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# ON KNOTS WITH TRIVIAL ALEXANDER POLYNOMIAL

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## Abstract

We use the 2-loop term of the Kontsevich integral to show that there are (many) knots with trivial Alexander polynomial which do not have a Seifert surface whose genus equals the rank of the Seifert form. This is one of the first applications of the Kontsevich integral to intrinsically 3-dimensional questions in topology.

Our examples contradict a lemma of Mike Freedman, and we explain what went wrong in his argument and why the mistake is irrelevant for topological knot concordance.


## 1. A question about classical knots

Our starting point is a wrong lemma of Mike Freedman in [**5**, Lemma 2], dating back before his proof of the 4-dimensional topological Poincaré conjecture. To formulate the question, we need the following:

**Definition 1.1.** A knot in 3-space has *minimal Seifert rank* if it has a Seifert surface whose genus equals the rank of the Seifert form.

Since the Seifert form minus its transpose gives the (non-singular) intersection form on the Seifert surface, it follows that the genus is indeed the smallest possible rank of a Seifert form. The formula which computes the Alexander polynomial in terms of the Seifert form shows that knots with minimal Seifert rank have trivial Alexander polynomial. Freedman's wrong lemma claims that the converse is also true. However, in the argument, he overlooks the problem that S-equivalence does *not*

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preserve the condition of minimal Seifert rank. It turns out that not just the argument, but also the statement of the lemma is wrong. This has been overlooked for more than 20 years, may be because none of the classical knot invariants can distinguish the subtle difference between trivial Alexander polynomial and minimal Seifert rank.

In the last decade, knot theory was overwhelmed by a plethora of new "quantum" invariants, most notably the HOMFLY polynomial (specializing to the Alexander and the Jones polynomials), and the Kontsevich integral. Despite their rich structure, it is not clear how strong these invariants are for solving open problems in low dimensional topology. It is the purpose of this paper to provide one such application.

**Theorem 1.2.** *There are knots with trivial Alexander polynomial which do not have minimal Seifert rank. More precisely, the* 2*-loop part of the Kontsevich integral induces an epimorphism* $\overline{Q}$ *from the monoid of knots with trivial Alexander polynomial, onto an* infinitely generated *abelian group, such that* $\overline{Q}$ *vanishes on knots with minimal Seifert rank.*

The easiest counterexample is shown in Figure 1, drawn using surgery on a clasper. Surgery on a clasper is a refined form of Dehn surgery (along an embedded trivalent graph, rather than an embedded link) which we explain in Section 5. Clasper surgery is an elegant way of drawing knots that amplifies the important features of our example suppressing irrelevant information (such as the large number of crossings of the resulting knot). For example, in Figure 1, if one pulls the central edge of the clasper out of the visible Seifert surface, one obtains an S-equivalence to a non-trivial knot with minimal Seifert rank.

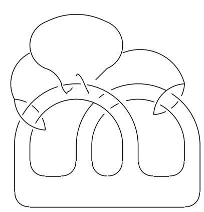


**Figure 1.** The simplest example, obtained by a clasper surgery on the unknot.

**Remark 1.3.** All of the above notions make sense for knots in homology spheres. Our proof of Theorem 1.2 works in that setting, too.

Since [**5**, Lemma 2] was the starting point of what eventually became Freedman's theorem which states that all knots with trivial Alexander polynomial are topologically slice, we should make sure that the above counterexamples to his lemma do not cause any problems in this important theorem. Fortunately, an argument independent of the wrong lemma can be found in [**6**, Theorem 7], see also [**7**, 11.7B]. However, it uses unnecessarily the surgery exact sequence and some facts from $L$-theory.

In an appendix, we shall give a more direct proof that Alexander polynomial 1 knots are topologically slice. We use no machinery, except for a single application of Freedman's main disk embedding theorem [**6**] in $D^4$. To satisfy the assumptions of this theorem, we employ a triangular base change for the intersection form of the complement of a Seifert surface in $D^4$, which works for all Alexander polynomial 1 knots. By Theorem 1.2, this base change does *not* work on the level of Seifert forms, as Freedman possibly tried to anticipate.

## 2. A relevant quantum invariant

The typical list of knot invariants that might find its way into a text book or survey talk on *classical* knot theory, would contain the Alexander polynomial, (twisted) signatures, (twisted) Arf invariants, and may be knot determinants. It turns out that all of these invariants can be computed from the homology of the infinite cyclic covering of the knot complement. In particular, they all vanish if the Alexander polynomial is trivial. This condition also implies that certain "non-commutative" knot invariants vanish, namely all those calculated from the homology of *solvable* coverings of the knot complement, like the Casson–Gordon invariants [**2**] or the von Neumann signatures of [**4**]. In fact, the latter are concordance invariants and, as discussed above, all knots with trivial Alexander polynomial are topologically slice.

Thus, it looks fairly difficult to study knots with trivial Alexander polynomial using classical invariants. Nevertheless, there are very natural topological questions about such knots like the one explained in the previous section. We do not know a classical treatment of that question, so we turn to quantum invariants.

One might want to use the Jones polynomial, which often distinguishes knots with trivial Alexander polynomial. However, it is not clear which knots it distinguishes, and which values it realizes, so the

Jones polynomial is of no help to this problem. Thus, we are looking for a quantum invariant that relates well to classical topology, has good realization properties, and is one step beyond the Alexander polynomial.

In a development starting with the Melvin–Morton–Rozansky conjecture and going all the way to the recent work of [**10**] and [**11**], the Kontsevich integral has been reorganized in a rational form $Z^{\mathrm{rat}}$ which is closer to the algebraic topology of knots. It is now a theorem (a restatement of the MMR Conjecture) that the "1-loop" part of the Kontsevich integral gives the same information as the Alexander polynomial [**1**, **17**].

The quantum invariant in Theorem 1.2 is the "2-loop" part $Q$ of the *rational invariant* $Z^{\mathrm{rat}}$ of [**11**]. We consider $Q$ as an invariant of Alexander polynomial 1 knots $K$ in integral homology spheres $M^3$, and summarize its properties:

- $Q$ takes values in the abelian group
$$\Lambda_\Theta := \frac{\mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}, t_3^{\pm 1}]}{(t_1 t_2 t_3 - 1, \quad \mathrm{Sym}_3 \times \mathrm{Sym}_2)}.$$
The second relations are given by the symmetric groups $\mathrm{Sym}_3$ which acts by permuting the $t_i$, and $\mathrm{Sym}_2$ which inverts the $t_i$ simultaneously.
- Under connected sums and orientation-reversing, $Q$ behaves as follows:
$$Q(M\#M', K\#K') = Q(M,K) + Q(M',K')$$
$$Q(M,-K) = Q(M,K) = -Q(-M,K).$$
- If one applies the augmentation map
$$\epsilon : \Lambda_\Theta \to \mathbb{Z}, \quad t_i \mapsto 1,$$
then $Q(M,K)$ is mapped to the Casson invariant $\lambda(M)$, normalized by $\lambda(S^3_{\text{Right Trefoil},+1}) = 1$.
- $Q$ has a simple behavior under surgery on *null claspers*, see Section 6.

All these properties are proven in [**10**] and in [**11**].

**Proposition 2.1** (Realization)**.** *Given a homology sphere $M^3$, the image of $Q$ on knots in $M$ with trivial Alexander polynomial is the subspace $\epsilon^{-1}(\lambda(M))$ of $\Lambda_\Theta$.*

**Remark 2.2.** The realization in the above proposition is concrete, not abstract. In fact, to realize the subgroup $\epsilon^{-1}(\lambda(M))$ one only needs (connected sums of) knots which are obtained as follows: Pick a standard Seifert surface $\Sigma$ of genus one for the unknot in $M$, and do a surgery along a clasper $G$ with one loop and two leaves which are meridians to the bands of $\Sigma$, just like in Figure 1. The loop of $G$ may intersect $\Sigma$ and these intersection create the interesting examples. Note that all of these knots are ribbons which implies unfortunately that the invariant $Q$ does *not* factor through knot concordance, even though it vanishes on knots of the form $K\# -K$.

Together with the following finiteness result, the above realization result proves Theorem 1.2, even for knots in a fixed homology sphere.

**Proposition 2.3** (Finiteness)**.** *The value of $Q$ on knots with minimal Seifert rank is the subgroup of $\Lambda_\Theta$, (finitely) generated by the three elements*

$$(t_1-1), \quad (t_1-1)(t_2^{-1}-1), \quad (t_1-1)(t_2-1)(t_3^{-1}-1).$$

*This holds for knots in 3-space, and one only has to add $\lambda(M)$ to all three elements to obtain the values of $Q$ for knots in a homology sphere $M$.*

**Corollary 2.4.** *If a knot $K$ in $S^3$ has minimal Seifert rank, then $Q(S^3,K)$ can be computed in terms of three Vassiliev invariants of degree* $3, 5, 5$.

The $Q$ invariant can be in fact calculated on many classes of examples. One such computation was done in [**8**]: The (untwisted) Whitehead double of a knot $K$ has minimal Seifert rank and $K \mapsto Q(S^3, \mathrm{Wh}(K))$ is a non-trivial Vassiliev invariant of degree 2.

**Remark 2.5.** Note that $K$ has minimal Seifert rank if and only if it bounds a certain grope of class 3. More precisely, the bottom surface of this grope is just the Seifert surface, and the second stages are embedded disjointly from the Seifert surface. However, they are allowed to intersect each other. So, this condition is quite different from the notion of a "grope cobordism" introduced in [**3**].

In a forthcoming paper, we will study related questions for boundary links. This is made possible by the rational version of the Kontsevich integral for such links recently defined in [**11**]. The analogue of knots with trivial Alexander polynomial are called *good boundary links*. In [**7**, 11.7C], this term was used for boundary links whose free cover has trivial homology. Unfortunately, the term was also used in [**5**] for a class of boundary links which should be rather called *boundary links of minimal Seifert rank*. This class of links is relevant because they form the atomic surgery problems for topological 4-manifolds, see Remark A.4. By Theorem 1.2, the two definitions of good boundary links in the literature actually differ substantially (even for knots). One way to resolve the "Schlamassel" would be to drop this term all together.

## 3. S-equivalence in homology spheres

We briefly recall some basic notions for knots in homology spheres. We decided to include the proofs because they are short and might not be well known for homology spheres, but we claim no originality. Let $K$ be a knot in a homology sphere $M^3$. By looking at the inverse image of a regular value under a map $M\backslash K \to S^1$, whose homotopy class generates

$$[M\backslash K, S^1] \cong H^1(M\backslash K;\mathbb{Z}) \cong H_1(K;\mathbb{Z}) \cong \mathbb{Z} \qquad \text{(Alexander duality in } M)$$

one constructs a *Seifert surface* $\Sigma$ for $K$. It is a connected oriented surface embedded in $M$ with boundary $K$. Note that *a priori* the resulting surface is not connected, but one just ignores the closed components. By the usual discussion about twistings near $K$, one sees that a collar of $\Sigma$ always defines the linking number zero pushoff of $K$.

To discuss uniqueness of Seifert surfaces, assume that $\Sigma_0$ and $\Sigma_1$ are both connected oriented surfaces in $M$ with boundary $K$.

**Lemma 3.1.** *After a finite sequence of "additions of tubes", i.e., ambient* 0*-surgeries,* $\Sigma_0$ *and* $\Sigma_1$ *become isotopic.*

*Proof.* Consider the following closed surface in the product $M \times I$ (where $I = [0,1]$):

$$\Sigma_0 \cup (K \times I) \cup \Sigma_1 \subset M \times I.$$

As above, relative Alexander duality shows that this surface bounds an connected oriented 3-manifold $W^3$, embedded in $M \times I$. By general position, we may assume that the projection $p : M \times I \to I$ restricts to a Morse function on $W$. Moreover, the usual dimension counts show that after an ambient isotopy of $W$ in $M \times I$, one can arrange for $p : W \to I$ to be an ordered Morse function, in the sense that the indices of the critical points appear in the same order as their values under $p$. This can be done relative to $K \times I \subset W$ since $p$ has no critical points there.

Consider a regular value $a \in I$ for $p$ between the index 1 and 2 critical points. Then, $\Sigma := p^{-1}(a) \subset M \times \{a\} = M$ is a Seifert surface for $K$. By Morse theory, $\Sigma$ is obtained from $\Sigma_0$ by

- A finite sequence of small 2-spheres $S_i$ in $M$ being born, disjoint from $\Sigma_0$. These correspond to the index 0 critical points of $p$.
- A finite sequence of tubes $T_k$, connecting the $S_i$ to (each other and) $\Sigma_0$. These correspond to the index 1 critical points of $p$.

Since $W$ is connected, we know that the resulting surface $\Sigma$ must be connected. In case there are no index 0 critical points, it is easy to see that $\Sigma$ is obtained from $\Sigma_0$ by additions of tubes. We will now reduce the general case to this case. This reduction is straight forward if the first tubes $T_i$ that are born have exactly one end on $S_i$, where $i$ runs through all index 0 critical points. Then, a sequence of applications of the *lamp cord trick* (in other words, a sequence of Morse cancellations) would show that up to isotopy, one can ignore these pairs of critical points, which include all index 0 critical points.

To deal with the general case, consider the level just after all $S_i$ were born and add "artificial" thin tubes (in the complement of the expected $T_k$) to obtain a connected surface. By the lamp cord trick, this surface is isotopic to $\Sigma_0$, and the $T_k$ are now tubes on $\Sigma_0$, producing a connected surface $\Sigma_0'$. Since by construction the tubes $T_k$ do not go through the artificial tubes, we can cut the artificial tubes to move from $\Sigma_0'$ back to $\Sigma$ (through index 2 critical points).

We can treat $\Sigma_1$ exactly as above, by turning the Morse function upside down, replacing index 3 by index 0, and index 2 by index 1 critical points. The result is a surface $\Sigma_1'$, obtained from $\Sigma_1$ by adding tubes, and such that $\Sigma$ is obtained from $\Sigma_1'$ by cutting other tubes.

Collecting the above information, we now have an ambient Morse function with only critical points of index 1 and 2, connecting $\Sigma_0$ and $\Sigma_1$ (rel $K$), and a middle surface $\Sigma$ which is tube equivalent to $\Sigma_0$ and $\Sigma_1$. The result follows. q.e.d.

The above proof motivates the definition of S-equivalence, which is the algebraic analogue, on the level of Seifert forms, of the geometric addition of tubes. Given a Seifert surface $\Sigma$ for $K$ in $M$, one defines the *Seifert form*

$$S_\Sigma : H_1\Sigma \times H_1\Sigma \to \mathbb{Z}$$

by the formula $S_\Sigma(a,b) := \mathrm{lk}(a,b^\downarrow)$. These are the usual linking numbers for circles in $M$ and $b^\downarrow$ is the circle $b$ on $\Sigma$, pushed slighly off the Seifert surface (in a direction given by the orientations). The down-arrow reminds us that in the case of $a$ and $b$ being the short and long curve on a tube, we are pushing $b$ *into* the tube, and hence the resulting linking number is one.

It should be clear what it means to "add a tube" to the Seifert form $S_\Sigma$: The homology increases by two free generators $s$ and $l$ (for "short" and "long" curve on the tube), and the linking numbers behave as follows:

$$\mathrm{lk}(s,s^\downarrow) = \mathrm{lk}(l,l^\downarrow) = \mathrm{lk}(l,s^\downarrow) = \mathrm{lk}(s,a^\downarrow) = 0,\ \mathrm{lk}(s,l^\downarrow) = 1 \quad \forall a \in H_1\Sigma.$$

Note that there is no restriction on the linking numbers of $l$ with curves on $\Sigma$, reflecting the fact that the tube can wind around $\Sigma$ in an arbitrary way.

Observing that isotopy of Seifert surfaces gives isomorphisms of their Seifert forms, we are lead to the following algebraic notion. It abstracts the necessary equivalence relation on Seifert forms coming from the non-uniqueness of the Seifert surface.

**Definition 3.2.** Two Seifert surfaces (for possibly distinct knots) are called *S-equivalent* if their Seifert forms become isomorphic after a finite sequence of (algebraic) additions of tubes.

## 4. Geometric basis for Seifert surfaces

It is convenient to discuss Seifert forms in terms of their corresponding matrices. So, for a given basis of $H_1\Sigma$, denote by $SM_\Sigma$ the matrix of linking numbers describing the Seifert form $S_\Sigma$. For example, the addition of a tube has the following effect on a Seifert matrix $SM$:

$$SM \mapsto \begin{pmatrix} SM & 0 & \rho \\ 0 & 0 & 1 \\ \rho^T & 0 & 0 \end{pmatrix}.$$

Here, we have used the short and long curves on the tube as the last two basis vectors (in that order). $\rho$ is the column of linking number of the long curve with the basis elements of $H_1\Sigma$ and $\rho^T$ is its transposed row. It is clear that, in general, this operation can destroy the condition of having minimal Seifert rank as defined in Definition 1.1. An important invariant of S-equivalence is the *Alexander polynomial*, defined by

$$\Delta_K(t) := \det(t^{1/2} \cdot SM - t^{-1/2} SM^T) \tag{1}$$

for any Seifert matrix $SM$ for $K$. One can check that this is unchanged under S-equivalence, it lies in $\mathbb{Z}[t^{\pm 1}]$ and satisfies the symmetry relations $\Delta_K(t^{-1}) = \Delta_K(t)$ and $\Delta_K(1) = 1$.

**Definition 4.1.** Let $\Sigma$ be a Seifert surface of genus $g$. The following basis of $H_1\Sigma$ will be useful.

- A *geometric basis* is a set of embedded simple closed curves $\{s_1, \ldots, s_g, \ell_1, \ldots, \ell_g\}$ on $\Sigma$ with the following geometric intersections
$$s_i \cap s_j = \emptyset = \ell_i \cap \ell_j, \text{ and } s_i \cap \ell_j = \delta_{i,j}$$
Note that the Seifert matrix $SM_\Sigma$ for a geometric basis always satisfies
$$SM_\Sigma - SM_\Sigma^T = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}.$$
- A *trivial Alexander basis* is a geometric basis such that the corresponding Seifert matrix can be written in terms of four blocks of $g \times g$-matrices as follows:
$$\begin{pmatrix} 0 & \mathbb{1} + U \\ U^T & V \end{pmatrix}.$$
Here, $U$ is an upper triangular matrix (with zeros on and below the diagonal), $U^T$ is its transpose, and $V$ is a symmetric matrix with zeros on the diagonal.
- A *minimal Seifert* basis is a trivial Alexander basis such that the matrices $U$ and $V$ are zero, so the Seifert matrix looks as simply as could be:
$$\begin{pmatrix} 0 & \mathbb{1} \\ 0 & 0 \end{pmatrix}.$$

By starting with a disk, and then adding tubes according to the matrices $U$ and $V$, it is clear that any matrix for a trivial Alexander basis can occur as the Seifert matrix for the unknot. The curves $s_i$ above are the short curves on the tubes, and $\ell_j$ are the long curves. The matrix $U$ must be lower triangular because the long curves can only link those short curves that are already present. The following lemma explains our choice of notation above:

**Lemma 4.2.** *Any Seifert surface has a geometric basis. Moreover,*

- *A knot has trivial Alexander polynomial if and only if there is Seifert surface with a trivial Alexander basis.*
- *A knot has minimal Seifert rank if and only if it has a Seifert surface with a minimal Seifert basis.*

*Proof.* By the classification of surfaces, they always have a geometric basis. If a knot has a trivial Alexander basis, then an elementary computation using Equation (1) implies that it has trivial Alexander polynomial. Finally, the Seifert matrix for a minimal Seifert basis obviously has minimal rank.

So, we are left with showing the two converses of the statements in our lemma. Start with a knot with trivial Alexander polynomial. Then, by Trotter's theorem [**23**], it is S-equivalent to the unknot, and hence its Seifert form is obtained from the empty form by a sequence of algebraic additions of tubes. Then, an easy induction implies that the resulting Seifert matrix $SM_\Sigma$ is as claimed, so we are left with showing that the corresponding basis can be chosen to be geometric on $\Sigma$. But since $SM_\Sigma$ - $SM_\Sigma^T$ is the standard (hyperbolic) form, we get a symplectic isomorphism of $H_1\Sigma$ which sends the given basis into a standard (geometric) one. Since the mapping class group realizes any such symplectic isomorphism, we see that the given basis can be realized by a geometric basis.

Finally, consider a Seifert surface with minimal Seifert rank. By assumption, there is a basis of $H_1\Sigma$ so that the Seifert matrix looks like

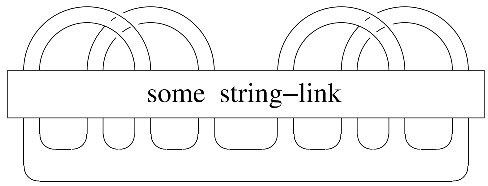
$$SM_\Sigma = \begin{pmatrix} 0 & A \\ 0 & B \end{pmatrix}.$$

Since $\Delta(1) = 1$, Equation (1) implies that $A$ must be invertible, and hence, there is a base change so that the Seifert matrix has the desired form

$$SM_\Sigma = \begin{pmatrix} 0 & \mathbb{1} \\ 0 & 0 \end{pmatrix}.$$

Just as above, one shows that this matrix is also realized by a geometric basis. q.e.d.

**Corollary 4.3.** *Every knot in $S^3$ with minimal Seifert rank $g$ can be constructed from a standard genus $g$ Seifert surface of the unknot,* *by tying the $2g$ bands into a 0-framed string link with trivial linking numbers.*




## 5. Clasper surgery

As we mentioned in Section 1, we can construct examples of knots that satisfy Theorem 1.2 using *surgery on claspers*. Since claspers play a key role in geometric constructions, as well as in realization of quantum invariants, we include a brief discussion here. For a reference on claspers[1] and their associated surgery, we refer the reader to [**14**, **15**, **16**] and also [**3**, **9**].

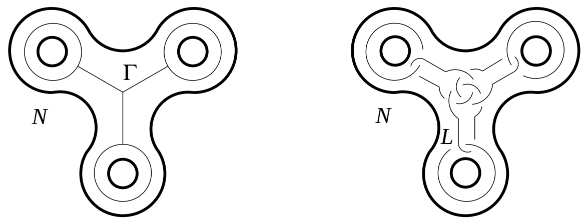
*Surgery* is an operation of cutting, twisting and pasting within the category of smooth manifolds. A low dimensional example of surgery is the well-known *Dehn surgery*, where we start from a framed link $L$ in a 3-manifold $M$, we cut out a tubular neighborhood of $L$, twist the boundary using the framing, and glue back. The result is a 3-dimensional manifold $M_L$.

Clasper surgery is entirely analogous to Dehn surgery, except that it is operated on claspers rather than links. A clasper is a thickening of a trivalent graph, and it has a preferred set of loops, called the leaves. The degree of a clasper is the number of trivalent vertices (excluding those at the leaves). With our conventions, the smallest clasper is a Y-clasper (which has degree one and three leaves), so we explicitly exclude struts (which would be of degree zero with two leaves).

A clasper of degree 1 is an embedding $G : N \to M$ of a regular neighborhood $N$ of the graph $\Gamma$ (with four trivalent vertices and six edges) into a 3-manifold $M$. Surgery on $G$ can be described by removing the genus 3 handlebody $G(N)$ from $M$, and regluing by a certain diffeomorphism of its boundary (which acts trivially on the homology of the boundary). We will denote the result of surgery by $M_G$. To explain the

[1]By clasper, we mean precisely the object called *clover* in [**9**]. For the sake of Peace in the World, after the Kyoto agreement of September 2001 at RIMS, we decided to follow this terminology.

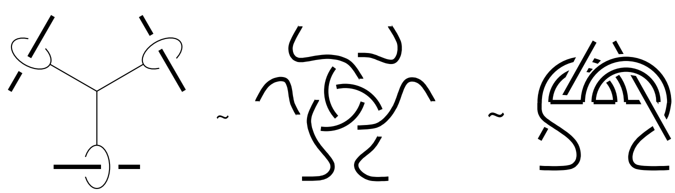
regluing diffeomorphism, we describe surgery on $G$ by surgery on the following framed six component link $L$ in $M$: $L$ consists of a 0-framed




Borromean ring and an arbitrarily framed three component link, the so-called *leaves* of $G$, see the figure above. The framings of the leaves reflect the prescribed neighborhood $G(N)$ of $\Gamma$ in $M$.

If one of the leaves is 0-framed and bounds an embedded disk disjoint from the rest of $G$, then surgery on $G$ does not change the 3-manifold $M$, because the gluing diffeomorphism extends to $G(N)$. In terms of the surgery on $L$, this is explained by a sequence of Kirby moves from $L$ to the empty link (giving a diffeomorphism $M_G \cong M$). However, if a second link $L'$ in $M \backslash G(N)$ intersects the disk bounding the 0-framed leaf of $L$, then the pairs $(M, L')$ and $(M_G, L')$ might not be diffeomorphic. This is the way how claspers act on knots or links in a fixed 3-manifold $M$, a point of view which is most relevant to this paper.

A particular case of surgery on a clasper of degree 1 (sometimes called a Y-move) looks locally as follows:



In general, surgery on a clasper $G$ of degree $n$ is defined in terms of simoultaneous surgery on $n$ claspers $G_1, \dots, G_n$ of degree 1. The $G_i$ are obtained from $G$ by breaking its edges and inserting 0-framed Hopf linked leaves as follows.

In particular, consider the clasper $G$ of degree 2 in Figure 1, which has two leaves and two edges. We can insert two pairs of Hopf links in the edges of $G$ to form two claspers $G_1$ and $G_2$ of degree 1, and describe the resulting clasper surgery on $G_1$ and $G_2$ by using twice the above figure on each of the leaves of $G$.

**Exercise 5.1.** Draw the knot which is described by surgery on a clasper of degree 2 in Figure 1.

It should be clear from the drawing why it is easier to describe knots by clasper surgery on the unknot, rather than by drawing them explicitly. Moreover, as we will see shortly, quantum invariants behave well under clasper surgery.

## 6. The $Q$ invariant

**6.1. A brief review of the $Z^{\mathrm{rat}}$ invariant.** The quantum invariant we want to use for Theorem 1.2 is the Euler-degree 2 part of the rational invariant $Z^{\mathrm{rat}}$ of [**11**]. In this section, we will give a brief review of the full $Z^{\mathrm{rat}}$ invariant. Hopefully, this will underline the general ideas more clearly, and will be a useful link with our forthcoming work. $Z^{\mathrm{rat}}$ is a rather complicated object; however, it simplifies when evaluated on Alexander polynomial 1 knots, as was explained in [**11**, Remark 1.6]. In particular, it is a map of monoids (taking connected sum to multiplication)

$$Z^{\mathrm{rat}} : \text{Alexander polynomial 1 knots} \longrightarrow \mathcal{A}(\Lambda),$$

where the range is a new algebra of diagrams with beads defined as follows. We abbreviate the ring of Laurent polynomials in $t$ as $\Lambda := \mathbb{Z}[t^{\pm 1}]$.

**Definition 6.1.** $\mathcal{A}(\Lambda)$ is the completed $\mathbb{Q}$-vector space generated by pairs $(G, c)$, where $G$ is a trivalent graph, with oriented edges and vertices and $c : \mathrm{Edges}(G) \to \Lambda$ is a $\Lambda$-coloring of $G$, modulo the relations: AS, IHX, Orientation Reversal, Linearity, Holonomy and Graph Automorphisms, (see Figure 2). $\mathcal{A}(\Lambda)$ is graded by the *Euler degree* (that is, the number of vertices of graphs) and the completion is with respect to this grading. $\mathcal{A}(\Lambda)$ is a commutative algebra with multiplication given by the disjoint union of graphs.

Notice that a connected trivalent graph $G$ has $2n$ vertices, $3n$ edges, and its Euler degree equals to $-2\chi(G)$, where $\chi(G)$ is the *Euler characteristic* of $G$. This explains the name "Euler degree".

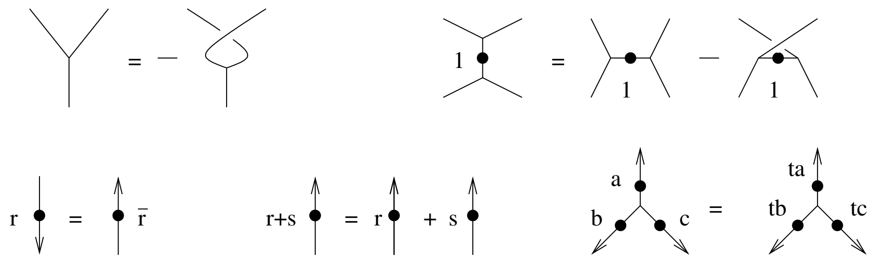
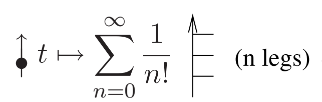



**Figure 2.** The AS, IHX, orientation reversal, linearity, and holonomy relations.

Where is the $Z^{\mathrm{rat}}$ invariant coming from? There is an important *hair* map

$$\mathrm{Hair} : \mathcal{A}(\Lambda) \longrightarrow \mathcal{A}(*),$$

which is defined by replacing a bead $t$ by an exponential of hair:




Here, $\mathcal{A}(*)$ is the completed (with respect to the *Vassiliev degree*, that is half the number of vertices) $\mathbb{Q}$-vector space spanned by vertex-oriented unitrivalent graphs, modulo the AS and IHX relations. It was shown in [**11**] that when evaluated on knots of Alexander polynomial 1, the Kontsevich integral $Z$ is determined by the rational invariant $Z^{\mathrm{rat}}$ by:

$$Z = \mathrm{Hair} \circ Z^{\mathrm{rat}}. \tag{2}$$

Thus, in some sense $Z^{\mathrm{rat}}$ is a *rational lift* of the Kontsevich integral. Note that although the Hair map above is not 1–1 [**20**], the invariants $Z$ and $Z^{\mathrm{rat}}$ might still contain the same information. The existence of the $Z^{\mathrm{rat}}$ invariant was predicted by Rozansky, [**21**], who constructed a rational lift of the colored Jones function, i.e., for the image of the Kontsevich integral on the level of the $\mathfrak{sl}_2$ Lie algebras. The $Z^{\mathrm{rat}}$ invariant was constructed in [**11**].

How can one compute the $Z^{\mathrm{rat}}$ invariant (and therefore, also the Kontsevich integral) on knots with trivial Alexander polynomial? This is a difficult question; however, $Z^{\mathrm{rat}}$ is a graded object, and in each degree, it is a *finite type invariant* in an appropriate sense. In order to explain this, we need to recall the *null move* of [**10**], which is defined in terms of surgery on a special type of clasper. Consider a knot $K$ in a homology sphere $M$ and a clasper $G \subset M \backslash K$ whose leaves are null homologous knots in the knot complement $X = M \backslash K$. We will call such claspers *null* and will denote the result of the corresponding surgery by $(M, K)_G$. Surgery on null claspers preserves the set of Alexander polynomial 1 knots. Moreover, by results of [**18**] and [**19**], one can untie every Alexander polynomial 1 knot via surgery on some null clasper, see [**10**, Lemma 1.3].

As usual, in the world of finite type invariants, if $G = \{G_1, \dots, G_n\}$ is a collection of null claspers, we set

$$[(M, K), G] := \sum_{I \subset \{0,1\}^n} (-1)^{|I|} (M, K)_{G_I},$$

where $|I|$ denotes the number of elements of $I$ and $(M, K)_{G_I}$ stands for the result of simultaneous surgery on $G_i$ for all $i \in I$. A *finite type invariant of null-type* $k$ by definition vanishes on all such alternating sums with $k < \deg(G) := \sum_{i=1}^n \deg(G_i)$.

**Theorem 6.2** ([**11**]). *$Z^{\mathrm{rat}}_{2n}$ is a finite type invariant of null-type $2n$.*

Furthermore, the degree $2n$ term (or *symbol*) of $Z^{\mathrm{rat}}_{2n}$ can be computed in terms of the equivariant linking numbers of the leaves of $G$, as we explain next. Fix an Alexander polynomial 1 knot $(M, K)$, and consider a *null homologous link* $C \subset X$ of two ordered components, where $X = M \backslash K$. The lift $\widetilde{C}$ of $C$ to the $\mathbb{Z}$-cover $\widetilde{X}$ of $X$ is a link. Since $H_1(\widetilde{X}) = 0$ (due to our assumption that $\Delta(M, K) = 1$) and $H_2(\widetilde{X}) = 0$ (true for $\mathbb{Z}$-covers of knot complements) it makes sense to consider the linking number of $\widetilde{C}$. Fix a choice of lifts $\widetilde{C}_i$ for the components of $C$. The equivariant linking number is the finite sum

$$\mathrm{lk}_{\mathbb{Z}}(C_1, C_2) = \sum_{n \in \mathbb{Z}} \mathrm{lk}(\widetilde{C}_1, t^n \, \widetilde{C}_2) \, t^n \ \in \mathbb{Z}[t^{\pm 1}] = \Lambda.$$

Shifting the lifts $\widetilde{C}_i$ by $n_i \in \mathbb{Z}$ multiplies this expression by $t^{n_1 - n_2}$. There is a way to fix this ambiguity by considering an arc-basing of $C$, that is a choice of disjoint embedded arcs $\gamma$ in $M \backslash (K \cup C)$ from a base point to each of the components of $C$. In that case, we can choose a lift of $C \cup \gamma$ to $\widetilde{X}$ and define the equivariant linking number $\mathrm{lk}_{\mathbb{Z}}(C_1, C_2)$. The result is independent of the lift of $C \cup \gamma$, but of course depends on the arc-basing $\gamma$.

It will be useful for computations to describe an alternative way of fixing the ambiguity in the definition of equivariant linking numbers.

Given $(M, K)$, consider a Seifert surface $\Sigma$ for $(M, K)$, and a link $C$ of two-ordered components in $M \backslash \Sigma$. We will call such links $\Sigma$-*null*. Notice that a $\Sigma$-null link is $(M, K)$-null, and conversely, every $(M, K)$-null link is $\Sigma$-null for some Seifert surface $\Sigma$ of $(M, K)$. Given a $\Sigma$-null link $C$ of two-ordered components, one can construct the $\mathbb{Z}$-cover $\widetilde{X}$ by cutting $X$ along $\Sigma$, and then putting $\mathbb{Z}$ copies of this fundamental domain together to obtain $\widetilde{X}$. It is then obvious that there are canonical lifts of $\Sigma$-null links which lie in one fundamental domain and using them, one can define the equivariant linking number of $C$ without ambiguity.

This definition of equivariant linking number agrees with the previous one if we choose basing arcs which are disjoint from $\Sigma$.

**Example 6.3.** Consider a standard Seifert surface $\Sigma$ for the unknot $\mathcal{O}$. Let $C_i$ be two meridians of the bands of $\Sigma$; thus, $(C_1, C_2)$ is $\Sigma$-null. If these bands are not dual, then $(\mathcal{O}, C_1, C_2)$ is an unlink and hence $\mathrm{lk}_{\mathbb{Z}}(C_1, C_2) = 0$. If the bands are dual, then this 3-component link is the Borromean rings. Recall that the Borromean rings are the Hopf link with one component Bing doubled (and the other one being $\mathcal{O}$). Then, one can pull apart that link, in the complement of $\mathcal{O}$, by introducing two intersections (of opposite sign) between $C_1$ and $C_2$, differing by the meridian $t$ to $\mathcal{O}$. This shows that in this case,

$$\mathrm{lk}_{\mathbb{Z}}(C_1, C_2) = t - 1.$$

In order to give a formula for the symbol of $Z^{\mathrm{rat}}_{2n}$, we need to recall the useful notion of a complete contraction of an $(M, K)$-null clasper $G$ of degree $2n$, [**10**, Section 3]. Let $G^{\mathrm{break}} = \{G_1, \ldots, G_{2n}\}$ denote the collection of degree 1 claspers $G_i$ which are obtained by inserting a Hopf link in the edges of $G$. Choose arcs from a fixed base point to the trivalent vertex of each $G^{nl}_i$, which allows us to define the equivariant linking numbers of the leaves of $G^{\mathrm{break}}$. Let $G^{nl} = \{G^{nl}_1, \ldots, G^{nl}_{2n}\}$ denote the collection of abstract unitrivalent graph obtained by removing the leaves of the $G_i$ (and leaving one leg, or univalent vertex, for each leave behind). Then, the *complete contraction* $\langle G \rangle \in \mathcal{A}(\Lambda)$ of $G$ is defined to be the sum over all ways of gluing pairwise the legs of $G^{nl}$, with the resulting edges of each summand labelled by elements of $\Lambda$ as follows: pick orientations of the edges of $G^{nl}$ such that pairs of legs that are glued are oriented consistently. If two legs $l$ and $l'$ are glued, with the orientation giving the order, then we attach the bead $\mathrm{lk}_{\mathbb{Z}}(l, l')$ on the edge created by the gluing.

The result of a complete contraction of a null clasper $G$ is a well-defined element of $\mathcal{A}(\Lambda)$. Changing the edge orientations is taken care of by the symmetry of the equivariant linking number as well as the orientation reversal relations. Changing the arcs is taken care by the holonomy relations in $\mathcal{A}(\Lambda)$.

Then, the complete contraction $\langle G\rangle \in \mathcal{A}(\Lambda)$ of a single clasper $G$ with $\Sigma$-null leaves is easily checked to be the sum over all ways of gluing pairwise the legs of $G^{nl}$, with the resulting edges of each summand labelled by elements of $\Lambda$ as follows: First, pick orientations of the edges of $G^{nl}$ such that pairs of legs that are glued are oriented consistently. If two legs $l$ and $l'$ are glued, with the orientation giving the order, then we attach the bead $\mathrm{lk}_{\mathbb{Z}}(l,l')$ on the edge created by the gluing. In addition, each internal edge $e$ of $G^{nl}$ is labelled by $t^n$, where $n\in\mathbb{Z}$ is the intersection number of $e$ with the Seifert surface $\Sigma$.

One can check directly that this way of calculating a complete contraction of a clasper $G$ with $\Sigma$-null leaves is a well-defined element of $\mathcal{A}(\Lambda)$: Changing the edge orientations is taken care of by the symmetry of the equivariant linking number as well as the orientation reversal relations. The holonomy relations in $\mathcal{A}(\Lambda)$ correspond beautifully to Figure 3 in which a trivalent vertex of $G$ is pushed through $\Sigma$.

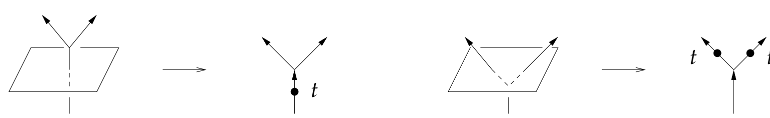



**Figure 3.** A surface isotopy that explains the holonomy relation.

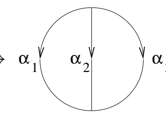
Finally, we can state the main result on calculating the invariant $Z^{\mathrm{rat}}$.

**Theorem 6.4** ([**11**, Theorem 4])**.** *If $(M,K)$ is a knot with trivial Alexander polynomial and $G$ is a collection of $(M,K)$-null claspers of degree $2n$, then*

$$Z^{\mathrm{rat}}_{2n}([(M,K),G]) = \langle G\rangle \in \mathcal{A}_{2n}(\Lambda).$$

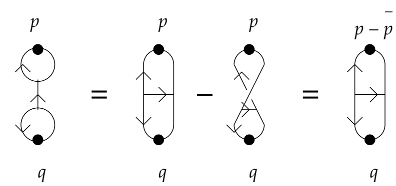
**6.2. A review of the $Q$ invariant.** We will be interested in $Q = Z^{\mathrm{rat}}_2$, the loop-degree 2 part of $Z^{\mathrm{rat}}$. It turns out that $Q$ takes values in a lattice $\mathcal{A}_{2,\mathbb{Z}}(\Lambda)$, that is the abelian subgroup of $\mathcal{A}(2,\Lambda)$ generated by integer multiples of graphs with beads. Lemma 6.5 (taken from [**12**, Lemma 5.9]) explains the definition of $\Lambda_\Theta$.

**Lemma 6.5.** *There is an isomorphism of abelian groups:*

(3) $\Lambda_\Theta \longrightarrow \mathcal{A}_{2,\mathbb{Z}}(\Lambda)$ *given by:* $\alpha_1\,\alpha_2\,\alpha_3 \mapsto$ 

.

*Proof.* Since $\mathrm{Aut}(\Theta) \cong \mathrm{Sym}_3 \times \mathrm{Sym}_2$, it is easy to see that the above map is well-defined. There are two trivalent graphs of degree 2, namely $\Theta$ and ⚭. Using the Holonomy Relation, we can assume that the labeling of the middle edge of ⚭ is 1. In that case, the IHX relation implies that




This shows that the map in question is onto. It is also easy to see that it is a monomorphism. q.e.d.

Let us define the *reduced* groups

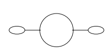
$$\widetilde{\mathcal{A}}(\Lambda) = \mathrm{Ker}(\mathcal{A}(\Lambda) \to \mathcal{A}(\phi))$$

induced by the augmentation map $\epsilon : \Lambda \to \mathbb{Z}$. Let $\widetilde{\Lambda}_\Theta := \mathrm{Ker}(\epsilon : \Lambda_\Theta \to \mathbb{Z})$. The proof of Lemma 6.5 implies that there is an isomorphism:

$$\widetilde{\Lambda}_\Theta \cong \widetilde{\mathcal{A}}_{2,\mathbb{Z}}(\Lambda).$$

### 6.3. Realization and finiteness.

*Proof of Proposition* 2.1. (Realization). Let us first assume that the ambient 3-manifold $M = S^3$. It is easy to see that $\widetilde{\Lambda_\Theta}$ is generated by $(t_1 - 1)t_2^n t_3^m$ for $n, m \in \mathbb{Z}$, so we only need to realize these values. Consider a standard genus one Seifert surface $\Sigma$ of an unknot with bands $\{\alpha, \beta\}$ and the clasper $G$



of degree 2 (with two leaves shown as ellipses above). Choose an embedding of $G$ into $S^3 \backslash \mathcal{O}$ in such a way that the two leaves are 0-framed meridians of the two bands of $\Sigma$ and the two internal edges of $G$ intersect $\Sigma$ algebraically $n$ respectively $m$ times. Then, $G$ is a $\Sigma$-null clasper and Theorem 6.4, together with Example 6.3 we get

$$Q(S^3, \mathcal{O}_G) = -Q([(S^3, \mathcal{O}), G]) = (1 - t_1)t_2^n t_3^m \in \widetilde{\Lambda}_\Theta.$$

The realization result follows for $M = S^3$. For the case of a general homology sphere $M$, use the behavior of $Q$ under connected sums. To show that the constructed knots are ribbon, we refer to [**13**, Lemma 2.1, Theorem 5], or [**3**, Theorem 4]. q.e.d.

Lemma 6.6 gives a clasper construction of all minimal Seifert rank knots. We first introduce a useful definition. Consider a surface $\Sigma \subset S^3$ and a clasper $G \subset S^3 \backslash \partial\Sigma$. We say that $G$ is $\Sigma$-*simple* if the leaves of $G$ are 0-framed meridians of the bands of $\Sigma$ and the edges of $G$ are disjoint from $\Sigma$.

**Lemma 6.6.** *Every knot in $S^3$ with minimal Seifert rank can be constructed from a standard Seifert surface $\Sigma$ of the unknot, by surgery on a disjoint collection of $\Sigma$-simple Y-claspers.*

*Proof.* The result follows by Lemma 4.3 and the fact, proven by Murakami–Nakanishi [**19**], that every string-link with trivial linking numbers can be untied by a sequence of Borromean moves. In terms of $\mathcal{O}$, these Borromean moves are $\Sigma$-simple Y-clasper surgeries (with the leaves being 0-framed meridians to the bands of $\Sigma$). q.e.d.

*Proof of Proposition* 2.3. (Finiteness) Consider a knot $K$ in $S^3$ with minimal Seifert rank. By Lemma 6.6, it is obtained from a standard Seifert surface $\Sigma$ of an unknot $\mathcal{O}$ by surgery on a disjoint collection $G$ of $\Sigma$-simple Y-claspers. The fact that $Q$ is an invariant of type 2 implies that

$$\begin{aligned} Q(S^3, K) &= -Q((S^3, \mathcal{O}) - (S^3, \mathcal{O})_G) \\ &= -\sum_{G' \subset G} Q([(S^3, \mathcal{O}), G']) + \sum_{G'' \subset G} Q([(S^3, \mathcal{O}), G'']), \end{aligned}$$

where the summation is over all claspers $G'$ and $G''$ of degree 1 and 2, respectively. The $Q([(S^3, \mathcal{O}), G''])$ terms can be computed by complete contractions and using Example 6.3, it follows that they contribute only summands of the form $(t_i - 1)$.

Next, we simplify the remaining terms, which are given by $\Sigma$-simple Y-claspers $G' \subset G$. Note that we can work modulo $\Sigma$-simple claspers of degree $> 1$ by the above argument. Using the Sliding Lemma [**10**, Lemma 2.5], we can move around all edges and finally put $G'$ into a standard position as in Figure 4.

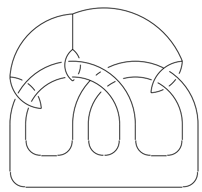


**Figure 4.** The remaining knots, possibly with half-twists (not shown) on the edges of the clasper.

We are reduced to $\Sigma$ of genus one because if the three leaves of $G'$ are meridians to three distinct bands of $\Sigma$, the unknot $\mathcal{O}$ would slip off the clasper altogether, i.e., surgery on the simplified $G'$ does not alter $\mathcal{O}$.

This means that we are left with a family of four examples, given by the various possibilities of the half-twists in the three edges of the the clasper in Figure 4. Let $\alpha$ and $\beta$ denote the two bands of the standard genus 1 surface $\Sigma$, and let $m_\alpha, m_\beta$ (resp. $\ell_\alpha, \ell_\beta$) denote the knots which are meridians (resp. longitudes) of the bands.

Let $G'$ denote the $\Sigma$-simple clasper of degree 1 as in Figure 4. It has 3 leaves $m_\alpha, m_\alpha$ and $\ell_\beta$.

**Claim 6.7.** We have

$$[(S^3, \mathcal{O}), G'] = [(S^3, \mathcal{O}), G''] + [(S^3, \mathcal{O}), G''']$$

modulo terms of degree 2, where $G''$ is a $\Sigma$-simple clasper with leaves $m_\alpha, m_\alpha, \ell_\alpha$ and $G'''$ is obtained from $G''$ by replacing the edge of $\ell_\alpha$ by one that intersects $\Sigma$ once.

*Proof of the claim.* Observe that $m_\beta$ is isotopic to $\ell_\alpha$ by an isotopy rel $\Sigma$. Use this isotopy to move the leaf $\ell_\beta$ of $G'$ near the $\alpha$ handle, and use the Cutting a Leaf lemma ([**10**, Lemma 2.4]) to conclude the proof. q.e.d.

Going back to the proof of Proposition 2.3, we may apply the Cutting a Leaf lemma once again to replace $G''$ by a $\Sigma$-simple clasper with leaves two copies of $m_\alpha$ together with a meridian of one copy of $m_\alpha$. For this clasper, the surface $\Sigma$ can slide off, and as a result surgery gives back the unknot. Work similarly for $G'''$, and conclude that $Q([(S^3, \mathcal{O}), G'])$

lies in the subgroup of $\Lambda_\Theta$ which is generated by the elements

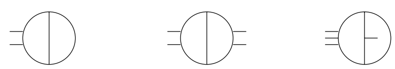
$$(t_1^{\epsilon_1}-1), \quad (t_1^{\epsilon_1}-1)(t_2^{\epsilon_2}-1), \quad (t_1^{\epsilon_1}-1)(t_2^{\epsilon_2}-1)(t_3^{\epsilon_3}-1)$$

for all $\epsilon_i = \pm 1$. Using the relations in $\Lambda_\Theta$, it is easy to show that this subgroup is generated by the three elements as claimed in Proposition 2.3. This concludes the proposition for knots in $S^3$.

In the case of a knot $K$ with minimal Seifert rank in a general homology sphere $M$, we may untie it by surgery on a collection of $\Sigma$-simple Y-claspers, $\Sigma$ a standard Seifert surface for the unknot $\mathcal{O}$. That is, we may assume that $(M,K) = (S^3,\mathcal{O})_G$ for some $\Sigma$-null clasper $G$ whose leaves are meridians of the bands of $\Sigma$ and have framing 0 or $\pm 1$. We can follow the previous proof to conclude our result. q.e.d.

*Proof of Corollary 2.4.* As we discussed previously, the rational invariant $Z^{\mathrm{rat}}$ determines the Kontsevich integral via Equation (2). It follows that $\mathrm{Hair} \circ Q$ is a power series of Vassiliev invariants. Although the Hair map is not 1–1, it is for diagrams with two loops, thus $\mathrm{Hair} \circ Q$ determines $Q$.

Consider the image of $t_1 - 1$, $(t_1-1)(t_2^{-1}-1)$ and $(t_1-1)(t_2-1)(t_3^{-1}-1)$ under the Hair map in $\mathcal{A}(*)$. It follows that the Vassiliev invariants of degree $3, 5$ and $5$ which separate the uni-trivalent graphs



determine the value of $Q$ on knots with minimal Seifert rank. q.e.d.

## Appendix A. Knots with trivial Alexander polynomial are topologically slice

A complete argument for this fact can be found in [**6**, Theorem 7], see also [**7**, 11.7B]. However, that argument uses unnecessarily the surgery exact sequence for the trivial as well as infinite cyclic fundamental group. Moreover, one needs to know Wall's surgery groups $L_i(\mathbb{Z}[\mathbb{Z}])$ for $i = 4, 5$.

We shall give a direct argument in the spirit of [**5**] but without assuming that the knot has minimal Seifert rank (which Freedman did assume indirectly). The simple new ingredient is the triangular base change, Lemma A.2. Note that at the time of writing [**5**], the topological disk embedding theorem was not known, so the outcome of the constructions below was much weaker than an actual topological slice.

The direct argument uses a single application of Freedman's main disk embedding theorem [**6**]. In [**6**], it is not stated in its most general form which we need here, so we really use the disk embedding theorem [**7**, 5.1B]. So, let us first recall this basic theorem. It works in any 4-manifold with *good* fundamental group, an assumption which up to day is not known to be really necessary. In any case, cyclic groups are known to be good which is all we need in this appendix. Note that the second assumption, on dual 2-spheres, is well known to be necessary. Without this assumption, the proof below would imply that every "algebraically slice" knot, i.e., a knot whose Seifert form has a Lagrangian, is topologically slice. This contradicts for example the invariants of [**2**]. A more direct reason that this assumption is necessary was recently given in [**22**]: In the absence of dual 2-spheres, there are non-trivial secondary invariants (in two copies of the group ring modulo certain relations), which are obstructions to a disk being homotopic to an embedding.

**Theorem A.1.** *[Disk embedding theorem [**7**, 5.1B]] Let $\Delta_j : (D^2, S^1) \to (N^4, \partial N)$ be continuous maps of disks which are embeddings on the boundary, and assume that all intersection and self-intersection numbers vanish in $\mathbb{Z}[\pi_1 N]$. If $\pi_1 N$ is good and there exist algebraically dual 2-spheres, then there is a regular homotopy (rel. boundary) which takes the $\Delta_j$ to disjoint (topologically flat) embeddings.*

The assumption on dual 2-spheres (which is an algebraic condition) means that there are framed immersions $f_i : S^2 \to N$ such that the intersection numbers in $\mathbb{Z}[\pi_1 N]$ satisfy

$$\lambda(f_i, \Delta_j) = \delta_{i,j}.$$

The following simple observation turns out to be crucial for Alexander polynomial 1 knots.

**Lemma A.2.** *There exist algebraically dual 2-spheres for $\Delta_i$ if and only if there exist framed immersions $g_i : S^2 \to N$ with*

$$\lambda(g_i, \Delta_i) = 1 \text{ and } \lambda(g_i, \Delta_j) = 0 \text{ for } i > j.$$

*So the matrix of intersection numbers of $g_i$ and $\Delta_j$ needs to have zeros only below the diagonal.*

*Proof.* Define $f_1 := g_1$, and then inductively

$$f_i := g_i - \sum_{k<i} \lambda(g_i, \Delta_k) f_k.$$

Then, one easily checks that $\lambda(f_i, \Delta_j) = \delta_{i,j}$. q.e.d.

**Remark A.3.** The disk embedding theorem is proven by an application of another embedding theorem [**7**, 5.1A], to the Whitney disks pairing the intersections among the $\Delta_i$. Thus, [**7**, Theorem 5.1A] might be considered as more basic. It sounds very similar to [**7**, Theorem 5.1B], except that the assumptions on trivial intersection and self-intersection numbers is moved from the $\Delta_i$ to the dual 2-spheres. Hence, one looses the information about the regular homotopy class of $\Delta_i$.

In most applications, one wants this homotopy information, hence we have stated Theorem 5.1B as the basic disk embedding theorem. However, in the application below, we might as well have used 5.1A directly, by interchanging the roles of $s_i$ and $\ell_i$.

The following proof will be given for knots (and slices) in $(D^4, S^3)$, but it works just as well in $(C^4, M^3)$ where $M$ is any homology sphere and $C$ is *the* contractible topological 4-manifold with boundary $M$.

*Proof of the appendix title.* Since the knot $K$ has trivial Alexander polynomial, Lemma 4.2 shows that we can choose a Seifert surface $\Sigma_1$ with a trivial Alexander basis $\{s_1, \ldots, s_g, \ell_1, \ldots, \ell_g\}$. Pick generically immersed disks $\Delta(s_j)$ (respectively $\Delta(\ell_j)$) in $D^4$ which bound ${s_j}^{\downarrow}$ (respectively $\ell_j$). So, these disks are disjoint on the boundary, and the intersection numbers satisfy

$$\Delta(s_i) \cdot \Delta(s_j) = \mathrm{lk}({s_i}^{\downarrow}, {s_j}^{\downarrow}) = \mathrm{lk}({s_i}^{\downarrow}, s_j) = 0 \quad \text{and}$$
$$\Delta(s_i) \cdot \Delta(\ell_j) = \mathrm{lk}({s_i}^{\downarrow}, \ell_j).$$

By Definition 4.1, the latter is a triangular matrix, which will turn out to be the crucial fact.

Now, we "push" the Seifert surface $\Sigma_1$ slightly into $D^4$ to obtain a surface $\Sigma \subset D^4$, and call $N$ the complement of (an open neighborhood of) $\Sigma$ in $D^4$. The basic idea of the proof is to use the disk embedding theorem in $N$ to show that $\Sigma$ can be ambiently surgered into a disk which will be a slice disk for our knot $K$.

To understand the 4-manifold $N$ better, note that by Alexander duality

$$H_1 N \cong H^2(\Sigma, \partial\Sigma) \cong \mathbb{Z} \text{ and } H_2 N \cong H^1(\Sigma, \partial\Sigma) \cong \mathbb{Z}^{2g}.$$

Moreover, a Morse function on $N$ is given by restricting the radius function on $D^4$. Reading from the center of $D^4$ outward, this Morse function has one critical point of index 0, one of index 1 (the minimum of $\Sigma$), and $2g$ critical points of index 2, one for each band of $\Sigma$. Together with the above homology information, this implies that $N$ is homotopy equivalent to a wedge of a circle and $2g$ 2-spheres.

To make the construction of $N$ more precise, we prefer to add an exterior collar $(S^3 \times [1, 1.5], K \times [1, 1.5])$ to $D^4$, i.e. we work with the knot $K$ in the 4-disk $D_{1.5}$ of radius 1.5. Then, the pushed in Seifert surface $\Sigma \subset D_{1.5}$ is just $(K \times [1, 1.5]) \cup \Sigma_1$. The normal bundle of $\Sigma_1$ in $D_{1.5}$ can then be canonically decomposed as

$$\nu(\Sigma_1, D_{1.5}) \cong \nu(S^3, D_{1.5}) \times \nu(\Sigma_1, S^3) =: \mathbb{R}_x \times \mathbb{R}_y.$$

Since $N^4$ is the complement of an open thickening of $\Sigma$ in $D_{1.5}$, we may assume that for points on $\Sigma_1$ the normal coordinates $x$ vary in the open interval $(0.9, 1.1)$, and $y$ in $(-\epsilon, \epsilon)$. Here, $\epsilon > 0$ is normalized so that for a curve $\alpha = \alpha \times 1 \times 0$ on $\Sigma_1$, one has

$$\alpha \times 1 \times -\epsilon = \alpha^{\downarrow} \quad \text{and} \quad \alpha \times 1 \times \epsilon = \alpha^{\uparrow}.$$

Note that by construction, the disks $\Delta(s_j)$ lie in $N$ and have their boundary $s_i{}^{\downarrow}$ in $\partial N$ and hence one can attempt to apply the disk embedding theorem to these disks. If we can do this successfully, then the $\Delta(s_j)$ may be replaced by disjoint embeddings and hence, we can surger $\Sigma$ into a slice disk for our knot $K$.

Let us check the assumptions in the disk embedding theorem: As mentioned above, $\pi_1 N \cong \mathbb{Z}$ is a good group. By construction, the (self-) intersections among the $\Delta(s_j)$ vanish algebraically, even in the group ring $\mathbb{Z}[\pi_1 N]$, because these disks lie in a simply connected part of $N$.

Finally, we need to check that the $\Delta(s_j)$ have algebraically dual 2-spheres. Note that this must be the place where the assumption on the Alexander polynomial is really used, since so far, we have only used that $K$ is "algebraically slice". We start with 2-dimensional tori $\mathcal{T}_i$ which are the boundaries of small normal bundles of $\Sigma$ in $D_{1.5}$, restricted to the curves $\ell_i$ in our trivial Alexander basis of $\Sigma_1$. More precisely,

$$\mathcal{T}_i := \ell_i \times S^1_t, \text{ where } S^1_t := [0.8, 1.2] \times \{-2\epsilon, 2\epsilon\} \cup \{0.8, 1.2\} \times [-2\epsilon, 2\epsilon]$$

in our normal coordinates introduced above. Note that $S^1_t$ is a (square shaped) meridian to $\Sigma$ and freely generates $\pi_1 N$. By construction, these $\mathcal{T}_i$ lie in our 4-manifold $N$. Moreover, they are disjointly embedded and dual to $\Delta(s_j)$ in the sense that the *geometric* intersections are

$$\mathcal{T}_i \cap \Delta(s_j) = (\ell_i \cap s_j) \times (0.8 \times -\epsilon) = \delta_{i,j}.$$

Hence, the $\mathcal{T}_i$ satisfy all properties of dual 2-spheres, except that they are not 2-spheres! However, we can use our disks $\Delta(\ell_i)$ with boundary $\ell_i$ as follows. First, remove collars $\ell_i \times (0.8, 1]$ from these disks (without changing their name) so that $\Delta(\ell_i)$ have boundary equal to the "long curve" $\ell_i \times 0.8$ on $\mathcal{T}_i$. Using two parallel copies of $\Delta(\ell_i)$, we can surger the $\mathcal{T}_i$ into 2-spheres $g_i$. These are framed because of our assumption that the $\ell_i$ are "untwisted", i.e., that $\operatorname{lk}(\ell_i, \ell_i{}^{\downarrow}) = 0$ (which is used only modulo 2). The equivariant intersection numbers are

$$\begin{aligned}\lambda(g_i, \Delta(s_j)) &= \delta_{i,j} + \Delta(\ell_i) \cdot \Delta(s_j)(1-t)\\ &= \delta_{i,j} + \operatorname{lk}(\ell_i, s_j{}^{\downarrow})(1-t) \;\in\; \mathbb{Z}[\pi_1 N] = \mathbb{Z}[t^{\pm 1}]\end{aligned}$$

because the single intersection point of $\Delta(s_i)$ with $\mathcal{T}_i$ remains and any geometric intersection point between $\Delta(\ell_i)$ and $\Delta(s_j)$ is now turned into exactly two (oppositely oriented) intersections of $g_i$ with $\Delta(s_j)$. These differ by the group element $t$ going around the short curve $S^1_t$ of $\mathcal{T}_i$. By our assumption on the linking numbers, the resulting 2-spheres $g_i$ satisfy the triangular condition from Lemma A.2 and can hence be turned into dual spheres for $\Delta(s_j)$.

Thus, we have checked all assumptions in the disk embedding theorem, and hence we may indeed surger $\Sigma$ to a slice disk for $K$ as planned. q.e.d.

**Remark A.4.** Recall that the topological surgery and s-cobordism theorems in dimension 4 (for all fundamental groups) are equivalent to certain "atomic" links being free slice [**7**, Chapter 12]. These atomic links are all boundary links with minimal Seifert rank in the appropriate sense. In particular, if the disk embedding theorem above was true for free fundamental groups, then the proof above (without needing our triangular base change) would show how to find free slices for all the atomic links. This shows how one reduces the whole theory to the disk embedding theorem for free fundamental groups.

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