otimes \Lambda^2 L^-) \oplus (L^- \otimes \Lambda^2 L^+) \tag{44}$$

then we see that the component of $v_1 \wedge v_2 \wedge v_n$ in $L^+ \otimes \Lambda^2 L^-$ is non-zero. But $v_1 \wedge v_2 \wedge v_n \in \Lambda^3 K^+ \subseteq \Lambda^3 K^+ + \Lambda^3 K^- = \Lambda^3 L^+ + \Lambda^3 L^-$, which means this component should be zero.

$\mathbf{r = 1}$. For any $1 < i < j$, we examine $v_1 \wedge v_i \wedge v_j$. The component in $L^+ \otimes \Lambda^2 L^-$ is $v_i^+ \wedge (v_j^- \wedge v_1) + v_j^+ \wedge (v_1 \wedge v_i^-)$. Since $v_i^+, v_j^+$ are linearly independent, this means $v_1 \wedge v_i^- = v_1 \wedge v_j^- = 0$. Thus $v_i^- = c_i v_1$, for some scalars $c_i$, $i > 1$. But then $v_1 \wedge v_i \wedge v_j = v_1 \wedge v_i^+ \wedge v_j^+$ lying in $L^- \otimes \Lambda^2 L^+$ in the decomposition of (44).

$\mathbf{r = 0}$. Suppose $K^+ \neq L^+$. Then we can assume without loss of generality that $v_1^- \neq 0$. If we consider the element $v_1 \wedge v_i \wedge v_j$, then the component in $L^+ \otimes \Lambda^2 L^-$ is $v_1^+ \wedge (v_i^- \wedge v_j^-) + v_i^+ \wedge (v_j^- \wedge v_1^-) + v_j^+ \wedge (v_1^- \wedge v_i^-)$. Since $v_1^+, v_i^+$, $v_j^+$ are linearly independent, we conclude that $v_1^- \wedge v_i^- = v_1^- \wedge v_j^- = 0$ and so $v_i^- = c_i v_1^-$, for suitable scalars $c_i$. Now let us replace each $v_i$ by $v_i - c_i v_1$. In other words we can assume that each $v_i \in L^+$ for $i > 1$. But now we can see that $v_1 \wedge v_i \wedge v_j$ has component $v_1^- \wedge v_i \wedge v_j$ in $L^- \otimes \Lambda^2 L^+$ and so $v_1^-$ would have to be zero.

The conclusion from these arguments is that either $r = n$, in which case $K^+ = L^-$, or, from the last case, that $K^+ = L^+$. Similarly, we see that $K^- = L^+$ or $L^-$. Finally we need to check that $\xi^+ \cup \xi^- = \eta^+ \cup \eta^-$ in case $K^+ = L^-$ and $K^- = L^+$. Using the orthogonal direct sum decomposition (44), we can write $u = u^+ + u^- + u^0, v = v^+ + v^- + v^0$, where $u^+, v^+ \in \Lambda^3 L^+$, $u^-, v^- \in \Lambda^3 L^-$ and $u^0, v^0 \in (L^+ \otimes \Lambda^2 L^-) \oplus (L^- \otimes \Lambda^2 L^+)$. Then we have

$$\xi^+ \cup \xi^-(u \wedge v) = \omega(u^+, v^-) - \omega(v^+, u^-)$$
$$\eta^+ \cup \eta^-(u \wedge v) = \omega(u^-, v^+) - \omega(v^-, u^+)$$

The skew-symmetry of the symplectic pairing implies that these are equal.

### *3.5. Proof of Proposition 1.4*

*Proof.* Under the assumptions of Corollary 1.4, we are given an embedded sphere $S \hookrightarrow M$ in an integral homology 3-sphere $M$ which separates $M$ into two components. Therefore, we have that $M$ is a connected sum of two integral homology 3-spheres $M_1, M_2$, along the separating sphere $S$, i.e., $M = M_1 \sharp M_2$. Furthermore, by assumption, the admissible surfaces $f_i : \Sigma_{g_i} \hookrightarrow M$ belong to different components of $M - S$. Recall the composite surface $f_1 \cup f_2 : \Sigma_{g_1+g_2} = \Sigma_{g_1} \cup_{\partial S - (D_1 \cup D_2)} \Sigma_{g_2} \hookrightarrow M$. Therefore, for $h_i \in \mathscr{T}(\Sigma_{g_i,1}) (i = 1, 2)$ we get an element $h_1 \cup h_2 \in \mathscr{T}(\Sigma_{g_1+g_2})$, and an isomorphism:

$$M_{h_1 \cup h_2} \simeq (M_1)_{h_1} \sharp (M_2)_{h_2}$$

Since $\mathscr{M}$ (resp., $\mathscr{A}(\phi)$) is a commutative algebra with multiplication the operation of connected sum on integral homology 3-spheres, (resp., the disjoint union of vertex oriented trivalent graphs) and since the isomorphism $\mathscr{G}^{as}_{3m}\mathscr{M} \simeq \mathscr{G}_m \mathscr{A}(\phi)$ preserves the algebra structures, the result follows. $\square$

*3.6. Proof of Theorem 5*

In this section we give a proof of Theorem 5. Since the proof combines several rather different techniques, for the convenience of the reader we separate it into several steps.

*Proof.* [of Theorem 5]

• **Step 1** The definition of the map $D_{f,m}$ of (16).
Set $D_{f,m} = \phi_f^{\mathcal{T}_g}$ from Corollary 2.12.

• **Step 2** $D_{f,m}$ is determined by $C_{f,m}$.

Indeed, equation (17) follows from Lemma 2.8 and the above definition of $D_{f,m}$. It remains to show that $\mathcal{G}_{2m}\mathcal{T}_g \otimes \mathbb{Q}$ is spanned by elements of the form $[x_1, [x_2, \ldots, [x_{2m-1}, x_{2m}]]]$ for $x_i \in \mathcal{T}_g$. This follows from several applications of the Jacobi identity: $[[a,b],c] = [a,[b,c]] - [b,[a,c]]$ for $a \in \mathcal{T}_g(n_1)$, $b \in \mathcal{T}_g(n_2), c \in \mathcal{T}_g(n_3)$ with $n_1 + n_2 + n_3 = 2m$.

In case $f : \Sigma \hookrightarrow M$ is an admissible Heegaard surface, $C_{f,m}$, and thus $D_{f,m}$, depends only on the associated Lagrangian pair $(L^+, L^-)$. In that case we will denote $D_{f,m}$ by $D_{L^\pm,m}$. Assume from now on that $f$ is an admissible Heegaard surface.

• **Step 3** $D_{L^\pm,m}$ satisfies the symmetry property of (18).

Indeed, Fig. 9 shows that $D_{f,m}(a^{-1}) = D_{\overline{f},m}(a)$ where $\overline{f}$ is the surface in the *orientation reversed* 3-manifold $\overline{M}$. Since the involution of reversing the orientation in the ambient 3-manifold is multiplication by $(-1)^m$ on $\mathcal{G}_m\mathcal{A}(\phi)$, the result follows.

• **Step 4** Assume now that $f$ is the standard genus $g$ Heegaard splitting of $S^3$. Then, for $g \geq 5m+1$, $D_{L^\pm,m}$ is onto.

This follows by Corollary 3.10 of Theorem 7 whose proof is given below. The proof of Theorem 7 and Corollary 3.10 given below is long and technical; furthermore it is logically independent from the rest of the proof of Theorem 5.

• **Step 5** The case of $m = 1$.

We now describe explicitly the map $D_{L^\pm,1}$. Assume that we are given an admissible Heegaard genus $g$ surface $f$. Recall first that $\mathcal{G}_m\mathcal{T}_g \otimes \mathbb{Q}$ is a finite dimensional, stable with respect to the genus, representation of $Sp(H)$. It follows by a theorem of Quillen [Qu] (see also [Ha1]) that it is a rational representation of $Sp(H_{\mathbb{Q}})$. It is a *very interesting* question to analyze the structure of the above representation. Motivated by the above question Morita [Mo5] developed a theory of *higher Johnson homomorphisms*, known to form a Lie algebra. The structure of this and related Lie algebras have been analyzed in the pioneering work of Morita [Mo2] and Hain [Ha2]. In case of $\mathcal{G}_2\mathcal{T}_g \otimes \mathbb{Q}$ the answer is known and we describe it here.

It turns out that for $g \geq 6$ we have the following decomposition as representations of $Sp(H_{\mathbb{Q}})$ [Mo2], [Ha2]:

$$\mathscr{G}_2\mathscr{T}_g \otimes \mathbb{Q} = V(0) + V(2\epsilon_2) \tag{45}$$

Recall from Sect. 2.3 that $\mathscr{T}_g(2) = \mathscr{K}_g$, thus we have that $\mathscr{G}_2\mathscr{T}_g = \mathscr{K}_g/\mathscr{T}_g(3)$. Morita [Mo2, Sec. 5] using his theory of *secondary characteristic classes* defined a group homomorphism $d_1 : \mathscr{K}_g \to \mathbb{Q}$ which vanishes on $\mathscr{T}_g(3)$, thus inducing a map $\mathscr{G}_2\mathscr{T}_g \otimes \mathbb{Q} \to \mathbb{Q}$. Furthermore, Morita [Mo2, Sect. 1] defined a higher version of Johnson's homomorphism $\tau_3 : \mathscr{K}_g \to V(2\epsilon_2)$, which also vanishes on $\mathscr{T}_g(3)$ thus inducing a map $\mathscr{G}_2\mathscr{T}_g \otimes \mathbb{Q} \to V(2\epsilon_2)$. The above maps are $Sp(H_{\mathbb{Q}})$ invariant, and stable, and realize the decomposition of (45) as a $Sp(H_{\mathbb{Q}})$ module. Moreover, Morita [Mo2] using a Heegaard splitting $f$, defined a map $q_f : V(2\epsilon_2) \to \mathbb{Q}$ and showed in his main result [Mo2, Theorem 6.1] that:

$$W_\lambda \circ D_{L^\pm,1} = -\frac{1}{24}d_1 - q_f \tag{46}$$

Note that the change in sign from [Mo2, Theorem 6.1] to the above equation is due to the fact that Morita uses the map $\mathscr{T}_g \to \mathbb{Q}$ to be $a \to \lambda(S^3_a) - \lambda(S^3)$; however we use the map $\mathscr{T}_g \to \mathbb{Q}$ to be $a \to \lambda(S^3) - \lambda(S^3_a)$. From this, it follows immediately that $W_\lambda \circ D_1 = -\frac{1}{24}d_1$, thus finishing step 5 and the proof of Theorem 5.

The proof of Theorem 7 and Corollary 3.10 occupies the rest of this section. The proof is long and technical, and consists of combinatorial as well as geometric topology arguments. We urge the reader to keep in mind the figures.

Let $f : \Sigma_g \hookrightarrow S^3$ be the standard genus $g$ Heegaard splitting of $S^3$, which we keep fixed for the rest of this section. We follow the notation and terminology of Sect. 2.2. For $L$ an $f$-compatible Lagrangian consider the maps $\mathscr{G}\Phi^L_f : \mathscr{G}\mathbb{Q}\mathscr{L}^L_g \to \mathscr{G}^L\mathscr{M}$ and $\phi^L_f : \mathscr{G}\mathscr{L}^L_g \otimes \mathbb{Q} \to \mathscr{G}\mathscr{A}^{conn}(\phi)$, which, for simplicity, we denote by $\Phi^L$ and $\phi^L$ respectively. Then, we have the following theorem:

**Theorem 7.** *Suppose* $g \geq 5n+1$. *Then there exists an f-compatible Lagrangian* $L \subseteq H_1(\Sigma_g)$ *so that* $\phi^L(\mathscr{G}_{3n}\mathscr{L}^L_g \otimes \mathbb{Q}) = \mathscr{G}_n\mathscr{A}^{conn}(\phi)$.

*Proof.* Due to the length of the proof, for the convenience of the reader we provide the proof in six (or perhaps seven) steps.

• **Step 0** A non proof.

We first give a "too good to be true" proof. $\mathscr{G}\Phi^L_f$ is a map of coalgebras and according to the results of [GL3] reviewed in Sect. 2.2, $\mathscr{G}\Phi^L_f$ is stably onto. If it *were* the case that $\mathscr{G}\Phi^L_f$ was a Hopf algebra map, the induced map on the primitives would be onto by a dimension count using the Poincaré-Birkhoff-

Witt theorem. Unfortunately, $\mathcal{G}\Phi^L_f$ does not preserve the product structure, see Remark 2.13.

• **Step 1** A reduction to "chord diagrams".

We begin by recalling the following definition: a degree $n$ chord diagram (on a cirle is a collection of $n$ chords with $2n$ distinct end points; for an example see Fig. 11. Note that a chord diagram can be thought of as a connected vertex oriented trivalent graph (with the counterclockwise orientation at each vertex), and furthermore this way a degree $n$ chord diagram gives rise to an element of $\mathcal{G}_n\mathcal{A}^{conn}(\phi)$. Note that a degree $n$ chord diagram has $2n$ *external* edges (the ones on the circle) and $n$ *internal* ones. Degree $n$ chord diagrams are rather special elements of $\mathcal{G}_n\mathcal{A}^{conn}(\phi)$, however we have the following claim:

**Claim 3.6.** *$\mathcal{G}_n\mathcal{A}^{conn}(\phi)$ is generated by chord diagrams as above.*

A proof, using the *IHX* relation, can be found in [GL2, Lemma 3.2].
With the notation of Sect. 2.2, we make the following:

**Claim 3.7.** *For every $n > 1$ there is an f-compatible Lagrangian L with the following property: For each degree n chord diagram $\Gamma$; there is an element $\xi^\Gamma \in \mathcal{L}^L_g(3n)$ such that:*

$$F(\phi^L(\xi^\Gamma)) \equiv \Gamma_b \bmod \mathcal{Y}_b \tag{47}$$

Using Lemma 2.9 the above claim implies that for each degree $n > 1$ chord diagram $\Gamma$, there is an element $\xi^\Gamma \in \mathcal{L}^L_g(3n)$ such that $\phi^L(\xi^\Gamma) = \Gamma_w$. This, together with Claim 3.6 implies Theorem 7 for $n > 1$.

Thus, we will prove Theorem 7 by proving the special case of Theorem 7 for $n = 1$, and proving Claim 3.7 for $n > 1$. Note that Theorem 7 is obvious for $n = 0$. In the rest of the proof, we will be working in the graded space $\mathcal{G}_{3n}\mathcal{A}_b(\phi)$ which is isomorphic to $\mathcal{G}^{as}_{3n}\mathcal{M}$.

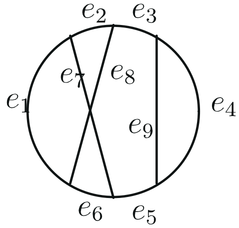



**Fig. 11.** A degree 3 chord diagram on a circle

• **Step 2** An important construction.

We begin with a construction which will be an important component of, as well as a warm-up for, the general construction. Let $\Sigma_2 = \partial H$ be the surface of genus two in $S^3$, where $H = T_1 \coprod T_2$ is the boundary connected sum of two solid tori in $\mathbb{R}^3$. We distinguish three simple closed curves on $\Sigma_2$. Let $a$ be a meridian in $\partial T_1$, i.e., the boundary of a meridian disk in $T_1$, and $c$ be a longitude in $\partial T_2$. Finally let $b$ be a band sum of two disjoint meridians in $\partial T_2$, where the band passes once, longitudinally, around $\partial T_1$. See Fig. 12. The orientations of these curves can be chosen arbitrarily for the moment. Let $\alpha, \beta, \gamma$ be the diffeomorphisms of $\Sigma_2$ defined by Dehn twists along $a, b, c$, respectively. Note that $\beta \in \mathscr{K}_2$, since $b$ bounds in $\Sigma_2$, but $\alpha$ and $\gamma$ do not lie in the Torelli group. However, there is a Lagrangian $L \subseteq H_1(\Sigma_2)$, compatible with the given embedding of $\Sigma_2$ in $S^3$, so that $\alpha, \beta, \gamma \in \mathscr{L}_2^L$. In fact we can just take $L$ to be the subgroup generated by the homology classes of $a$ and $c$.

Consider the element $[[\alpha, \beta], \gamma] \in (\mathscr{L}_2^L)_3$. Then, we have the following:

• **Step 3** $\phi^L[[\alpha, \beta], \gamma]$ is a generator of $\mathscr{G}_3^{as}\mathscr{M} \cong \mathscr{G}_3\mathscr{A}(\phi) = \mathscr{G}_1\mathscr{A}^{conn}(\phi) \cong \mathbb{Q}$. Thus theorem 7 holds for $n = 1$.

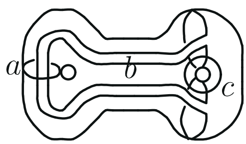



**Fig. 12.** A genus 2 handlebody $\Sigma_2$ together with 3 curves $a, b, c$ on it

*Proof*. Consider the element $(1-\alpha)(1-\beta)(1-\gamma) \in I = (I\mathscr{L}_2^L)^3$. Then $\Phi^L((1-\alpha)(1-\beta)(1-\gamma))$ represents the same linear combination in $\mathscr{M}$ as $[S^3, K]$, where the 3-component link $K$ is constructed as follows. Take three concentric copies of $\Sigma_2$ in $\mathbb{R}^3$ and place $a$ in the outer copy, $b$ in the middle copy and $c$ in the inner copy. Then $K$ is given by these three disjoint curves in $\mathbb{R}^3$. See Fig. 13. We refer the reader to [ GL3] for the explanation of this.

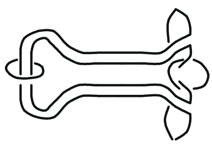


**Fig. 13.** After taking 3 concentric copies of the surface of Fig. 12, and placing $a, b, c$ on each copy in that order, we arrive at the 3-component Borromean link $K$ shown in the figure above

Note that $K$ is just the Borromean rings and so represents a generator of $\mathscr{G}_3^{as}\mathscr{M}$. The following Claim 3.8 completes the proof of this step.

**Claim 3.8.** *We have*:

$$\Phi^L(1 - [[\alpha, \beta], \gamma]) = \Phi^L((1-\alpha)(1-\beta)(1-\gamma))$$

To prove the claim recall the following identity from Lemma 2.8:

$$\begin{aligned} 1 - [[\alpha, \beta], \gamma] \equiv\ & (1-\alpha)(1-\beta)(1-\gamma) - (1-\beta)(1-\alpha)(1-\gamma) \\ & - (1-\gamma)(1-\alpha)(1-\beta) + (1-\gamma)(1-\beta)(1-\alpha) \bmod I^4 \end{aligned}$$

Now for any permutation $\alpha_1, \alpha_2, \alpha_3$ of $\alpha, \beta, \gamma$, $\Phi^L((1-\alpha_1)(1-\alpha_2)(1-\alpha_3)) = [S^3, K']$, where $K'$ is the 3-component link defined by placing $a, b, c$ in concentric copies of $\Sigma_2$, just as above, except that they are placed in the permuted order. But it is easy to see that, for any *non-trivial* permutation, the resulting link $K'$ is trivial. See Fig. 14. Thus, from the above formula we see that $\Phi^L(1 - [[\alpha, \beta], \gamma]) = \Phi^L((1-\alpha)(1-\beta)(1-\gamma))$. This concludes the proof of the claim and of Step 3. □

• **Step 4** The definition of the $f$-admissible Lagrangian $L$.

Now let $\Gamma$ be any chord diagram with $2n$ vertices. We associate to $\Gamma$ a Heegaard surface $\Sigma_\Gamma \subseteq \mathbb{R}^3$ as follows. Choose an embedding of $\Gamma \subseteq \mathbb{R}^3$. Then, at every vertex $v$ of $\Gamma$, place a copy $\Sigma(v)$ of $\Sigma_2$ and, for every edge $e$ of $\Gamma$ with vertices $v_e, v'_e$, take a connected sum of $\Sigma(v_e)$ with $\Sigma(v'_e)$ using a *tube* $T(e)$ running along the edge $e$. See Fig. 15.

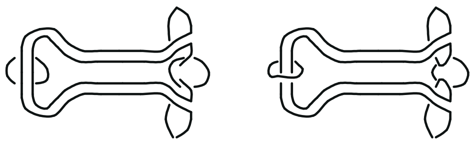


**Fig. 14.** The links shown above associated to any nontrivial permutation of $a, b, c$ are trivial

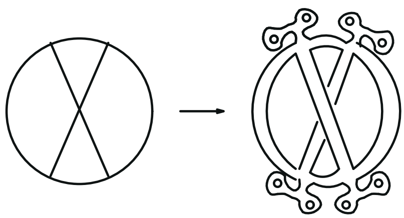


**Fig. 15.** For the chord diagram $\Gamma$ shown on the left, the construction of a handlebody $\Sigma_\Gamma$ on the right

Note that the resulting surface $\Sigma_\Gamma$ will have genus $g = 5n + 1$. In each $\Sigma(v)$ we have three copies $a(v), b(v), c(v)$ of the curves $a, b, c$ in $\Sigma_2$ and we can assume they avoid the holes where the tubes $\{T(e)\}$ meet $\Sigma(v)$. Now label the edges of $\Gamma$ with labels $a, b$ or $c$, so that all the internal edges have label $b$ and the external edges are labeled alternately $a$ and $c$ as we go round the external circle. Thus the three edges incident to any vertex have all different labels. See Fig. 16. Now for each edge $e$ that connects the vertices $v_e$ and $v'_e$ we take a band sum of the curves $x(v_e)$ and $x(v'_e)$, where $x \in \{a, b, c\}$ is the label of $e$, using a band which travels along the boundary of the tube $T(e)$. We will denote this band-sum curve by $\hat{e}$. Note that there are $6n$ labeled curves $a(v), b(v), c(v)$ in $\Sigma_\Gamma$ which, after the above mentioned band sum, yield $3n$ curves $\hat{e}$ in $\Sigma_\Gamma$.

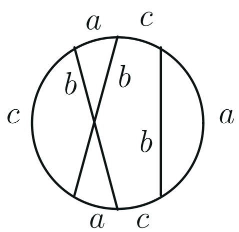



**Fig. 16.** A labeling of the edges of a degree 3 chord diagram with labels $a, b, c$

Let $\gamma_e$ be the diffeomorphism of $\Sigma_\Gamma$ defined by a Dehn twist along $\hat{e}$. Notice that $\gamma_e \in \mathscr{K}_g$ if $e$ has label $b$, since it can be arranged, by taking our connected sums correctly, that $\hat{e}$ bounds in $\Sigma_\Gamma$. See Fig. 17.

If we define our *Lagrangian* $L$ to be generated by the homology classes in each $\Sigma(v)$ represented by $a(v)$ and $c(v)$ and, in addition, the meridians of the tubes $T(e)$, then $L$ is $f$-compatible and each $\gamma_e \in \mathscr{L}_g^L$. It will be a very

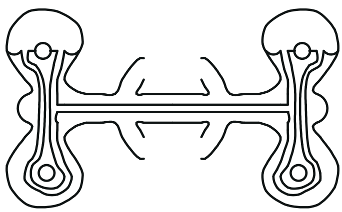


**Fig. 17.** The connected sum of two $b$ labeled curves is a bounding curve. Thus Dehn surgery along it represents an element of $\mathscr{K}_g$

important observation that whenever $\hat{e}_i$ and $\hat{e}_j$ are disjoint- for example if the two edges $e_i$ and $e_j$ have the same label, or if $e_i$ has label $a$ and $e_j$ has label $c$- then $\gamma_{e_i}$ and $\gamma_{e_j}$ commute.

• **Step 5** Verification of Claim 3.7 for $n \neq 1$.

We begin with some preliminaries. Choose any ordering $\mathscr{I} = \{e_1, \cdots, e_{3n}\}$ of the edges of $\Gamma$. We now consider the element $\xi_{\mathscr{I}}^\Gamma = (1 - \gamma_{e_1}) \cdots (1 - \gamma_{e_{3n}}) \in I^{3n}$, where $I \stackrel{\text{def}}{=} I\mathscr{L}_g^L$, and its image $\Phi^L(\xi_{\mathscr{I}}^\Gamma) \in \mathscr{G}_{3n}^L \mathscr{M}$. Recall from Sect. 2.2 that there is a map $\mathscr{G}_{3n}^L \mathscr{M} \to \mathscr{G}_{3n}^{as} \mathscr{M}$ (induced by an inclusion map $\mathscr{F}^L_{3n}\mathscr{M} \subseteq \mathscr{F}^{as}_{3n}\mathscr{M}$); we will thus identify $\mathscr{G}^L_{3n}\mathscr{M}$ with its image in $\mathscr{G}^{as}_{3n}\mathscr{M}$. With this identification in mind, we will now describe the element $\Phi^L(\xi^\Gamma_{\mathscr{I}}) \in \mathscr{G}^{as}_{3n}\mathscr{M}$. Choose $3n$ concentric copies of $\Sigma_\Gamma$ in $\mathbb{R}^3$ and consider the curve $\hat{e}_i$ placed in the $i$-th copy of $\Sigma_\Gamma$ (counting from the outer to the inner). These curves define a $3n$ component link $K$ in $S^3$ and we have that $[S^3, K] = \Phi^L(\xi^\Gamma_{\mathscr{I}})$. If $e_i, e_j, e_k (i < j < k)$ are the edges incident to a vertex $v$, then the three curves $\hat{e}_i, \hat{e}_j, \hat{e}_k$ form a Borromean link if $e_i, e_j, e_k$ have labels $a, b, c$, in that order, but form a *trivial link* if the labels are in any other order. We will call $v$ $\mathscr{I}$*-proper* in the former case and $\mathscr{I}$*-improper* in the latter case. We call an ordering $\mathscr{I}$, or the associated element $\xi^\Gamma_{\mathscr{I}}$ proper if all vertices are $\mathscr{I}$-proper, otherwise improper.

Now recall the isomorphism $F' : \mathscr{G}_n\mathscr{A}_b(\phi) \to \mathscr{G}^{as}_{3n}\mathscr{M}$ of equation (28) of Sect. 2.2. Comparing the definition of this map with our description of $K$, we see that $\Phi^L(\xi^\Gamma_{\mathscr{I}}) = F'(\Gamma_{\mathscr{I}})$ where $\Gamma_{\mathscr{I}}$ is the graph obtained from $\Gamma$ by splitting every *improper* vertex $v$ into three univalent vertices. Observing that $F'(\Gamma_{\mathscr{I}}) \equiv \begin{cases} F'(\Gamma) & \bmod F'(\mathscr{Y}_b) \quad \text{if } \mathscr{I} \text{ is proper} \\ 0 & \bmod F'(\mathscr{Y}_b) \quad \text{if } \mathscr{I} \text{ is improper} \end{cases}$ (which follows from the fact that each trivalent graph with at least one univalent vertex lies in $\mathscr{Y}_b$, see the proof of Lemma 2.9), it follows that

$$\Phi^L(\xi^\Gamma_{\mathscr{I}}) \equiv \begin{cases} F'(\Gamma) & \bmod F'(\mathscr{Y}_b) \quad \text{if } \mathscr{I} \text{ is proper} \\ 0 & \bmod F'(\mathscr{Y}_b) \quad \text{if } \mathscr{I} \text{ is improper} \end{cases} \tag{48}$$

For each label $x \in \{a, b, c\}$, let $x_i$ denote the edges with label $x$, in any order. Fix an initial ordering $\mathscr{I}_0 = \{a_1, \cdots, a_n, b_1, \cdots, b_n, c_1, \cdots, c_n\}$ of the edges of $\Gamma$. To simplify the notation we will also write $x_i$ when we really mean $\gamma_{x_i}$. This should cause no confusion.

**Claim 3.9.** *There is a commutator $C$ in the $e_i$ such that*

$$\xi^\Gamma_{\mathscr{I}_0} \equiv 1 - C + \text{improper terms} \mod I^{3n+1}$$

Note that (48) and the above claim imply that $\Phi^L(\xi^\Gamma_{\mathscr{I}}) = F'(\Gamma_b) \bmod F'(\mathscr{Y}_b)$. Using the definition of $F'$ (see (28)) and the definition of $\phi^L$ this imples that $F(\phi^L(\xi^\Gamma_{\mathscr{I}})) = \Gamma_b \bmod \mathscr{Y}_b$ which finishes the proof of Step 5.

*Proof*. [of Claim 3.9] Choose any $a_r$ and $b_s$ which are incident. Then, using (20), we have

$$\begin{aligned} 1 - \xi^\Gamma_{\mathscr{I}_0} \equiv & \pm \prod_{i \neq r}(1 - a_i)(1 - [a_r, b_s]) \prod_{j \neq s}(1 - b_j) \prod (1 - c_i) \\ & \pm \prod_{i \neq r}(1 - a_i)(1 - b_s)(1 - a_r) \prod_{j \neq s}(1 - b_j) \prod (1 - c_i) \mod I^{3n+1} \end{aligned}$$

The second term on the right side is improper since $b_s$ precedes $a_r$. Next choose some $a_t$ which is incident to $b_s$. Then we have

$$\prod_{i\neq r}(1-a_i)(1-[a_r,b_s])\prod_{j\neq s}(1-b_j)\prod(1-c_i)\equiv$$
$$\prod_{i\neq r,t}(1-a_i)(1-[a_t,[a_r,b_s]])\prod_{j\neq s}(1-b_j)\prod(1-c_i)$$
$$-\prod_{i\neq r,t}(1-a_i)(1-[a_r,b_s])(1-a_t)\prod_{j\neq s}(1-b_j)\prod(1-c_i)\quad \bmod I^{3n+1}$$

The second term on the right side can be further expanded by applying (20) to $1-[a_r,b_s]$ and we obtain, $\bmod I^{3n+1}$, a sum of two improper terms (since $b_s$ precedes $a_t$ in both terms). We next look for some $b_u$ which is incident to $a_t$ and, if it exists we can, in the same manner, write

$$\prod_{i\neq r,t}(1-a_i)(1-[a_t,[a_r,b_s]])\prod_{j\neq s}(1-b_j)\prod(1-c_i)\equiv$$
$$\prod_{i\neq r,t}(1-a_i)(1-[[a_t,[a_r,b_s]],b_u])\prod_{j\neq s,u}(1-b_j)\prod(1-c_i)$$
$$+\ \text{improper terms}\quad \bmod I^{3n+1}$$

Continuing in this way we eventually reach a point where we can write, after renumbering:

$$\xi^{\Gamma}_{\mathscr{I}_0}\equiv\pm\prod_{i\leq p}(1-a_i)(1-C')\prod_{j\leq q}(1-b_j)\prod(1-c_i)$$
$$+\ \text{improper terms}\quad \bmod I^{3n+1}$$

for some $p,q<n$ (actually $p=q$) and $C'$ is a commutator in which each $a_i, i>p$ and $b_j, j>q$ appears once and no $a_i, b_j$ with $i\leq p,\ j\leq q$ is incident to any $a_i, b_j$ with $i>p,\ j>q$. Thus $C'$ commutes with every $b_j$ and we have:

$$\xi^{\Gamma}_{\mathscr{I}_0}\equiv\pm\prod_{i\leq p}(1-a_i)\prod_{j\leq q}(1-b_j)(1-C')\prod(1-c_i)$$
$$+\ \text{improper terms}\quad \bmod I^{3n+1}$$

We now play the same game with $c_i, b_j$ and $C'$ that we just played with $a_i, b_j$. We will then eventually arrive at

$$\xi^{\Gamma}_{\mathscr{I}_0}\equiv\pm\prod_{i\leq p}(1-a_i)(1-C'')\prod_{j\leq q}(1-b_j)\prod_{i\leq r}(1-c_i)\tag{49}$$
$$+\ \text{improper terms}\quad \bmod I^{3n+1}$$

for some new $q$ and $r$ and new commutator $C''$ involving all the $a_i, b_i, c_i$ not involved in the other terms on the right side, and no $b_i$ or $c_i$ in $C''$ is incident to any not in $C''$. We now play the game again with the $a_i, b_i$ and $C''$. Going back and forth like this, we eventually arrive at the point where we have an equation of the form (49) where none of the edges in $C''$ are incident to any of the edges outside $C''$. But, since $\Gamma$ is connected, this is impossible unless there are no edges in $C''$. $\square$

• **Step 6** The Lagrangian $L$ and the Heegaard surface $\Sigma_\Gamma$ do not depend on the choice of the chord diagram $\Gamma$.
The equivalence (under isotopy) of the embeddings is not hard to see, assuming that we always choose the "natural" embedding associated with each chord diagram, namely embed the external circle to coincide with the standard circle in $\mathbb{R}^3$ and embed the internal edges as straight lines between the endpoints on the external circle, introducing a small undercrossing or overcrossing when necessary to avoid intersecting another internal edge. See Fig. 15 again. By sliding the ends of the tubes associated to the internal edges around on the torus associated to the external circle we can construct an isotopy between the embeddings associated to any two chord diagrams with the same number of edges. See Fig. 18.

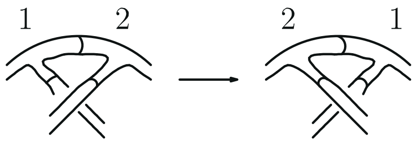



**Fig. 18.** An isotopy of the handlebody on the left to the handlebody on the right via a handle slide

What happens to $L$ under such an isotopy? The generators of $L$ are either curves on the $\Sigma(v)$ or meridians on the tubes associated to internal edges. In either case the isotopy preserves the curve and so there is no problem. This finishes the proof of Theorem 7. $\square$

We now discuss the analogous result for the Torelli group $\mathcal{T}_g$ and Johnson's group $\mathcal{K}_g$ instead of the Lagrangian group $\mathcal{L}^L_g$. Recall from Sect. 2.2 the maps $\phi^{\mathcal{J}}_f$ for $\mathcal{J} = \mathcal{T}_g, \mathcal{K}_g, \mathcal{L}^L_g$, for our fixed Heegaard splitting $f$, and the $f$-compatible Lagrangian $L$ of Theorem 7. For simplicity, we drop the dependence on $f$ from the notation of the above maps. We now have the following corollary:

**Corollary 3.10.** *Assuming $f$ to be the standard genus $g$ Heegaard splitting of $S^3$, the maps $\phi^K$ and $\phi^T$ are onto for $g \geq 5n+1$.*

*Proof.* Suppose $\Gamma$ is a connected trivalent vertex oriented graph of degree $n$. We have constructed above an element $C \in (\mathcal{L}^L_g)_{3n}$ such that $\Phi^L(1-C) = \Gamma \in \mathcal{G}^{as}_{3n}\mathcal{M}$. Recall that $C$ is a formal commutator in the elements $a_i, b_i, c_i$ for $1 \leq i \leq n$. Also recall that $b_i \in \mathcal{K}_g$ and so, since $\mathcal{K}_g$ is normal in $\mathcal{L}^L_g$, it follows that $C \in (\mathcal{K}_g)_n$. Since $\phi^K$ (resp., $\phi^L$) is induced by $\Phi^K$ (resp., $\Phi^L$) it follows by Lemma 2.14 that $\phi^L(C) = \phi^K(C) = \Phi^L(1-C)$ $= \Gamma$, which proves the corollary for $\mathcal{K}_g$.

For $\mathscr{T}_g$ we need to recall from Sect. 2.3 the fact that $\mathscr{K}_g = \mathscr{T}_g(2)$, where $\mathscr{T}_g(2)$ is the second quotient of the rational central series of $\mathscr{T}_g$. Since $C \in (\mathscr{K}_g)_n \subseteq \mathscr{T}_g(2n)$, we have $C^r \in (\mathscr{T}_g)_{2n}$ for some $r$. This and equation (21) show that $1 - C^r \equiv r(1 - C) \bmod (I\mathscr{K}_g)^{n+1}$ and so $\phi^T(C^r) = \phi^K(C^r) = $ v$\Phi^K(1 - C^r) = \Phi^K(r(1 - C)) = r\Gamma$ and we are done. $\square$

**3.7. A Gusarov group for homology spheres.** As an application of Theorem 7 in this section we prove Theorem 6. Recall the map $\tau_n : \mathcal{O}_n \to \mathscr{G}^{as}_{3n}\mathscr{M}$ of (19). First we observe the well-known fact that $\tau_n(\mathcal{O}_n)$ lies in the subspace of primitive elements of $\mathscr{G}^{as}\mathscr{M}$, which follows from

$$\begin{aligned}\Delta(S^3 - M) &= \Delta(S^3) - \Delta(M) = S^3 \otimes S^3 - M \otimes M \\ &= -(S^3 - M) \otimes (S^3 - M) + S^3 \otimes (S^3 - M) + (S^3 - M) \otimes S^3\end{aligned}$$

and the fact that if $S^3 - M \in \mathscr{F}^{as}_{3n}\mathscr{M}$ then $(S^3 - M) \otimes (S^3 - M) \equiv 0$ in $\mathscr{G}^{as}_{3n}(\mathscr{M} \otimes \mathscr{M})$.

With respect to the standard genus $g$ Heegaard splitting of $S^3$, we have a map $\mathscr{T}_g \hookrightarrow \mathbb{Q}\mathscr{T}_g \to \mathscr{M}$, using (2). It is easy to show that for every nonnegative integer $n$, the above map induces a well defined map $\mathscr{T}_g/\mathscr{T}_g(2n+1) \to \mathscr{E}_n$, thus, by restriction, we get a map $\sigma_n : \mathscr{G}_{2n}\mathscr{T}_g \to \mathscr{E}_n$. It is easy to show that $\sigma_n$ is a group homomorphism, and that the following diagram commutes:

$$\begin{array}{ccc} \mathscr{G}_{2n}\mathscr{T}_g & \longrightarrow & \mathscr{G}_{2n}\mathscr{T}_g \otimes \mathbb{Q} \\ \sigma_n \big\downarrow & & \big\downarrow \phi^T \\ \mathcal{O}_n & \xrightarrow{\tau_n} & \mathscr{G}^{as}_{3n}\mathscr{M} \end{array}$$

By its definition, $\tau_n$ is one-to-one. Furthermore, by Corollary 3.10, Im $\phi^T$ is the space of primitive elements, if $g \geq 5n + 1$.

We can now prove that $\mathcal{O}_n$ is a group. Suppose $\alpha \in \mathcal{O}_n$ – then $\tau_n(\alpha)$ is primitive and so $\tau_n(\alpha) \in$ Im $\phi^T$. So there are non-zero integers $k, l$ such that $\tau_n(k\alpha) = \phi^T(l\beta)$, for some $\beta \in \mathscr{G}_{2n}\mathscr{T}$. We can now compute:

$$\tau_n(k\alpha + \sigma_n(-l\beta)) = \phi^T(l\beta) + \phi^T(-l\beta) = 0$$

Since $\tau_n$ is one-one, we see that $(k-1)\alpha + \sigma_n(-l\beta)$ is an (additive) inverse for $\alpha$. Thus we have shown that $\mathcal{O}_n$ is an abelian group.

Now $\tau_n$ induces a linear map $\mathcal{O}_n \otimes \mathbb{Q} \to \mathscr{G}^{as}_{3n}\mathscr{M}$ which is one-to-one (since $\tau_n$ is one-to-one) and onto the subspace of primitive elements (since $\phi^T$ is onto). This conlcudes the proof of Theorem 6.

## 4. Results for the subgroups $\mathscr{K}_g$, $\mathscr{L}^L_g$ of the mapping class group

**4.1. Cocycles for $\mathscr{K}_g$.** In this section we discuss some similar constructions of cohomology classes for the Johnson subgroup $\mathscr{K}_g$, discussed in Sect. 2.3.

Following [GL3], recall from Sect. 2.2 that given an admissible surface $f : \Sigma_g \hookrightarrow M$, the induced map $\Phi_f^K : \mathbb{Q}\mathcal{K}_g \to \mathcal{M}$ maps the powers $(I\mathcal{K}_g)^m$ into $\mathcal{F}_m^K\mathcal{M}$. Moreover, it was shown that $\mathcal{F}_m^K\mathcal{M} \subseteq \mathcal{F}_{2m}^T\mathcal{M} = \mathcal{F}_{3m}^{as}\mathcal{M}$ thus inducing a map of associated graded spaces $\mathcal{G}_m^K\mathcal{M} \to \mathcal{G}_{2m}^T\mathcal{M} \simeq \mathcal{G}_{3m}^{as}\mathcal{M} \simeq \mathcal{G}_m\mathcal{A}(\phi)$, where the last isomorphism is given by the fundamental theorem of [LMO], as was explained in the introduction. In Remark 2.13 we pointed out that the composite map $\mathcal{G}_m^K\mathcal{M} \to \mathcal{G}_{3m}^{as}\mathcal{M}$ is onto, a fact that we will not use here. If we combine Corollary 2.5, for $G = \mathcal{K}_g, q = 2$, with $\Phi_f^K((I\mathcal{K}_g)^m)$, we obtain cocycles $C_{f,m}^K \in C^m(\mathcal{K}_g/\mathcal{K}_g(2); \mathcal{G}_m\mathcal{A}(\phi))$. Unlike the case of the Torelli group, we *cannot* conclude that the pull-back of $[C_{f,m}^K]$ to $H^m(\mathcal{K}_g; \mathcal{G}_m\mathcal{A}(\phi))$ is trivial.

The little that we can say is summarized in the following proposition.

**Proposition 4.1.**
• $C_{f,m}^K$ *is multilinear and* $\Gamma_g$*-equivariant, i.e., satisfies the following property* (*for* $\alpha_i \in \mathcal{K}_g/\mathcal{K}_g(2), h \in \Gamma_g$):

$$C_{hf,m}^K(h_*\alpha_1, \ldots, h_*\alpha_m) = C_{f,m}^K(\alpha_1, \ldots, \alpha_m) \tag{50}$$

• *If* $f$ *is an admissible Heegaard surface, then* $C_{f,m}^K$ *depends only on the* "*Lagrangians*" $L^+, L^- \subseteq \pi/\pi(3)$, *where* $\pi = \pi_1(\Sigma_g)$ *and* $L^\pm = \mathrm{Ker}\{i_\pm \circ f_* : \pi/\pi(3) \to \pi_1(M_\pm)/\pi_1(M_\pm)(3)\}$.

*Remark 4.2*. Let $\Gamma_g^K$ denote the quotient group $\Gamma_g/\mathcal{K}_g$. Then, from the work of Johnson, we have a short exact sequence:

$$1 \to U \to \Gamma_g^K \to Sp(H) \to 1$$

$\Gamma_g^K$ acts on $\mathcal{K}_g/\mathcal{K}_g(2)$ by conjugation, and on $\pi/\pi(3)$ by definition. It therefore acts on the Lagrangian pair in $\pi/\pi(3)$ and the equivariance property of $C_{f,m}^K$ is really an $\Gamma_g^K$-equivariance. Moreover the subgroup of $\Gamma_g^K$ which preserves the Lagrangian pair acts trivially.

*Proof*. The multilinearity and $\Gamma_g$-equivariance follows exactly as for the Torelli group.

The proof of the second assertion follows the same lines as the proof of the analogous result in Theorem 2 for the Torelli group. We need the following analogue of Lemma 3.1.

**Lemma 4.3.** *Suppose that* $Q$ *is a handlebody. Set* $\theta = \pi_1(Q), \pi = \pi_1(\partial Q)$ *and* $L = \mathrm{Ker}\{\pi/\pi(3) \to \theta/\theta(3)\}$. *If* $\alpha$ *is an automorphism of* $\pi/\pi(3)$ *such that* $\alpha(L) = L$, *then there exists a diffeomorphism* $h$ *of* $Q$ *such that* $(h|\partial Q)_* = \alpha$ (*modulo inner automorphisms*).

If we assume this lemma, then the rest of the proof proceeds as in the proof of Theorem 2 with the following changes.

The analogue of Lemma 3.2 is established with $H_1(\Sigma)$ replaced by $\pi/\pi(3)$. The proof only needs to be modified by observing that $g_\epsilon$ belongs to $\mathscr{K}_g$, since Johnson's result says that an element $g \in \Gamma_g$ belongs to $\mathscr{K}_g$ if and only if it induces the identity on $\pi/\pi(3)$.

The analogue of Lemma 3.3, with $H_1(\Sigma)$ replaced again by $\pi/\pi(3)$, is established by the same proof, with the extra observation that we may choose $h \in \mathscr{K}_g$ using the result of Morita [Mo3] that any integral homology 3-sphere is of the form $S^3_h$ for some $h \in \mathscr{K}_{g'}$, for some $g'$.

*Proof of Lemma 4.3.* Before we begin, lift $\alpha$ to an automorphism $F/F(3)$, where $F = \pi_1(\Sigma_0)$ is a free group. Recall the classical fact that the induced automorphism $\alpha_*$ on $H_1(\Sigma)$ is symplectic and so we can apply Lemma 3.1 to find a diffeomorphism $g$ of $Q$ so that $(g|\partial Q)_* = \alpha_*$ on $H_1(\Sigma)$. Thus we can assume that $\alpha_* =$ identity. The effect of $\alpha$ on $F/F(3)$ is measured by $\tau(\alpha)$, where $\tau$ is the Johnson homomorphism. We will now use Morita's Lemma 3.4. We can choose a set of generators $\{x_1, \dots, x_g, y_1, \dots, y_g\}$ for $F$ so that $L$ is normally generated by $\{y_1, \dots, y_g\}$. Then Morita's lemma says that $\alpha$ is induced by a diffeomorphism of $\Sigma_0$ if and only if $\tau(\alpha)$ belongs to the subgroup $W = \mathrm{Ker}\{\Lambda^3 H \to \Lambda^3 H/\Lambda^3 L\}$. Recall the definition of $\tau(\alpha)$. For any $h \in F/F(3)$ we can write $\alpha(h) = h\lambda(h)$. The assignment $h \to \lambda(h)$ defines a homomorphism $\lambda : H \to \Lambda^2 H \simeq F(2)/F(3)$. Now consider the element

$$\tau(\alpha) = \sum_i (x_i \otimes \lambda(y_i) - y_i \otimes \lambda(x_i)) \in \Lambda^3 H \subseteq H \otimes \Lambda^2 H \tag{51}$$

The inclusion $\Lambda^3 H \subseteq H \otimes \Lambda^2 H$ is defined by

$$a \wedge b \wedge c \to a \otimes (b \wedge c) + b \otimes (c \wedge a) + c \otimes (a \wedge b)$$

Now it is easy to see that $W = W' \cap \Lambda^3 H$, where $W' \subseteq H \otimes \Lambda^2 H$ is the kernel of the projection $H \otimes \Lambda^2 H \to (H/L) \otimes \Lambda^2(H/L)$. The condition that $\alpha(L) = L$ implies that $\lambda(h) \in \mathrm{Ker}\{\Lambda^2 H \to \Lambda^2(H/L)\}$. Remembering that $L$ is generated by $\{y_i\}$, we see that the first terms in equation (51) lie in $H \otimes \mathrm{Ker}\{\Lambda^2 H \to \Lambda^2(H/L)\}$ while the second terms lie in $L \otimes \Lambda^2 H$. Thus $\tau(\alpha) \in W$. □

This completes the proof of Proposition 4.1. □

**4.2. Cocycles for $\mathscr{L}^L_g$.** Given an admissible surface $f : \Sigma \hookrightarrow M$, and an $f$-compatible Lagrangian $L$, recall from Sect. 2.2 (see also [GL3]) the Lagrangian subgroup $\mathscr{L}^L_g \subseteq \Gamma_g$ of the mapping class group and the associated map $\Phi^L_f : \mathscr{L}^L_g \to \mathscr{M}$. It is proved in [GL3] that $\mathscr{F}^{as}_m \mathscr{M}$ is the union of the images $\Phi_f(I\mathscr{L}^L_g)^m)$ over all $f$ and $f$- admissible Lagrangians $L$. Recall also that $\mathscr{F}^{as}_m \mathscr{M}$ is a 3-step filtration and that $\mathscr{F}^{as}_{3m}\mathscr{M}/\mathscr{F}^{as}_{3m+1}\mathscr{M} \simeq \mathscr{G}_m\mathscr{A}(\phi)$. The general results of Sect. 2.1 yield the following.

**Proposition 4.4.** *Let $f : \Sigma_g \hookrightarrow M$ be an admissible surface, $L$ a $f$-compatible Lagrangian and $m$ be a nonnegative integer.*

• *There exist cocycles*

$$\begin{aligned} C^L_{f,3m} &\in C^{3m}(\mathscr{L}^L_g/\mathscr{L}^L_g(2);\mathscr{G}_m\mathscr{A}(\phi)) && \textit{for all } m \\ C^L_{f,3m-1} &\in C^{3m-1}(\mathscr{L}^L_g/\mathscr{L}^L_g(3);\mathscr{G}_m\mathscr{A}(\phi)) && \textit{for odd } m, \textit{ and} \\ C^L_{f,3m-2} &\in C^{3m-2}(\mathscr{L}^L_g/\mathscr{L}^L_g(4);\mathscr{G}_m\mathscr{A}(\phi)) && \textit{for even } m. \end{aligned}$$

• *The pullback of* $C^L_{f,3m}$ *to* $C^{3m}(\mathscr{L}^L_g/\mathscr{L}^L_g(3);\mathscr{G}_m\mathscr{A}(\phi))$ *under the projection* $\mathscr{L}^L_g/\mathscr{L}^L_g(2) \to \mathscr{L}^L_g/\mathscr{L}^L_g(3)$ *is a coboundary if m is even. The pullback of* $C^L_{f,3m-1}$ *to* $C^{3m-1}(\mathscr{L}^L_g/\mathscr{L}^L_g(4);\mathscr{G}_m\mathscr{A}(\phi))$ *under the projection* $\mathscr{L}^L_g/\mathscr{L}^L_g(3) \to \mathscr{L}^L_g/\mathscr{L}^L_g(4)$ *is a coboundary if m is odd.*

*Proof*. The first statement follows from Corollary 2.5 and the paragraph preceding Proposition 2.6. The second statement follows from Proposition 2.6. □

## 5. Discussion

In this section we discuss the results of the present paper in comparison with the work of Hain [Ha2] and Morita [Mo6], which has been a source of motivation and inspiration for our results.

**5.1. Finite type invariants of knots and integral homology 3-spheres.** One of the results of the present paper is the construction of a cocycle $C_{f,m} : \otimes^{2m} U \to \mathscr{G}_m\mathscr{A}(\phi)$ given an admissible surface $f : \Sigma \hookrightarrow M$, see Theorem 1. There is a well known (dictionary) correspondence between invariants of integral homology 3-spheres and invariants of knots. For several statements using the above dictionary, see [Ha2]. We caution the reader however, that the above mentioned dictionary is helpful in stating results, but *not necessarily* in proving them.

In this section we discuss a related map after we replace integral homology 3-spheres by knots and admissible surfaces by admissible braids. Let $\sigma \in B_n$ be a braid whose associated permutation is transitive, i.e., whose closure is a knot: such a braid will be called *admissible*. Let $\mathscr{A}(S^1)$ be the vector space over $\mathbb{Q}$ on the set of admissible trivalent graphs with additional univalent vertices that lie on a circle, divided out by the *AS* and *IHX* relations, see [B-N]. Using the definition of the map of equation (5), and replacing $\mathscr{M}$ (the vector space over $\mathbb{Q}$ of integral homology 3-spheres) by $\mathscr{K}$ (the vector space over $\mathbb{Q}$ of oriented knots in $S^3$), $\mathscr{F}^{as}_*\mathscr{M}$ by the Vassiliev filtration $\mathscr{F}_*\mathscr{K}$, $\mathscr{A}(\phi)$ by $A(S^1)$, admissible surfaces by admissible braids, and the fundamental theorem of finite type invariant s of integral homology 3-spheres by the fundamental theorem of finite type invariants of knots, we can define a map:

$$C_{\sigma,m} : \otimes^m (P_n/P_n(2)) \to \mathscr{G}_m\mathscr{A}(S^1) \tag{52}$$

One can show that the above map coincides with the following one: recall first that the abelianization $P_n/P_n(2)$ of the pure braid group is a free abelian group in generators $x_{ij}$ with $i, j = 1, 2, \ldots, n$ and relations $x_{ii} = 0$, $x_{ij} = x_{ji}$. Thus the tensor algebra $T(P_n/P_n(2))$ is a free (noncommutative) algebra in generators $x_{ij}$ for $1 \leq i < j \leq n$. Monomials in this algebra are represented in the left hand side of Fig. 20 and will be called chord diagrams on $n$ vertical strands. The map $C_{\sigma,m} : \otimes^m(P_n/P_n(2)) \to \mathscr{G}_m\mathscr{A}(S^1)$ defined above coincides with the map that closes a degree $m$ monomial (thought of as a degree $m$ vertical chord diagram on $n$ strands) to a chord diagram on $S^1$. The above mentioned closure of course depends on the admissible braid $\sigma$ but only in a mild way: one can show that it depends only on the image of the associated permutation.

As we discussed above, in the case of knots, the map (52) is well understood. This is due to the fact that there is a nice presentation of the pure braid group, and the fact that the $I$-adic completion of the rational group ring of the pure braid group $P_n$ is equal to a quotient of the tensor algebra $T(P_n/P_n(2))$ modulo the ideal generated by the 4-term relation.

In the case of the Torelli group though, this is *not* the case. To begin with, it is still unknown whether the Torelli group is finitely presented. On the other hand, the $I$-adic completion of the rational group ring of the Torelli group has been recently calculated by Hain [Ha2] using the *transcendental* theory of Mixed Hodge Structures. No combinatorial proof of the result is known. The map $C_{L^\pm,m}$ of equation (5) may help us understand the structure of the Torelli group in a combinatorial way, and, in the other direction, help us understand the space of finite type invariants of integral homology 3-spheres. It may also be a first step in understanding Hain's calculation.

We can now give the following dictionary between the case of knots and integral homology 3-spheres, summarized in the following table:

| | Knots | 3-Manifolds |
|---|---|---|
| $G$ | $P_n$ | $\mathscr{T}_g$ |
| $G/G(2)$ | $Free(x_{ij})_{i<j}$ | $U$ |
| Admissible objects | braids | surfaces |
| Graphical interpretation | Fig. 20 | Fig. 19 |
| Cocycles | $\mathscr{C}_{\sigma,m}$ | $\mathscr{C}_{f,m}$ |
| Chord diragrams | $\mathscr{A}(S^1)$ | $\mathscr{A}(\phi)$ |

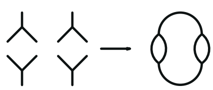


**Fig. 19.** On the left, 4 trivalent vertices, and on the right a particular closing to a trivalent graph

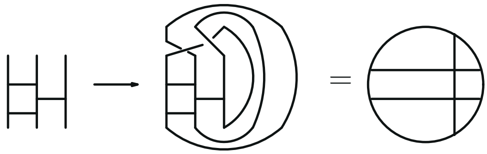


**Fig. 20.** On the left chord diagrams on 3 vertical strands, and on the right the resulting a chord diagram on $S^1$ ontained by closing the chord diagram on the left

**5.2. Comparison with the results of Morita** In this section we review briefly a very recent and important paper [Mo6] of Morita. A common problem addressed in both Morita's recent paper and ours is the one of constructing cocycles in various subgroups of the mapping class group. Morita [Mo6, Theorem p.3] uses a map[1] $\rho_1 : \Gamma_g \to \frac{1}{2}U \rtimes Sp(H)$, to construct for each $i$ an $Sp(H)$-invariant element $\beta_i \in H^{2i}(\frac{1}{2}U, \mathbb{Q})$ with the property that, if $\bar{\beta}_i$ is the class in $H^{2i}(U \rtimes Sp(H); \mathbb{Q})$ naturally associated to $\beta_i$, then $[\rho_1^*(\bar{\beta}_i)] = e_i \in H^{2i}(\Gamma_g, \mathbb{Q})$, where $e_i$ is a certain cohomology class studied by Mumford and Morita. By definition, the classes $\beta_i$ have the following properties:

- They are defined on the *cocycle* level.
- The pullback $[\rho_1^*(\bar{\beta}_i)]$ represents cohomology classes in the full mapping class group.
- They are $Sp(H)$ invariant cocycles.
- The pullback cocycles in $\mathcal{K}_g$ vanish.
- The coefficients of the cocycles are rational numbers.

On the other hand, the cocycles $C_{L^\pm,m}$ that we defined in Theorems 1 and 2 have the following properties:

- They are cocycles.
- They are defined in the abelianization of the Torelli group.
- The pullback of these cocycles to the Torelli group $\mathcal{T}_g$ represent trivial cohomology classes.
- They depend on a choice of Lagrangian pair $(L^+, L^-)$ and thus are only $GL(L^+)$ invariant, and not $Sp(H)$ invariant.
- The pullback of these cocycles to $\mathcal{K}_g$ vanish.
- The coefficients of these cocycles are the finite dimensional vector spaces of manifold weight systems $\mathcal{G}_m\mathcal{A}(\phi)$.

## 6. An epilogue or a beginning?

We end this paper with the following question. Recall from Theorem 5 the construction of a linear map $D_m : (\mathcal{G}_{2m}\mathcal{T}_g \otimes \mathbb{Q})^{Sp_g} \to \mathcal{G}_m\mathcal{A}^{conn}(\phi)$. This map is stable with respect to the genus and, for $m = 1$ it was shown to be a vector

[1]recall that we denote $U$ by $\Lambda^3 H/H$

space isomorphism (of one dimensional vector spaces). The authors now ask the following question:

**Question 1.** Is the map $D_m$ stably an onto for $m = $ odd?

Note that a positive answer would connect several different areas together. We hope to come back to the above question in the near future.

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