THE COLORED HOMFLYPT FUNCTION IS q-HOLONOMIC

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Abstract

We prove that the HOMFLYPT polynomial of a link colored by partitions with a fixed number of rows is a q-holonomic function. By specializing to the case of knots colored by a partition with a single row, it proves the existence of an (a,q) superpolynomial of knots in 3-space, as was conjectured by string theorists. Our proof uses skew-Howe duality that reduces the evaluation of web diagrams and their ladders to a Poincaré–Birkhoff–Witt computation of an auxiliary quantum group of rank the number of strings of the ladder diagram. The result is a concrete and algorithmic web evaluation algorithm that is manifestly q-holonomic.

Contents

1. Introduction

1.1. The colored Jones polynomial

The best-known quantum invariant of a knot or link L in 3-space is the Jones polynomial J_L, which, when properly normalized, is a Laurent polynomial in a variable

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q with integer coefficients. Jones's [29] discovery of this polynomial marked the birth of quantum topology, and shortly afterward, a plethora of quantum invariants of knots and links were discovered by Reshetikhin and Turaev (see [51] and also the books [46], [54]).

Although Jones's definition of the Jones polynomial came from the von Neumann algebras and their subfactors, a connection between the Jones polynomial and the simplest non-Abelian simple Lie algebra, $\mathfrak{sl}_2$, and its representations was soon discovered. More precisely, given a simple Lie algebra $\mathfrak{g}$, an irreducible (finite-dimensional) representation V (usually called a *color* in the physics literature), and a knot K, the theory of ribbon categories (see [51], [54]) defines an invariant $J_K^{\mathfrak{g}}(V) \in \mathbb{Z}[q^{\pm 1}]$. The original construction of this invariant was a rational function in a fractional power of q, and a normalization of this invariant was shown in [35] to be an element of $\mathbb{Z}[q^{\pm 2}]$. The Reshetikhin–Turaev construction extends to framed, oriented links as well, each component of which is colored by an irreducible representation of $\mathfrak{g}$.

By specializing to $\mathfrak{sl}_2$ and using the well-known fact that there is one irreducible representation h_n of $\mathfrak{sl}_2$ of dimension $n+1$ for every natural number n, it follows that a knot K gives rise to a sequence of polynomials $J_K^{\mathfrak{sl}_2}(h_n) \in \mathbb{Z}[q^{\pm 1}]$ for $n = 0, 1, 2, \ldots$. This sequence, although infinite, satisfies some finiteness property which in particular implies that it is determined by finitely many initial terms. (The number of initial terms depends on the knot though.) More precisely, it was proven by the first and third authors [19] that for every knot K there exists a recursion

$$\begin{aligned} c_d(q^n,q)J_K^{\mathfrak{sl}_2}(h_{n+d}) + c_{d-1}(q^n,q)J_K^{\mathfrak{sl}_2}(h_{n+d-1}) + \cdots + c_0(q^n,q)J_K^{\mathfrak{sl}_2}(h_n) \\ = 0 \qquad (1) \end{aligned}$$

for all $n \in \mathbb{N}$, where $d \in \mathbb{N}$, $c_j(u,v) \in \mathbb{Q}[u^{\pm 1}, v^{\pm 1}]$ for all $j = 0, \ldots, d$, and $c_d \neq 0$. In this article, $\mathbb{N}$ denotes the set of all nonnegative integers. The recursion depends on the knot, and although it is not unique, it can be chosen canonically.

Aside from the above-mentioned finiteness statement, the importance of this minimal recursion (often called the $\hat{A}$-*polynomial*) is not a priori clear. Keeping in mind that $\mathrm{PSL}(2,\mathbb{C})$ is the isometry group of orientation-preserving isometries of 3-dimensional hyperbolic space, we see that there are at least two connections between the $\hat{A}$-polynomial and hyperbolic geometry.

(a) Specializing the coefficients of the above recursion to $q = 1$ is conjectured to recover the defining polynomial for the $\mathrm{SL}(2,\mathbb{C})$-character variety of the knot complement, restricted to the boundary torus of the knot complement. This so-called *AJ conjecture* is one link between the colored Jones polynomial and the geometry of $\mathrm{SL}(2,\mathbb{C})$ representations (see [16], [36]).

(b) Such a recursion can be used to numerically compute several terms of the asymptotics of the colored Jones polynomial at complex roots of unity, a fascinating story that connects quantum topology to hyperbolic geometry and number theory. For a sample of computations, the reader may consult [17] or [22].

Returning to recursion relations, we note that sequences that satisfy a recursion relation of the form (1) are q-holonomic, a key concept introduced by Zeilberger [58]. The literature shows that q-holonomic functions enjoy several closure properties. A key theorem of Wilf–Zeilberger [57, Theorem 5.1] is that a multisum of a q-proper hypergeometric term (where we sum all but one variable) is q-holonomic. This theorem and the fact that quantum knot invariants are multisums of q-proper hypergeometric terms (coming from structure constants of corresponding quantum groups) explain why the quantum knot invariants are q-holonomic functions.

Converting the above statement into a theorem and a proof requires additional work. To begin with, one needs to consider functions of several variables. For instance the $\mathfrak{sl}_3$-colored Jones polynomial of a knot or the $\mathfrak{sl}_2$-colored Jones polynomial of a 2-component link is a function of two discrete variables. A definition of q-holonomic functions of several variables was given by Sabbah [52], using the language of homological algebra. Sabbah used a theory of Hilbert dimension for modules over rings generated by q-commuting variables and proved a key Bernstein inequality. A survey of Zeilberger's and Sabbah's work was given by the first and third authors [20], where detailed proofs and examples of q-holonomic functions were discussed. A summary of the main definitions and properties of q-holonomic functions is given in Section 4.

1.2. The colored HOMFLYPT polynomial

Shortly after the discovery of the Jones polynomial, five groups independently discovered a 2-variable polynomial, the HOMFLYPT polynomial W, that takes values in the ring $\mathbb{Q}(q)[x^{\pm 1}]$ (see [12], [48]). Turaev [53] showed that the latter unifies the quantum link invariants $J_L^{\mathfrak{sl}_n}(h_1, \dots, h_1)$, where $h_1 = \mathbb{C}^n$ is the defining representation of $\mathfrak{sl}_n$, as follows: for every $n \geq 2$ and every framed, oriented link L whose components are colored by $\mathbb{C}^n$, we have

$$\tilde{J}_L^{\mathfrak{sl}_n}(h_1, \dots, h_1) = W_L|_{x=q^n}.$$

Here $\tilde{J}_L$ is a normalized version of J_L (see Section 2).

Let $\mathcal{P}$ denote the set of partitions $\lambda = (\lambda_1, \lambda_2, \dots)$, where $\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$ is a decreasing sequence of nonnegative natural numbers, all but finitely many zero. As usual, a partition is presented by a Young diagram. Let $\mathcal{P}_{n-1}$ be the set of partitions with at most $n-1$ rows. Irreducible representations of $\mathfrak{sl}_n$ are parameterized by partitions in $\mathcal{P}_{n-1}$, and we will identify a partition $\lambda \in \mathcal{P}_{n-1}$ with its corresponding irreducible $\mathfrak{sl}_n$-module (which has highest weight λ; see [15]). With this identification, the partition h_a, which has one row and a boxes, is the ath symmetric power of h_1, and the partition e_a, which has one column and a boxes, is the ath external power of $h_1 = e_1$.

Wenzl [56], generalizing Turaev's result, showed that the $\mathfrak{sl}_n$-quantum link invariants interpolate a 2-variable function in the following sense. If L is an oriented, framed link with r ordered components and the λ_i's are partitions with at most ℓ rows for $i = 1, \dots, r$, then there exists a 2-variable colored HOMFLYPT function $W_L(\lambda_1, \dots, \lambda_r) \in \mathbb{Q}(q)[x^{\pm 1}]$ such that for all natural numbers n with $n \geq \ell + 1$ we have

$$\tilde{J}_L^{\mathfrak{sl}_n}(\lambda_1, \dots, \lambda_r) = W_L(\lambda_1, \dots, \lambda_r)|_{x=q^n}.$$

A detailed definition of the HOMFLYPT polynomial and its colored version in terms of the HOMFLYPT polynomial of cables of the link is given in [41] and [42].

1.3. Statement of our results

The set $\mathcal{P}$ of all partitions has an involution defined by $\lambda \mapsto \lambda^\dagger$ which transposes columns and rows of a partition. The map $\iota_\ell : \mathbb{N}^\ell \to \mathcal{P}_\ell$ given by

$$\iota_\ell(n_1, \dots, n_\ell) = (\lambda_1, \dots, \lambda_\ell) \in \mathcal{P}_\ell, \quad \text{where } \lambda_i = \sum_{j=1}^{\ell-i+1} n_j,$$

is a bijection, and so is $\iota_\ell^\dagger : \mathbb{N}^\ell \to \mathcal{P}_\ell^\dagger$ (where $\mathcal{P}_\ell^\dagger$ is the set of all partitions with at most ℓ columns) defined by $\iota_\ell^\dagger(n_1, \dots, n_\ell) = (\iota_\ell(n_1, \dots, n_\ell))^\dagger$.

THEOREM 1.1
Suppose L is an oriented, framed link with r ordered components and ℓ a nonnegative integer. Then, the functions

$$W_L \circ (\iota_\ell)^r : \mathbb{N}^{r\ell} \to \mathbb{Q}(q)[x^{\pm 1}], \qquad W_L \circ (\iota_\ell^\dagger)^r : \mathbb{N}^{r\ell} \to \mathbb{Q}(q)[x^{\pm 1}]$$

are q-holonomic.

COROLLARY 1.2
For a framed, oriented knot K colored with partitions with a single row, the sequence $W_K(h_a)$ for $a = 0, 1, 2, \dots$ is q-holonomic.

Some special cases of the above corollary are known. See Cherednik [7] for the case of torus knots, Wedrich [55] for the case of 2-bridge knots, and Kawagoe [32] for some 2-bridge knots and some pretzel knots.

On the set of all functions from $\mathbb{N}$ to $\mathbb{Q}(x, q)$ define two operators L, M by

$$(\mathsf{L}f)(a) = f(a+1), \qquad (\mathsf{M}f)(a) = q^a f(a).$$

Then $\mathsf{LM} = q\mathsf{ML}$, and a recurrence for f has the form $Pf = 0$, where

$$P = \sum_{j=0}^{d} c_j(q,x,M)L^j, \quad c_j(q,x,M) \in \mathbb{Z}[q,x,M].$$

When nonzero recurrence for f exists, there are many of them, and there is a unique one, up to sign, such that (i) d is minimal, (ii) the total degree in q, x, M, L is minimal, and (iii) all the integer coefficients of P are coprimes (see [16], [36]). For a knot K, we denote such a minimal recurrence for $W_K(h_a)$ by $A_K(M,L,x,q)$.

Physicists have conjectured the existence of the 4-variable polynomial (see, e.g., [2], [24]) and have further conjectured that, when we set $q = 1$, the corresponding 3-variable polynomial $A_K(M,L,x,1)$ is equal, after some universal (i.e., knot-independent) change of variables, to a 3-variable polynomial that comes out of knot contact homology (see [1], [45]). In the physics literature, $A_K(\mathsf{M},\mathsf{L},Q,1)$ is often called the *Q-deformed A-polynomial* of a knot, and it appears in string theory in the geometry of spectral curves, topological strings, matrix models, and M-theory dualities. There is a lot of literature on this polynomial following the pioneering work of Gukov, Fuji, Stosic, Sułkowski, and others. For a detailed discussion, see [2], [9], [10], [13], [14], [19], [23], [24], and [25].

Remark 1.3
The proof of Theorem 1.1 implies that the function $\mathbb{N} \times \mathbb{N}^{r\ell} \to \mathbb{Z}[q^{\pm 1}]$ defined by

$$(n, \vec{m}) \in \mathbb{N} \times \mathbb{N}^{r\ell} \mapsto \big(W_L \circ (\iota_\ell)^r\big)(\vec{m})|_{x=q^n}$$

is q-holonomic in all $r\ell + 1$ variables. The latter was conjectured in [21].

1.4. An example
Suppose that K is the right-handed trefoil (see Figure 6) with 0 framing. Define

$$\begin{aligned} a_0 &= x\mathsf{M}^6(x^2\mathsf{M}^2 - 1)(\mathsf{M}^4 - q^6x^2), \\ a_1 &= q(q^8\mathsf{M}^2x^4 - x^4q^4 + \mathsf{M}^6q^2x^4 + \mathsf{M}^6x^4 - \mathsf{M}^6q^6x^2 \\ &\quad - \mathsf{M}^6q^2x^2 - \mathsf{M}^8x^2 + \mathsf{M}^{10})(\mathsf{M}^4 - q^4x^2), \\ a_2 &= -x^5q^6(q^4\mathsf{M}^2 - 1)(\mathsf{M}^4 - x^2q^2). \end{aligned}$$

Then for all $m \geq 0$,

$$a_2W_K(h_{m+2}) + a_1W_K(h_{m+1}) + a_0W_K(h_m) = 0. \tag{2}$$

Remark 1.4
In [13], a conjectural formula for the colored HOMFLYPT function $W_K(h_m)$ was

given for the case in which K is the left-handed trefoil (and other torus knots). Based on this conjectural formula, the authors of [13], using a computer program of Zeilberger [47], found a recurrence formula for $W_K(h_m)$, which is different from (2) since another normalization was used. In Appendix C, we will give a proof of (2).

1.5. Plan of the proof

The quantum group invariants require familiarity with category theory, the representation theory of quantum groups, and an understanding of the accompanying graphical notation. In Section 2 we discuss three categories $n\mathbf{Rep}_\wedge$, $n\mathbf{Web}$, and $n\mathbf{Lad}$ which are related to representations of quantum groups as well as to a diagrammatic description of links and their invariants. In Section 3 we discuss how to unify the $\mathfrak{sl}_n$-link invariants to one that is independent of n. In Section 4 we discuss the basic definitions, examples, and properties of q-holonomic functions. In Section 5 we give the proof of Theorem 1.1. The proof is concrete and algorithmic, with a detailed example for the case of the right-handed trefoil given in Section 3.7. We summarize the steps here, using the notation of the proof.

(a) We start with a braid word representative β whose closure $\mathrm{cl}(\beta)$ is the link L. The corresponding braid has m strands and a fixed number of letters. For the trefoil, this is given in (36).

(b) The link is now given by joining to the braid the bottom and top part of the closure consisting of cup/cap diagrams, respectively. We replace the bottom part by a monomial in some operators E_i, the braid word by a product of Lusztig braid operators $T_i(b)^{\pm 1}$ defined in Section 3.4, and the top part by a monomial in some operators F_j. For the trefoil, this is given in (37).

(c) Each operator $T_i(b)^{\pm 1}$ is a sum (over the integers) of operators E_i and F_j (see (29a)–(29b)).

(d) The operators E_i and F_j satisfy the quantum group q-commutation relations given in (14a)–(14d), and using those we can sort the above expressions by moving all the E's to the right and all the F's to the left.

(e) The fact that the operators E_i annihilate the last bit 1_ϑ, corresponding to the projection onto a highest weight determined from the link diagram, adds a product of delta functions in our sum.

(f) The requirement that all weights appearing in the sum be positive introduces Heaviside functions into the sum, as explained in the proof of Proposition 5.2.

(g) In this way, we obtain a multidimensional sum over the integers, whose summand is a product of extended q-binomial coefficients of linear forms (with integer coefficients) of the summation variables times a sign raised to a linear form of the summation variables. These sums are always terminating. For the trefoil, this 6-dimensional sum is given in (39).

(h) We show in Section 5.1 that such multisums are q-holonomic.

Hidden in the above algorithm is the quantum skew-Howe duality (see [5]), which allows us to compute colored $\mathfrak{sl}_n$-invariants by evaluating ladder diagrams in $2m$ strands by using an auxiliary quantum group based on the Lie algebra $\mathfrak{gl}_{2m}$. Steps (c)–(e) are exactly a Poincaré–Birkhoff–Witt computation on $\mathfrak{gl}_{2m}$.

To avoid any confusion or misunderstanding, in an earlier article, the first author [18] reduced the q-holonomicity of the colored HOMFLYPT polynomial to the q-holonomicity of the evaluation of Murakami–Ohtsuki–Yamada (MOY) graphs and observed that the latter would follow from the existence of a q-holonomic evaluation algorithm for MOY graphs. Unfortunately, such an algorithm based on simplifications of MOY graphs or web diagrams is yet to be found.

1.6. Computations and questions

With regard to the computation of the 4-variable polynomial of a knot, there are several formulas for the HOMFLYPT polynomial of some links in the literature colored by partitions with one row (see, e.g., [26], [27], [32], [44]). These formulas are manifestly q-holonomic, as follows by the fundamental theorem of Wilf–Zeilberger theory. Using these formulas and Wilf–Zeilberger theory, one can sometimes compute the 4-variable knot polynomial. For sample computations for the case of twist knots and some torus knots, see [44].

The next question is inaccessible with our methods. A positive answer would be useful in the study of Labastida–Mariño–Ooguri–Vafa (LMOV) (also known as Bogomol'nyi–Prasad–Sommerfield (BPS)) invariants of links (see [34]). First, using linearity extend the colored HOMFLYPT function to the case when the color of each link component is a $\mathbb{Z}$-linear combination of Young diagrams. Let $p_a = \sum_{k=0}^{a}(-1)^k(k, 1^{a-k})$. Note that $(k, 1^{a-k})$ is a hook partition with one row with k boxes and one column with $a-k$ boxes.

Question 1.5

Is it true that the HOMFLYPT polynomial of a knot colored by p_a is a q-holonomic function of a?

2. Categories, links, and their invariants

Throughout the article, $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$ denote the set of nonnegative integers, the set of integers, and the set of rational numbers, respectively. We emphasize that our $\mathbb{N}$ contains 0. Also n will denote an integer greater than or equal to 2. We will denote by $\mathbb{Q}(q^{1/n})$ the field of rational functions in an indeterminate $q^{1/n}$ and denote by $\mathbb{Q}(q)$ its subfield generated by $q = (q^{1/n})^n$. Also $\mathbb{Z}[q^{\pm 1}] \subset \mathbb{Q}(q)$ will denote the ring of Laurent polynomials in q with integer coefficients.

In this section we will discuss three categories $n\mathbf{Rep}_\wedge$, $n\mathbf{Web}$, and $n\mathbf{Lad}$ which are connected by functors

$$n\mathbf{Lad} \xrightarrow{\Psi_n} n\mathbf{Web} \xrightarrow{\Gamma_n} n\mathbf{Rep}_\wedge. \tag{3}$$

A ring homomorphism $f : \mathbb{Q}(q^{1/n}) \to \mathbb{Q}(q^{1/n})$ (thought of as a homomorphism from the empty set to the empty set) is the multiplication by a scalar, and we denote this scalar by $\mathrm{ev}(f) \in \mathbb{Q}(q^{1/n})$. These categories are intimately related to diagrammatic descriptions of framed tangles and of quantum groups.

2.1. The quantized enveloping algebras $\mathrm{U}_q(\mathfrak{gl}_n)$ *and* $\mathrm{U}_q(\mathfrak{sl}_n)$

Consider the lattice $\mathbb{Z}^n$ with the standard Euclidean inner product $\langle\cdot,\cdot\rangle$, and consider the root vectors

$$\alpha_i = (0,\dots,0,1,-1,0,\dots,0) \in \mathbb{Z}^n,$$

with 1 on the ith position. The quantized enveloping algebra $\mathrm{U}_q(\mathfrak{gl}_n)$ is the associative algebra over $\mathbb{Q}(q)$ generated by L_i, $i = 1,\dots,n$, and E_i, F_i, $i = 1,\dots,n-1$, subject to the relations

$$\begin{aligned}
&L_a L_b = L_{a+b}, \qquad L_0 = 1,\\
&L_a E_j = q^{a_j - a_{j+1}} E_j L_a, \qquad L_a F_j = q^{a_{j+1}-a_j} F_j L_a,\\
&E_i^{(2)} E_{i+1} - E_i E_{i+1} E_i + E_{i+1} E_i^{(2)} = 0 = F_i^{(2)} F_{i+1} - F_i F_{i+1} F_i + F_{i+1} F_i^{(2)},\\
&E_i F_j - F_j E_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q - q^{-1}},\\
&E_i E_j = E_j E_i, \qquad F_i F_j = F_j F_i \quad \text{for } |i-j| > 1.
\end{aligned}$$

Here $L_a = L_1^{a_1} \cdots L_n^{a_n}$ for $a = (a_1,\dots,a_n) \in \mathbb{Z}^n$, $K_i = L_i L_{i+1}^{-1}$, and

$$E_i^{(r)} = E_i^r/[r]!, F_i^{(r)} = F_i^r/[r]!, \quad \text{where } [r]! := \prod_{j=1}^{r} \frac{q^j - q^{-j}}{q - q^{-1}}.$$

There is a structure of a Hopf algebra on $\mathrm{U}_q(\mathfrak{gl}_n)$ with the coproduct and the antipode (see, e.g., [6], [28], [38]).

The quantized enveloping algebra $\mathrm{U}_q(\mathfrak{sl}_n)$ is the subalgebra of $\mathrm{U}_q(\mathfrak{gl}_n)$ generated by $E_i, F_i, K_i^{\pm 1}$, $i = 1,\dots,n-1$. Then $\mathrm{U}_q(\mathfrak{sl}_n)$ inherits a Hopf algebra structure from that of $\mathrm{U}_q(\mathfrak{gl}_n)$.

A weight of $\mathrm{U}_q(\mathfrak{gl}_n)$ (resp., $\mathrm{U}_q(\mathfrak{sl}_n)$) is an element $a \in \mathbb{Z}^n$ (resp., an element $a \in \mathbb{Z}^n$ such that $\sum_i a_i = 0$). A $\mathrm{U}_q(\mathfrak{sl}_n)$-module V is called a *weight module* (or, perhaps better, a *weighted module*) if $V = \bigoplus_a V_{[a]}$, where each a is a $U_q(\mathfrak{sl}_n)$-weight and

$$V_{[a]} = \{ v \in V \mid K_i(v) = q^{\langle \alpha_i, a\rangle} v \}.$$

For a partition $\lambda = (l_1, \dots, l_\ell)$ with $l_1 \geq l_2 \geq \cdots \geq l_\ell > 0$ we call $\ell = \text{length}(\lambda)$ the length of λ and $|\lambda| = \sum_i l_i$ the weight of λ. Denote by $\lambda^\dagger$ the conjugate of λ, which is the partition whose Young diagram is the transpose of that of λ. For a thorough treatment of partitions, see [39]. Finite-dimensional irreducible weight $U_q(\mathfrak{sl}_n)$-modules are parameterized by partitions $\lambda \in \mathcal{P}_{n-1}$, that is, partitions of length at most $n-1$ (see, e.g., [6], [28]). For every $\lambda \in \mathcal{P}_{n-1}$ denote by V_λ the corresponding irreducible weight $U_q(\mathfrak{sl}_n)$-module.

2.2. The category of $U_q(\mathfrak{sl}_n)$-modules and link invariants

The category n**Rep** of finite-dimensional weight $U_q(\mathfrak{sl}_n)$-modules is a *ribbon category* (see [54]), where the braiding comes from the universal R-matrix. To be precise, one needs to extend the ground field to $\mathbb{Q}(q^{1/n})$ so that the braiding and the ribbon element can be defined.

By the theory of ribbon categories, for a framed, oriented link L in 3-space with r ordered components and r objects $V_1, \dots, V_r$ of n**Rep**, one can define an invariant

$$J_L^{\mathfrak{sl}_n}(V_1, \dots, V_r) \in \mathbb{Q}(q^{1/n}).$$

If $\lambda_1, \dots, \lambda_r \in \mathcal{P}_{n-1}$, we use the notation

$$J_L^{\mathfrak{sl}_n}(\lambda_1, \dots, \lambda_r) = J_L^{\mathfrak{sl}_n}(V_{\lambda_1}, \dots, V_{\lambda_r}).$$

It is known that a properly normalized version of $J_L^{sl_n}(V_1, \dots, V_r)$ belongs to $\mathbb{Z}[q^{\pm 2}]$ (see [35], [40]). A special case of this integrality phenomenon is the following. Let ℓ_{ij} be the linking number between the ith and the jth components of L, with ℓ_{ii} being the framing of the ith component. Define

$$\tilde{J}_L^{\mathfrak{sl}_n}(\lambda_1, \dots, \lambda_r) = q^{\frac{1}{n}\sum_{i,j} \ell_{ij}|\lambda_i||\lambda_j|} J_L^{\mathfrak{sl}_n}(\lambda_1, \dots, \lambda_r). \tag{4}$$

Then we have

$$\tilde{J}_L^{\mathfrak{sl}_n}(\lambda_1, \dots, \lambda_r) \in \mathbb{Z}[q^{\pm 1}]. \tag{5}$$

Not only is $\tilde{J}_L^{\mathfrak{sl}_n}(\lambda_1, \dots, \lambda_r)$ a Laurent polynomial in q, but it also enjoys the following stability (with respect to the rank n) property.

PROPOSITION 2.1 (see [56])

There exists an invariant $W_L(\lambda_1, \dots, \lambda_r) \in \mathbb{Q}(q)[x^{\pm 1}]$ such that, for any n greater than the length of any of the λ_j's, we have

$$W_L(\lambda_1,\dots,\lambda_r)|_{x=q^n} = \tilde{J}_L^{\mathfrak{sl}_n}(\lambda_1,\dots,\lambda_r).$$

Usually, W_L is called the *colored HOMFLYPT function*. The theorem was first proved by Wenzl, using quantum group theory. For a detailed proof using skein theory, see [37, Theorem 11.4.18]. The theorem also follows from our proof of Theorem 1.1 below. For the simplest case, when all partitions have one box, Proposition 2.1 was first proved by Turaev [53].

Remark 2.2
The integrality (5) shows that the polynomial $P = W_L(\lambda_1,\dots,\lambda_r) \in \mathbb{Q}(q)[x^{\pm 1}]$ has the property that $P|_{x=q^n} \in \mathbb{Z}[q^{\pm 1}]$ for all integers $n > 1$. Such a polynomial $P \in \mathbb{Q}(q)[x^{\pm 1}]$ is called *q-integral* and is studied in [4, Section 2.3].

Remark 2.3
Our $W_L(\lambda_1,\dots,\lambda_r)$ is equal to $P(L * (Q_{\lambda_1},\dots,Q_{\lambda_r}))$ in the notation of [42, Section 6], with our q and x equal to, respectively, s and v^{-1} there.

2.3. Properties of the colored HOMFLYPT polynomial
Let Λ be the free $\mathbb{Q}$-vector space with basis the set $\mathcal{P}$ of all Young diagrams, including the empty one. Suppose that L is a framed, oriented link with r ordered components. The invariant $W_L(\lambda_1,\dots,\lambda_r)$ can be extended to a $\mathbb{Q}$-multilinear map

$$W_L : \Lambda^r \to \mathbb{Q}(q)[x^{\pm 1}].$$

There is a $\mathbb{Q}$-algebra structure on Λ which makes it isomorphic to the algebra of symmetric functions (see, e.g., [39]). Under this isomorphism, a Young diagram λ is mapped to the Schur function S_λ corresponding to λ.

We collect here some well-known properties of the quantum invariant W_L.

PROPOSITION 2.4
Let L be a framed, oriented link in the 3-space with k ordered components.

(a) *Suppose that L' is the same L with the components renumbered by a permutation σ of $\{1,\dots,k\}$. Then*

$$W_{L'}(\lambda_1,\dots,\lambda_r) = W_L(\lambda_{\sigma 1},\dots,\lambda_{\sigma r}).$$

(b) *Suppose that ΔL is the result of replacing the first component of L by two copies of its parallel pushoff (using the framing). Then*

$$W_{\Delta L}(\lambda_1',\lambda_1'',\lambda_2,\dots,\lambda_r) = W_L(\lambda_1'\lambda_1'',\lambda_2,\dots,\lambda_r).$$

(c) *We have*

$$W_L(\lambda_1,\dots,\lambda_r) = W_L(\lambda_1^\dagger,\dots,\lambda_r^\dagger)|_{q\to -q^{-1}}. \tag{6}$$

Parts (a) and (b) follow from the corresponding properties for J_L (see [54]). While (a) is trivial, (b) follows from the hexagon equation of the braiding in the braided category. Part (c) is well known and has been discussed in many papers (see, e.g., [34, (4.41)]). For completeness, we give proofs of parts (b) and (c) in Appendix B.

2.4. *The category* $n\mathbf{Rep}_\wedge$

Let e_a be the partition whose Young diagram is a column with a boxes; that is, $e_a = (1^a)$ in the standard notation of partitions. The $\mathrm{U}_q(\mathfrak{sl}_n)$-module V_{e_a} with $1 \le a \le n-1$ is called a *fundamental* $\mathrm{U}_q(\mathfrak{sl}_n)$*-module*. We also use e_0 to denote the empty Young diagram, which corresponds to the trivial $\mathrm{U}_q(\mathfrak{sl}_n)$-module.

Let $n\mathbf{Rep}_\wedge$ be the full subcategory of $n\mathbf{Rep}$ whose objects are those isomorphic to tensor products of the fundamental $\mathrm{U}_q(\mathfrak{sl}_n)$-modules. Then $n\mathbf{Rep}_\wedge$ inherits a ribbon category structure from $n\mathbf{Rep}$.

The advantage of $n\mathbf{Rep}_\wedge$ is that it has a remarkable presentation using planar diagrams called spider webs, which are described in the next section. Since $n\mathbf{Rep}$ is the idempotent completion of $n\mathbf{Rep}_\wedge$, we do not lose much by working with $n\mathbf{Rep}_\wedge$.

2.5. *The category* $n\mathbf{Web}$

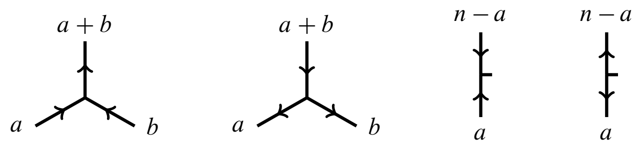
We describe here the category $n\mathbf{Web}$ of $\mathfrak{sl}_n$-webs, following Cautis, Kamnitzer, and Morrison [5]. Recall that a pivotal monoidal category is a category with tensor products and a coherent notion of duality in which the double dual functor is naturally isomorphic to the identity. The morphisms and the relations among morphisms of such categories afford a diagrammatic description using planar diagrammatics. They are essentially equivalent to the description of the Temperley–Lieb algebra for $n = 2$, Kuperberg's [33] spider webs (for $n = 3$), and the planar algebras of Jones [30]. They are also closely related to the MOY graphs [43]. Standard references for pivotal categories include [54, Chapter XI], [31], and [11, Chapter 4.7].

An *n-web* is a compact subset Z of the horizontal strip $\mathbb{R}\times[0,1]$ with additional data satisfying (i)–(iii).

(i) Each connected component of X is either an oriented circle or a directed graph (i.e., a finite 1-dimensional CW-complex) where the degree of each vertex is 1, 2, or 3. Every circle component and every edge is labeled by an integer in $[1, n-1]$.

(ii) The set ∂Z of univalent vertices of Z is in the union of the top and bottom lines of the strip, and $Z \setminus \partial Z$ is in the interior of the strip.

(iii) Up to isotopy there are two types of trivalent vertices and two types of bivalent vertices as in the following figure (with labeling of edges attached to the vertex):



(7)

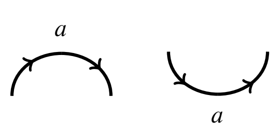
The third and the fourth graphs depict bivalent vertices but not trivalent vertices, because the small tag there is not officially an edge. The tag provides a distinguished side and makes the bivalent vertices not rotationally symmetric.

We will declare isotopic webs to be equal. Let $\partial_- Z = (i_1^{\varepsilon_1}, \dots, i_k^{\varepsilon_k})$, where $i_1, \dots, i_k$ are the labels of the edges ending on the bottom line listed from left to right, and let $\varepsilon_j = +$ if the orientation at the jth ending point is upward and $\varepsilon_j = -$ otherwise. One defines $\partial_+ Z$ exactly the same way, using the top line instead of the bottom line.

The category $n\mathbf{Web}$ is the pivotal monoidal $\mathbb{Q}(q^{1/n})$-linear category whose objects are sequences in the symbols $\{1^\pm, \dots, (n-1)^\pm\}$. Given objects a, b of $n\mathbf{Web}$, the set of morphisms $\mathrm{Hom}_{n\mathbf{Web}}(a, b)$ is the set of $\mathbb{Q}(q^{1/n})$-linear combinations of n-webs Z such that $\partial_- Z = a$ and $\partial_+ Z = b$, subject to certain local relations described in [5, Section 2.2]. In [5], our $n\mathbf{Web}$ is denoted by $\mathcal{S}p(\mathrm{SL}_n)$. The tensor product $Z_1 \otimes Z_2$ is obtained by placing Z_2 to the right of Z_1. The composition $Z_1 Z_2$ is the result of placing Z_1 atop Z_2, after an isotopy to make the top ends of Z_2 match the bottom ends of Z_1.

For example, the first diagram in (7) represents a morphism from $a^+ \otimes b^+ = (a^+, b^+) \to (a+b)^+$, and the second one represents a morphism from $a^- \otimes b^- \to (a+b)^-$.

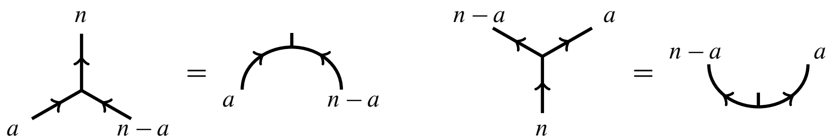
The monoidal unit $n\mathbf{Web}$ is the empty sequence. The planar isotopy condition implies that the object a^+ is dual to the object a^-. The cap and cup morphisms



(8)

give rise to maps $a^+ \otimes a^- \to \emptyset$ and $\emptyset \to a^- \otimes a^+$ that realize this duality.

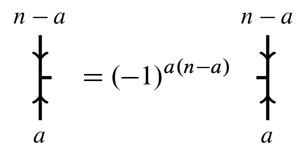
For simplicity we allow diagrams to carry labels of 0 and n with the understanding that n-labeled edges connected to a trivalent vertex should be deleted and replaced by a tag as in the cap and cup diagrams:



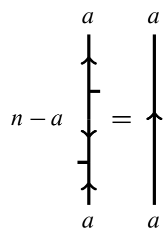
(9)

and the remaining edges and loops labeled 0 or n should be deleted. Note that the cap and cup diagrams coming from the duality a^+ with a^- arising from the pivotal structure do not require tags.

The following are consequences of the relations among generators of $\mathfrak{sl}_n$-webs:



(10)



(11)

Remark 2.5
The tags appearing in n-webs (which do not appear in [43]) play an important role in keeping track of the fact that, while $(V_{e_a})^*$ is isomorphic to $V_{e_{n-a}}$, this isomorphism is not canonical. The tags in $\mathfrak{sl}_n$-webs keep track of these isomorphisms and contribute signs that would have otherwise been missed by wrongly identifying the dual of a^+ with $(n-a)^+$.

2.6. *An equivalence between* n**Web** *and* n**Rep**$_\wedge$

The main result of [5] is the construction of an equivalence, which is a $\mathbb{Q}(q^{1/n})$-linear pivotal functor,

$$\Gamma_n : n\mathbf{Web} \to n\mathbf{Rep}_\wedge$$

defined on objects by $\Gamma_n(a^+) = V_{e_a}$ and $\Gamma_n(a^-) = (V_{e_a})^*$. The ribbon structure of $n\mathbf{Rep}_\wedge$ can be pulled back to make $n\mathbf{Web}$ a ribbon category. In particular, we have a braiding $X_{a,b} : a \otimes b \to b \otimes a$ for any two objects a, b of $n\mathbf{Web}$. For simple objects $a, b \in [1, n-1]$ we use the diagrams with crossings as in Figure 1 to denote the braiding $X_{a,b}$ and its inverse $X_{b,a}^{-1}$. The braiding allows us to introduce crossings in diagrams representing morphisms of $n\mathbf{Web}$.

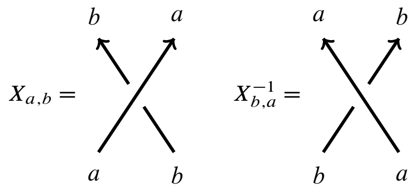


Figure 1. The braiding $X_{a,b}$ (left) and its inverse $X_{b,a}^{-1}$.

Suppose that D is a link diagram in the plane in general position with respect to the height function, whose components are labeled by integers in $[0, n-1]$. Then D defines a morphism in the category $n\mathbf{Web}$ from $\emptyset$ to $\emptyset$. Since $\mathrm{Hom}_{n\mathbf{Web}}(\emptyset,\emptyset) = \mathbb{Q}(q^{1/n})$, the morphism D is determined by the scalar $\mathrm{ev}(D) \in \mathbb{Q}(q^{1/n})$. The equivalence Γ_n shows that this scalar $\mathrm{ev}(D)$ is equal to the invariant $J_L^{\mathfrak{sl}_n}(e_{a_1},\dots,e_{a_k})$; that is,

$$J_L^{\mathfrak{sl}_n}(e_{a_1},\dots,e_{a_k}) = \mathrm{ev}(D), \tag{12}$$

where L is the framed link whose blackboard diagram is D and $a_1,\dots,a_k$ are the labels of the components of L.

2.7. *The ladder category*

We give the definition of the ladder category $\mathbf{Lad}_m$, which is a diagrammatic presentation of Lusztig's idempotent form $\dot{\mathrm{U}}_q(\mathfrak{gl}_m)$ of the quantum group $\mathrm{U}_q(\mathfrak{gl}_m)$. Typically, $\dot{\mathrm{U}}_q(\mathfrak{gl}_m)$ is regarded as a $\mathbb{Q}(q)$-algebra where the unit is replaced by a system of mutually orthogonal idempotents 1_a indexed by the weight lattice of $\mathfrak{gl}_m$. Using the quantum skew-Howe duality, Cautis, Kamnitzer, and Morrison [5] showed that there is a braided monoidal functor from the ladder category to the category $n\mathbf{Web}$. We explain how to use this result to calculate quantum $\mathrm{U}_q(\mathfrak{sl}_n)$-invariants of links using ladders.

A ladder Z with m sides is a uni-trivalent graph drawn in the strip $\mathbb{R}\times[0,1]$, with

(i) m parallel vertical lines running from the bottom line to the top line of the strip, oriented upward;

(ii) some number of oriented horizontal lines in the interior of the strip $\mathbb{R}\times[0,1]$, called *steps*, connecting adjacent sides; and

(iii) a labeling of each interval (steps or segments of sides) by integers such that the signed sum of the labels at each trivalent vertex is zero. Here the sign of each incoming vertex is positive, and the sign of each outgoing vertex is negative.

Let $\partial_- Z$ (resp., $\partial_+ Z$) be the sequence of labels appearing on the bottom (resp., top) edge of the strip. Then $\partial_- Z, \partial_+ Z \in \mathbb{Z}^m$ are considered as weights of $\mathrm{U}_q(\mathfrak{gl}_m)$. For example, see Figure 2.

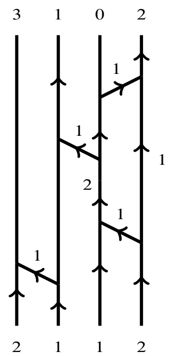
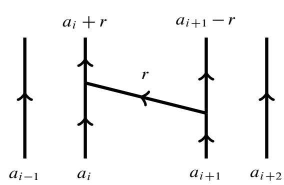


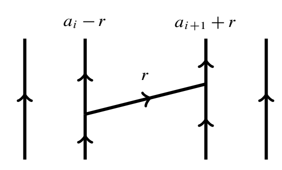
Figure 2. A morphism in $\mathbf{Lad}_4$.

The category $\mathbf{Lad}_m$ is the $\mathbb{Q}(q)$-linear category whose set of objects is $\mathbb{Z}^m$. Given two objects a, b, the morphisms $\mathrm{Hom}_{\mathbf{Lad}_m}(a, b)$ is the set of all $\mathbb{Q}(q)$-linear combinations of ladders Z with m sides such that $\partial_- Z = a$ and $\partial_+ Z = b$, subject to the relations described in (14a)–(14e) below.

The composition of morphisms is given by the vertical concatenation of ladders. Note that $\mathbf{Lad}_m$ does not have dual objects and hence is not pivotal.

For an object $a = (a_1, \dots, a_m)$ of $\mathbf{Lad}_m$, for i such that $1 \le i \le m-1$, and for $r \in \mathbb{N}$, let $E_i^{(r)} 1_a$ and $F_i^{(r)} 1_a$ denote the following ladders:

$E_i^{(r)} 1_a := \cdots$  $\cdots \in \mathrm{Hom}_{\mathbf{Lad}_m}(a, a + r\alpha_i),$

$F_i^{(r)} 1_a := \cdots$  $\cdots \in \mathrm{Hom}_{\mathbf{Lad}_m}(a, a - r\alpha_i).$

Here and in what follows, we draw the steps of a ladder by using slightly slanted lines instead of horizontal lines such that the orientation of the step is upward. With this convention we do not have to mark the orientation in a ladder diagram, since all segments are oriented upward.

By comparing the sequences at the end of these ladders it is clear that

$$E_i^{(r)} 1_a = 1_{a+r\alpha_i} E_i^{(r)} = 1_{a+r\alpha_i} E_i^{(r)} 1_a,$$
$$F_i^{(r)} 1_a = 1_{a-r\alpha_i} F_i^{(r)} = 1_{a-r\alpha_i} F_i^{(r)} 1_a. \tag{13}$$

When the specific weight is clear we will write E_i instead of $E_i 1_a$ and F_i instead of $F_i 1_a$. For example, $F_i^{(r)} E_j^{(s)} 1_a$ means $F_i^{(r)} 1_{a+s\alpha_j} E_j^{(s)} 1_a$.

With this convention, the relations of the morphisms of $\mathbf{Lad}_m$ are given by

$$E_i^{(r)} F_i^{(s)} 1_a = \sum_{t=0}^{\min(r,s)} \begin{bmatrix} \langle a, \alpha_i\rangle + r - s \\ t \end{bmatrix} F_i^{(s-t)} E_i^{(r-t)} 1_a, \tag{14a}$$

$$E_i^{(r)} F_j^{(s)} 1_a = F_j^{(s)} E_i^{(r)} 1_a \quad \text{if } i \neq j, \tag{14b}$$

$$E_i^{(r)} E_j^{(s)} 1_a = E_j^{(s)} E_i^{(r)} 1_a \quad \text{if } |i-j| > 1, \text{ and likewise for the } F\text{'s}, \tag{14c}$$

$$E_i^{(s)} E_i^{(r)} 1_a = \begin{bmatrix} r+s \\ r \end{bmatrix} E_i^{(r+s)} 1_a, \quad \text{and likewise for the } F\text{'s}, \tag{14d}$$

$$E_i E_j E_i 1_a = (E_i^{(2)} E_j + E_j E_i^{(2)}) 1_a$$
$$\text{if } |i-j| = 1, \text{ and likewise for the } F\text{'s}, \tag{14e}$$

for all $r, s \in \mathbb{N}$, $1 \leq i \leq m-1$, and $a \in \mathbb{Z}^m$.

Recall that $\langle a, \alpha_i\rangle = a_i - a_{i+1}$ is the standard inner product on $\mathbb{Z}^m$, and the quantum integers and factorial and binomial coefficients are defined by

$$[r] = \frac{q^r - q^{-r}}{q - q^{-1}}, \quad r \in \mathbb{Z}, \tag{15a}$$

$$[r]! = \prod_{k=1}^{r} [k], \quad r \geq 0, \tag{15b}$$

$$\begin{bmatrix} r \\ s \end{bmatrix} = \begin{cases} \frac{\prod_{k=r-s+1}^{r} [k]}{[s]!} & r, s \in \mathbb{Z}, s \geq 0, \\ 0 & s < 0. \end{cases} \tag{15c}$$

Remark 2.6

If k is a field and $\mathcal{C}$ is a k-linear category, then it gives rise to an algebra $A(\mathcal{C})$ whose underlying vector space is the direct sum of all Hom spaces $\bigoplus_{a,b} \operatorname{Hom}(a,b)$. The product of $x \in \operatorname{Hom}(b,a)$ and $y \in \operatorname{Hom}(b',a')$ is defined to be zero unless $b = a'$, in which case the product is defined to be the composite xy. In general, $A(\mathcal{C})$ is a k-algebra without unit. Since the relations (14a)–(14e) are the defining relations of Lusztig's idempotent algebra $\dot{\mathrm{U}}_q(\mathfrak{gl}_m)$, $A(\mathbf{Lad}_m) \cong \dot{\mathrm{U}}_q(\mathfrak{gl}_m)$.

2.8. The Schur quotient, the highest weight ϑ, and evaluation

Fix positive integers m and n. The Schur quotient $n\mathbf{Lad}_m$ is defined to be the $\mathbb{Q}(q^{1/n})$-linear category with set of objects all $a = (a_1, \dots, a_m) \in \mathbb{Z}^m$ such that $a \in [0,n]^m$, that is, $0 \le a_i \le n$ for all i. The algebra of morphisms of $n\mathbf{Lad}_m$ is the quotient of the algebra of morphisms of $\mathbf{Lad}_m$, with ground field extended to $\mathbb{Q}(q^{1/n})$, by the two-sided ideal generated by all 1_a's with $a \notin [0,n]^m$. For example, $E_i^{(r)} 1_a$ is always 0 in $n\mathbf{Lad}_m$ when $r > n$.

Let

$$\vartheta(n,m) := (n^m, 0^m) \in \mathbb{Z}^{2m}, \tag{16}$$

often abbreviated by ϑ. Considered as an object of $n\mathbf{Lad}_{2m}$, ϑ is a *highest weight element for* $n\mathbf{Lad}_{2m}$, in the sense that for every $i = 1, \dots, 2m-1$, we have

$$E_i 1_\vartheta = 0, \qquad 1_\vartheta F_i 1_{\vartheta+\alpha_i} = 0. \tag{17}$$

This is because $\vartheta + \alpha_i$ has entries outside $[0,n]$. It follows that the algebra of endomorphisms of ϑ is isomorphic to the ground field $\mathbb{Q}(q^{1/n})$. In other words, we have an evaluation map

$$\mathrm{ev}_{n,m} : \mathrm{Hom}_{n\mathbf{Lad}_{2m}}(\vartheta,\vartheta) \xrightarrow{\simeq} \mathbb{Q}(q^{1/n}), \qquad x = \mathrm{ev}_{n,m}(x) 1_\vartheta. \tag{18}$$

2.9. Braiding for ladders

The category $n\mathbf{Lad}_m$ does not have a tensor product and hence is not a monoidal category. However, $n\mathbf{Lad} := \bigoplus_{m=1}^{\infty} n\mathbf{Lad}_m$ is monoidal. This category does not have duals, since all webs are directed upward. But it is a braided monoidal category, as follows. The objects of $n\mathbf{Lad}$ are sequences $a = (a_1, \dots, a_m)$ of integers $a_i \in [0,n]$. Given two objects $a = (a_1, \dots, a_m)$ and $b = (b_1, \dots, b_p)$, $\mathrm{Hom}_{n\mathbf{Lad}}(a,b) = \mathrm{Hom}_{n\mathbf{Lad}_m}(a,b)$ if $p = m$ and 0 otherwise.

The tensor product of objects $a \otimes b$ is the horizontal concatenation of a and b from left to right. A similar convention is used for morphisms.

In [5, Section 6] it is shown that $n\mathbf{Lad}$ admits a braided monoidal category structure; that is, it has a braiding, which is a system of natural isomorphisms $X_{a,b} : a \otimes b \to b \otimes a$ satisfying the hexagon equations (see [51], [54]). The braiding for $n\mathbf{Lad}$ is constructed using Lusztig's [38] braid elements. We also use the diagrams with crossings in Figure 1 to denote the braiding $X_{a,b}$ and its inverse $X_{a,b}^{-1}$ in the category $n\mathbf{Lad}_m$.

When β is a braid on m strands and $a = (a_1, \dots, a_m) \in \mathbb{Z}^m$, let $\beta 1_a \in \mathrm{Hom}_{n\mathbf{Lad}_m}(a, \beta(a))$ be the morphism described in Figure 3. Here $\beta(a)$ is obtained from a by applying the permutation corresponding to the braid β. For example, $\sigma_i 1_a$ and $\sigma_i^{-1} 1_a$, where σ_i, σ_i^{-1} are the ith standard braid generator and its inverse, are depicted in Figure 3.

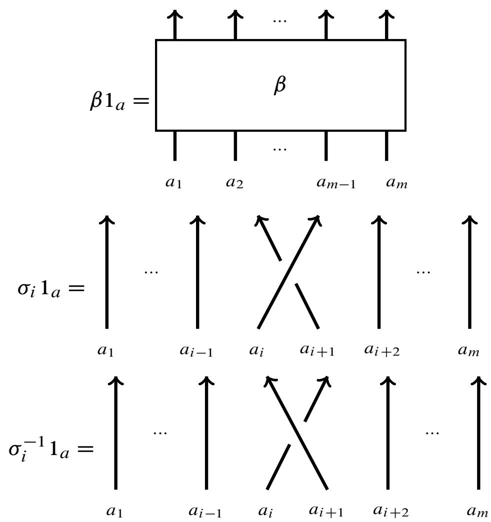


Figure 3. The morphisms $\beta 1_a$, $\sigma_i 1_a$, and $\sigma_i^{-1} 1_a$.

Then $\sigma_i^{\pm 1} 1_a \in \mathrm{Hom}_{n\mathbf{Lad}_m}(a, \sigma_i(a))$. We record here the formula for the braidings from [5]:

$$\sigma_i 1_a = (-1)^{a_i + a_i a_{i+1}} q^{a_i - \frac{a_i a_{i+1}}{n}} \sum_{\substack{r,s\geq 0\\ s-r=a_i-a_{i+1}}} (-q)^{-s} E_i^{(r)} F_i^{(s)} 1_a, \tag{19}$$

$$\sigma_i^{-1} 1_a = (-1)^{a_i + a_i a_{i+1}} q^{-a_i + \frac{a_i a_{i+1}}{n}} \sum_{\substack{r,s\geq 0\\ s-r=a_i-a_{i+1}}} (-q)^{s} E_i^{(r)} F_i^{(s)} 1_a. \tag{20}$$

Note that the right-hand sides are finite sums, since $F_i^{(r)}$ and $E_i^{(r)}$ are 0 for $r > n$. Also $\sigma_i^{-1} 1_a$ is obtained from $\sigma_i 1_a$ by the involution $q \to q^{-1}$.

Remark 2.7

Originally, Lusztig [38, Section 5.2.1] defined the braiding and its inverses using triple product formulas. The simplification of Lusztig's formulas to double products in (19)–(20) was first observed for $q = 1$ by Chuang and Rouquier [8]. For general q, a proof of this simplification can be found in [5, Lemma 6.1.1].

2.10. *From ladders to webs*

In [5, Section 5] it is proved that there is a $\mathbb{Q}(q^{1/n})$-linear functor

$$\Psi_{n,m} : n\mathbf{Lad}_m \to n\mathbf{Web}$$

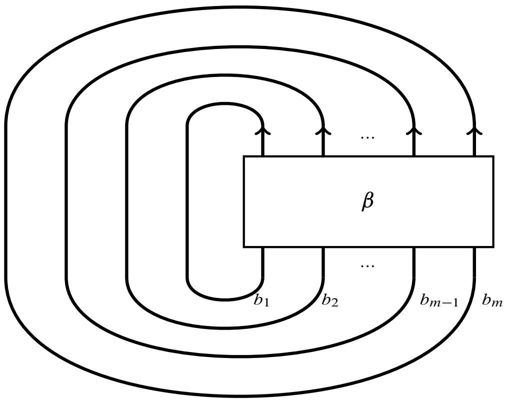


Figure 4. The standard closure of a braid β with four strands.

defined as follows. For an object $a = (a_1, \dots, a_m)$ of $n\mathbf{Lad}_m$, $\Psi_{n,m}(a)$ is obtained from a by deleting 0's and n's from a and converting k to k^+. For a morphism f of $n\mathbf{Lad}_m$ which is a ladder, $\Psi_{n,m}(f)$ is the same f considered as an n-web, using the convention about labelings 0 and n. This means edges connected to the label 0 should be deleted from the diagrams, and those connected to the label n should be truncated to the "tags" depicted in the last two diagrams in (7) as explained in (9). The existence of $\Psi_{n,m}$ is a consequence of the quantum skew-Howe duality.

The functors $\Psi_{n,m} : n\mathbf{Lad}_m \to n\mathbf{Web}$, with all m, piece together to give a functor $\Psi_n : n\mathbf{Lad} \to n\mathbf{Web}$. By [5, Theorem 6.2.1], Ψ_n is a braided monoidal functor.

Suppose that β is a braid on m strands. We view β as a diagram with crossings in the standard plane with strands oriented upward. Let $\mathrm{cl}(\beta)$ be the link diagram obtained by closing β in the standard way (see Figure 4), and let $L = L(\beta)$ be the corresponding framed link. Assume L has r ordered components which are labeled by integers $a_1, \dots, a_r \in [1, n-1]$. Let $a = (a_1, \dots, a_r)$. Let $b_1, \dots, b_m$ be the induced labeling of strands of β from left to right (at the bottom of β). Of course, each b_i is one of the a_j's.

Let $\mathrm{Lcl}(\beta, a)$, called the *ladder closure of* β, be the endomorphism of $\vartheta(n, m)$ in the category $n\mathbf{Lad}_{2m}$ given by the ladder described in Figure 5. Here the labels of the strands of the braids are $b_1, \dots, b_m$, which are determined by the labels $a_1, \dots, a_r$ of the link L. All the dashed vertical lines of the m left-hand sides are labeled by n, while all the dashed vertical lines of the m right-hand sides are labeled by 0. Then

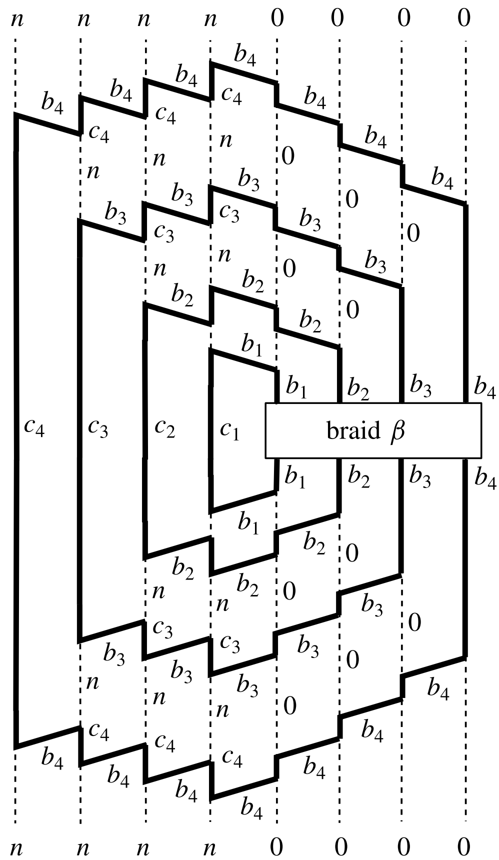


Figure 5. The ladder closure of a braid β with four strands with labels. Here $c_i = n - b_i$.

the remaining labels are uniquely determined by the rule that the signed sum at every trivalent vertex is 0.

PROPOSITION 2.8
We have

$$\mathrm{ev}_{n,m}\big(\mathrm{Lcl}(\beta, a)\big) = J_L(e_{a_1}, \ldots, e_{a_r}).$$

Proof
Let L denote the closure of β. We have that L is a link colored by a. Identities (10) and (11) show that

$$\Psi_n(L, a) = \mathrm{Lcl}(\beta, a).$$

Since Ψ_n is a $\mathbb{Q}(q^{1/n})$-linear braided functor, we have

$$\begin{aligned}\mathrm{ev}\big(\mathrm{Lcl}(\beta,a)\big) &= \mathrm{ev}\big(\Psi_n(L,a)\big)\\ &= J_L(e_{a_1},\dots,e_{a_r}),\end{aligned}$$

where the second identity follows from (12). □

3. Introducing the variable $x = q^n$

Proposition 2.8 allows one to calculate the quantum $\mathfrak{sl}_n$-invariant of a link L for each fixed $n \geq 2$. In this section, we introduce an algebra that allows us to unify the quantum $\mathfrak{sl}_n$-invariants of links into Laurent polynomials of a variable $x = q^n$.

3.1. Free associative algebra on E_i, F_j

Let

$$\mathcal{X}_m = \{E_1,\dots,E_{m-1},F_1,\dots,F_{m-1}\}, \tag{21}$$

and let $\mathfrak{A}_m$ be the free associative unital $\mathbb{Q}(q)$-algebra generated by $\mathcal{X}_m$. For $i = 1,\dots,m-1$, define the divided powers by

$$E_i^{(r)} := E_i^r/[r]! \in \mathfrak{A}_m, \qquad F_i^{(r)} := F_i^r/[r]! \in \mathfrak{A}_m,$$

where $[r]!$ is given by (15b). A $\mathbb{Q}(q)$-basis of $\mathfrak{A}_m$ can be described as follows. For $Y = (Y_1,\dots,Y_r) \in (\mathcal{X}_m)^r$ and $k = (k_1,\dots,k_r) \in \mathbb{N}^r$ define

$$Y^{(k)} := Y_1^{(k_1)} Y_2^{(k_2)} \cdots Y_r^{(k_r)}.$$

Then the set of all $Y^{(k)}$, where $Y_i \neq Y_{i+1}$ and $k_i \geq 1$, along with $k = \emptyset$, is a $\mathbb{Q}(q)$-basis of $\mathfrak{A}_m$.

Note that, for each $a \in \mathbb{Z}^m$, $Y^{(k)} 1_a$ is a morphism in the category $n\mathbf{Lad}_m$. For $a, b \in \mathbb{Z}^m$ and $n > 1$, define the $\mathbb{Q}(q)$-linear map

$$p^n_{a,b} : \mathfrak{A}_m \to \mathrm{Hom}_{n\mathbf{Lad}_m}(a,b), \qquad p^n_{a,b}(Y^{(k)}) = 1_a Y^{(k)} 1_b.$$

The algebra $\mathfrak{A}_m$ admits a natural $\mathbb{Z}^m$-grading, called *weight*, defined by

$$\mathrm{w}(F_i) = -\alpha_i, \qquad \mathrm{w}(E_i) = \alpha_i.$$

Observe that $p^n_{a,b}(Y^{(k)}) = 0$ unless $a = b + \mathrm{w}(Y^{(k)})$.

Let I_s be the two-sided ideal of $\mathfrak{A}$ generated by $E_i^{(r)}, F_i^{(r)}$, with $i = 1,\dots,m-1$ and $r \geq s$. It is clear that $I_{s+1} \subset I_s$. Let $\widehat{\mathfrak{A}}_m$ be the completion of $\mathfrak{A}_m$ with respect to the nested sequence of ideals I_s. Since $p^n_{a,b}(I_s) = 0$ if $s > n$, we can extend $p^n_{a,b}$ to a map, also denoted by $p^n_{a,b}$,

$$p^n_{a,b} : \widehat{\mathfrak{A}}_m \to \mathrm{Hom}_{n\mathbf{Lad}_m}(a,b).$$

3.2. Convention on negative powers

The divided powers $E_i^{(r)}$ and $F_i^{(r)}$ are defined for nonnegative integers r. It is convenient to extend them to negative powers by the following convention. For $r < 0$, $a \in \mathbb{Z}^m$, we use the following convention:

$$E_i^{(r)} = F_i^{(r)} = 0 \quad \text{in } \widehat{\mathfrak{A}}_m,$$

$$E_i^{(r)} 1_a = F_i^{(r)} 1_a = 0 \quad \text{in } \mathbf{Lad}_m.$$

With the above convention, (14a), (14b), and (14d) can be rewritten in the following form. For all $r, s \in \mathbb{Z}$ and $i \neq j$, we have the following identities in $\mathbf{Lad}_m$:

$$E_i^{(r)} F_i^{(s)} 1_a = \sum_{t \in \mathbb{Z}} \begin{bmatrix} \langle a, \alpha_i \rangle + r - s \\ t \end{bmatrix} F_i^{(s-t)} E_i^{(r-t)} 1_a, \tag{22}$$

$$E_i^{(r)} F_j^{(s)} 1_a = F_j^{(s)} E_i^{(r)} 1_a, \tag{23}$$

$$E_i^{(s)} E_i^{(r)} 1_a = \begin{bmatrix} r+s \\ r \end{bmatrix} E_i^{(r+s)} 1_a, \qquad F_i^{(s)} F_i^{(r)} 1_a = \begin{bmatrix} r+s \\ r \end{bmatrix} F_i^{(r+s)} 1_a. \tag{24}$$

3.3. Evaluation

Fix positive integers n, m. Recall that ϑ given by (16) is an object of $n\mathbf{Lad}_{2m}$, and recall the evaluation map (18). This gives rise to an evaluation map

$$\mathrm{ev}_n : \widehat{\mathfrak{A}}_{2m} \to \mathbb{Q}(q^{1/n}), \qquad \mathrm{ev}_n(x) := \mathrm{ev}_{n,m}\big(p^n_{\vartheta,\vartheta}(x)\big). \tag{25}$$

Given a monomial z in E_i, F_j, the element $\mathrm{ev}_n(z)$ can be calculated by a simple algorithm moving each divided power in E_i appearing in z to the right by using (22) and (23). Note that we are not moving divided powers of E_i past divided powers of E_j. Since the E_i's annihilate the weight 1_ϑ, all that remains after sliding all the E_i's to the right is a sum of products of the quantum binomials produced from the application of (22). For details see the example in Section 3.7 and Proposition 5.2.

Suppose that $Y = (Y_1, \dots, Y_k) \in (\mathcal{X}_{2m})^k$ and $b = (b_1, \dots, b_k) \in \mathbb{Z}^k$. There is an easy case when $\mathrm{ev}_n(Y^{(b)}) = 0$, namely, when $1_\vartheta Y^{(b)} 1_\vartheta$ factors through a weight with a negative component. The weight of $Y^{(b)}$ is denoted by

$$\mathrm{w}(Y^{(b)}) = \big(\mathrm{w}_1(Y^{(b)}), \dots, \mathrm{w}_{2m}(Y^{(b)})\big) \in \mathbb{Z}^{2m}.$$

We say $Y^{(b)}$ has *negative* weight if $\mathrm{w}_j(Y^{(b)}) < 0$ for some j with $m < j \le 2m$. For an index i, $1 \le i \le k$, define the ith tail $\mathrm{Tail}_i(Y, b)$ by

$$\mathrm{Tail}_i(Y, b) = Y_i^{(b_i)} Y_{i+1}^{(b_{i+1})} \cdots Y_k^{(b_k)}.$$

We say (Y, b) is *tail-negative* if there is an index i, $1 \le i \le k$, such that $\mathrm{Tail}_i(Y, b)$ has negative weight.

LEMMA 3.1
Suppose that (Y, b) is tail-negative. Then $\mathrm{ev}_n(Y^{(b)}) = 0$ for all n.

Proof
Note that $Y^{(b)} 1_\vartheta$ factors through $\mathrm{Tail}_i(Y, b) 1_\vartheta \in \mathrm{Hom}_{\mathbf{Lad}^n_{2m}}(\mu, \vartheta)$, where

$$\mu = \mathrm{w}\big(\mathrm{Tail}_i(Y, b)\big) + \vartheta.$$

Suppose that $\mathrm{w}_j(\mathrm{Tail}_i(Y, b)) < 0$ for some $j > m$ and $1 \le i \le k$. We have $\mu_j = \mathrm{w}(\mathrm{Tail}_i(Y, b)) + \vartheta_j = \mathrm{w}_j(\mathrm{Tail}_i(Y, b)) < 0$. By definition, $\mathrm{Tail}_i(Y, b) 1_\vartheta = 0$ in $\mathbf{Lad}^n_{2m}$. Hence, $Y^{(b)} 1_\vartheta = 0$ in $\mathbf{Lad}^n_{2m}$. □

The tail-negative condition can be characterized by the function

$$\mathcal{H}(Y, b) := \prod_{j=m+1}^{2m} \prod_{i=1}^{k} \mathrm{He}\Big(\mathrm{w}_j\big(\mathrm{Tail}_i(Y, b)\big)\Big), \tag{26}$$

where

$$\mathrm{He}(x) = \begin{cases} 1 & \text{if } x \ge 0, \\ 0 & \text{if } x < 0, \end{cases} \tag{27}$$

denotes the Heaviside function. Note that

$$\mathcal{H}(Y, b) = \begin{cases} 0 & \text{if } (Y, b) \text{ is tail-negative}, \\ 1 & \text{otherwise}. \end{cases} \tag{28}$$

3.4. Braiding in $\widehat{\mathfrak{A}}$
Suppose that $a = (a_1, \ldots, a_m) \in \mathbb{Z}^m$ and $1 \le i \le m - 1$. Let

$$T_i(a) = (-1)^{a_i + a_i a_{i+1}} q^{a_i} \sum_{s \in \mathbb{Z}} (-q)^{-s} E_i^{(s + a_{i+1} - a_i)} F_i^{(s)} \in \widehat{\mathfrak{A}}, \tag{29a}$$

$$T_i^{-1}(a) = (-1)^{a_i + a_i a_{i+1}} q^{-a_i} \sum_{s \in \mathbb{Z}} (-q)^{s} E_i^{(s + a_{i+1} - a_i)} F_i^{(s)} \in \widehat{\mathfrak{A}}. \tag{29b}$$

Recall that we use the convention $E_i^{(r)} = F_i^{(r)} = 0$ if $r < 0$. Note that $T_i^{-1}(a)$ is obtained from $T_i(a)$ by the involution $q \to q^{-1}$. From (19) and (20) it follows that, for $\varepsilon = \pm 1$,

$$\sigma_i^{\varepsilon} 1_a = q^{-\varepsilon \frac{a_i a_{i+1}}{n}} T_i^{\varepsilon}(a) 1_a \quad \text{in } n\mathbf{Lad}_m. \tag{30}$$

3.5. Special functions

Let $Y = (Y_1, \dots, Y_k) \in (\mathcal{X}_m)^k$. A function $H : \mathbb{Z}^r \to \widehat{\mathfrak{A}}_m$ is called Y*-special* if

$$H(a) = \sum_{s \in \mathbb{Z}^t} (-1)^{g_1(a,s)} q^{g_2(a,s)} Y^{(f(a,s))}, \tag{31}$$

where

- $g_1 : \mathbb{Z}^{r+t} \to \mathbb{Z}$ is quadratic, that is, given by a polynomial with integer coefficients of total degree at most 2,
- $g_2 : \mathbb{Z}^{r+t} \to \mathbb{Z}$ is linear, and
- $f : \mathbb{Z}^{r+t} \to \mathbb{Z}^k$ is affine such that $f(a, \cdot) : \mathbb{Z}^t \to \mathbb{Z}^k$ is injective for every $a \in \mathbb{Z}^k$.

The injectivity property ensures that the right-hand side of (31) defines an element in $\widehat{\mathfrak{A}}_m$. The next lemma is easy to verify.

LEMMA 3.2

(a) *The functions* $T_i, T_i^{-1} : \mathbb{Z}^m \to \widehat{\mathfrak{A}}_m$ *given by (29a) and (29b) are* (E_i, F_i)*-special.*

(b) *Suppose that* $f : \mathbb{Z}^k \to \mathbb{Z}^r$ *is a linear function. Then the function* $H : \mathbb{Z}^k \to \widehat{\mathfrak{A}}_m$ *given by*

$$H(a) = Y^{(f(a))}$$

is Y*-special.*

(c) *If* H' *is* Y'*-special and* H'' *is* Y''*-special, then* $H'H''$ *is* $Y' \times Y''$*-special.*

3.6. Unifying the $\mathfrak{sl}_n$-link invariant

For $a = (a_1, \dots, a_r) \in \mathbb{Z}^r$, let $\|a\|_\infty$ be the usual norm defined by $\|a\|_\infty = \max_i |a_i|$.

PROPOSITION 3.3

Suppose that L *is a framed, oriented link in* 3*-space with* r *ordered components which is the closure of a braid with* m *strands. Then there exist a sequence* Y *of letters in* $\mathcal{X}_{2m}$ *and a* Y*-special function* $H : \mathbb{Z}^r \to \widehat{\mathfrak{A}}_{2m}$ *such that, for all integers* $a_1, \dots, a_r \in [0, n-1]$*, we have*

$$\tilde{J}_L^{\mathfrak{sl}_n}(e_{a_1}, \dots, e_{a_r}) = \mathrm{ev}_n\big(H(a_1, \dots, a_r)\big). \tag{32}$$

Moreover, $(Y, f(a,s))$ *is tail-negative whenever* $\|s\|_\infty > \|a\|_\infty$*. Here* $f(a,s)$ *is the function appearing in the presentation (31) of* H*.*

Proof

Let L be the closure of a braid $\beta \in B_m$ as in Figure 4, and let $a = (a_1, \dots, a_r) \in \mathbb{N}^r$.

Suppose that $1^m \otimes \beta$, the braid (in B_{2m}) obtained by adding m straight strands to the left of β, has a presentation

$$1^m \otimes \beta = \sigma_{i_1}^{\varepsilon_1} \cdots \sigma_{i_t}^{\varepsilon_t}, \quad i_j \in \{m+1, \dots, 2m-1\}, \varepsilon_j \in \{\pm 1\} \text{ for } j = 1, \dots, t, \tag{33}$$

where σ_i is the ith standard generator of the braid group (see Figure 1). Here t is the number of crossings of β.

Write $b = (b_1, \dots, b_m)$ and $c = (c_m, c_{m-1}, \dots, c_1)$ to denote the sequences of labels labeling the ladder closure $\mathrm{Lcl}(\beta, a)$ as in Figure 5, so that $c_i = n - b_i$ and each b_i is one of $(a_1, \dots, a_r)$. The horizontal lines at the bottom and the top of the braid β decompose $\mathrm{Lcl}(\beta)$ into three morphisms in $n\mathbf{Lad}_{2m}$:

$$\mathrm{Lcl}(\beta, a) = \mathrm{Cap}_m(a)(1_c \otimes \beta 1_b)\mathrm{Cup}_m(a).$$

Each part can be written in a form that does not depend on n.

Indeed, the lower morphism $\mathrm{Cup}_m(a)$ is a composition of $F_i^{(b_j)}$ for various i, j. Hence,

$$\mathrm{Cup}_m(a) = \mathsf{V}_m(a) 1_{\vartheta(n,m)},$$

where $\mathsf{V}_m(a) \in \mathfrak{A}_{2m}$ is the product of several $F_i^{(b_j)}$'s. Explicitly,

$$\mathsf{V}_m(a) = \overrightarrow{\prod_{k \in [1,m]}} \Big[\Big(\overleftarrow{\prod_{i \in [1,k-1]}} (F_{m+k}^{(b_k)} F_{m-k}^{(b_k)}) \Big) F_m^{(b_k)} \Big],$$

where

$$\overrightarrow{\prod_{i \in [1,k]}} x_i := x_1 x_2 \cdots x_k, \qquad \overleftarrow{\prod_{i \in [1,k]}} x_i := x_k x_{k-1} \cdots x_1.$$

Then, as a function of a, $\mathsf{V}_m(a)$ is a special function (see Lemma 3.2).

Similarly, the top morphism

$$\mathrm{Cap}_m(a) = \Lambda_m(a) 1_{c \otimes b} = 1_{\vartheta(n,m)} \Lambda_m(a) 1_{c \otimes b}$$

is a special function. Explicitly,

$$\Lambda_m(a) = \overleftarrow{\prod_{k \in [1,m]}} \Big[E_m^{(b_k)} \overrightarrow{\prod_{i \in [1,k-1]}} (E_{m+k}^{(b_k)} E_{m-k}^{(b_k)}) \Big].$$

Now consider the middle morphism $1_c \otimes \beta 1_b$. Using (33) and (30), we have

$$1_c \otimes \beta 1_b = q^{-\frac{1}{n} \sum_{ij} \ell_{ij} a_i a_j} z_1(a) z_2(a) \cdots z_t(a) 1_{c \otimes b},$$

where

$$z_j(a) = T_{i_j}^{\varepsilon_j}\big(\sigma_{i_j+1}\cdots\sigma_{i_t}(c,b)\big).$$

Using (29a) and (29b) for $T_i^{\pm 1}(b)$, we see that z_j is a special function. Let

$$H(a) = \Lambda_m(a) z_1(a) z_2(a)\cdots z_t(a) \mathsf{V}_m(a). \tag{34}$$

Then $H : \mathbb{Z}^k \to \widehat{\mathfrak{A}}_{2m}$ is a product of special functions and, hence, is a special function (see Lemma 3.2). By (30), we have

$$\mathrm{Lcl}(\beta, a) = q^{-\frac{1}{n}\sum_{ij}\ell_{ij}a_i a_j} 1_\vartheta H(a) 1_\vartheta.$$

Applying the evaluation map ev_n to both sides and using Proposition 2.8 and the normalization (4) of $\tilde{J}_L$ for the left-hand side, we obtain that

$$\tilde{J}_L(e_{a_1},\ldots,e_{a_k}) = \mathrm{ev}_n\big(H(a)\big).$$

This proves (32).

Let us have a closer look at the formula of H. By (29a) and (29b), z_j has the form

$$z_j = \sum_{s_j\in\mathbb{Z}} (-1)^{g_j(a,s_j)} q^{\varepsilon_j h_j(a,s_j)} E_{i_j}^{(f_j(a,s_j))} F_{i_j}^{(s_j)}, \tag{35}$$

where g_j is a quadratic function and h_j, f_j are linear functions. From (34), it follows that H has a presentation (31), where $s = (s_1,\ldots,s_t)$, and

$$Y^{(f(a,s))} = \Lambda_m(a)\Big(\overrightarrow{\prod_{j\in[1,t]}} E_{i_j}^{(f_j(a,s_j))} F_{i_j}^{(s_j)}\Big)\mathsf{V}_m(a).$$

Assume that $\|s\|_\infty > \|a\|_\infty$; that is, there is l such that $|s_l| > \|a\|_\infty$. We can assume that $s_l > 0$, since otherwise $s_l < 0$ and the factor $F_{i_l}^{(s_l)}$ on the right-hand side of (35) is 0. We will show that the i_lth component of the weight of $F_{i_l}^{(s_l)} z$ is negative, where

$$z = \Big(\overrightarrow{\prod_{j\in[l+1,t]}} E_{i_j}^{(f_j(a,s_j))} F_{i_j}^{(s_j)}\Big)\mathsf{V}_m(a).$$

This will prove that $(Y, f(a,s))$ is tail-negative, since $i_l > m$. Note that

$$\mathrm{w}(z) = \mathrm{w}\big(z_{l+1}(a)\cdots z_t(a)\mathsf{V}_m(a)\big) = (c_m - n,\ldots,c_1 - n, b'),$$

where b' is a permutation of b. Since $\|b'\|_\infty = \|b\|_\infty = \|a\|_\infty$, we have

$$\mathrm{w}_{i_l}(F_{i_l}^{(s_l)} z) = -s_l + (b')_{i_l} < 0,$$

which completes the proof of the proposition. □

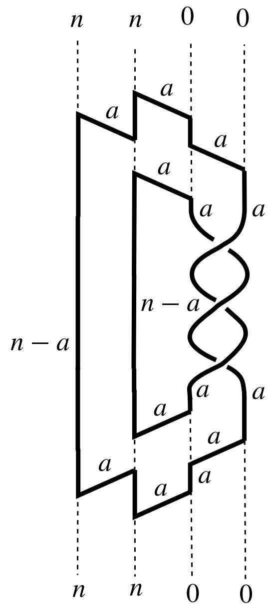


Figure 6. The ladder closure of braid $\beta = \sigma_1^3$.

Remark 3.4
Our evaluation algorithm should be closely related to the variant of skew-Howe duality defined for so-called *doubled Schur algebras* in [50] and [49].

3.7. *An example: The trefoil knot*

Before we proceed further, let us illustrate Proposition 3.3 by computing the invariant of the trefoil and draw some useful conclusions regarding the q-holonomicity of the invariant.

We take

$$\beta = \sigma_1^3 = \sigma_1\sigma_1\sigma_1, \qquad m = 2, \qquad \vartheta = (n, n, 0, 0). \tag{36}$$

Then $L = \mathrm{cl}(\beta)$ is the right-handed trefoil knot, colored by $a \in \mathbb{N} \cap [0, n-1]$ (see Figure 6).

By Proposition 3.3 and (34), we obtain that $\tilde{J}_{3_1}^{\mathfrak{sl}_n}(e_a) = \mathrm{ev}_n(H(a))$, where

$$H(a) = E_2^{(a)} E_1^{(a)} E_3^{(a)} E_2^{(a)} (T_3)^3 F_2^{(a)} F_3^{(a)} F_1^{(a)} F_2^{(a)}, \tag{37}$$

where $T_3 = T_3(n-a, n-a, a, a)$. Using (29a), we replace each occurrence of T_3 by a sum over the integers and obtain the triple sum formula

$$H(a) = \sum_{s_1,s_2,s_3\in\mathbb{Z}} (-q)^{s_1+s_2+s_3} E_2^{(a)} E_1^{(a)} E_3^{(a)} E_2^{(a)} (E_3^{(s_1)} F_3^{(s_1)} E_3^{(s_2)} F_3^{(s_2)} \times E_3^{(s_3)} F_3^{(s_3)}) F_2^{(a)} F_3^{(a)} F_1^{(a)} F_2^{(a)}. \tag{38}$$

This is an explicit form of special function for H.

Next, we use the commutation rules given in (22)–(24) to sort the expression of $H(a)1_\vartheta$, moving all the E's to the right and all the F's to the left. Every time we move $E_i^{(r)}$ (from the left) past an $F_i^{(s)}$ (from the right), we obtain a 1-dimensional sum over the integers. Then, use (17) to add some delta functions in the sum. Finally, (53), which is explained later in the proof of Proposition 5.2, tells us to add Heaviside functions $\mathrm{He}(k)$ (see Section 4). Doing so, we eventually get the following formula for the quantum $\mathfrak{sl}_n$-invariant of the trefoil colored by e_a (the details are given in Appendix A):

$$\begin{aligned}\tilde{J}_L^{\mathfrak{sl}_n}(e_a) = \begin{bmatrix} n \\ a \end{bmatrix} \sum_{s\in\mathbb{Z}^6} &(-q)^{-(s_1+s_2+s_3)} \mathrm{He}(a-s_1)\mathrm{He}(a-s_2)\mathrm{He}(a-s_3) \\ &\times \mathrm{He}(a+s_1+s_2-s_4)\mathrm{He}(a+s_2+s_3-s_5)\mathrm{He}(\tau) \\ &\times \begin{bmatrix} s_2+s_1 \\ s_4 \end{bmatrix}\begin{bmatrix} s_2+s_3 \\ s_5 \end{bmatrix}\begin{bmatrix} \tau+s_2+s_6 \\ s_6 \end{bmatrix}\begin{bmatrix} s_1+s_2-s_4 \\ s_1 \end{bmatrix}\begin{bmatrix} \tau \\ s_1+s_2-s_4 \end{bmatrix} \\ &\times \begin{bmatrix} s_2+s_3-s_5 \\ s_3 \end{bmatrix}\begin{bmatrix} \tau \\ s_2+s_3-s_5 \end{bmatrix}\begin{bmatrix} a \\ a-\tau \end{bmatrix}\begin{bmatrix} n-\tau \\ a \end{bmatrix},\end{aligned} \tag{39}$$

where $\tau = s_1+s_2+s_3-s_4-s_5-s_6$ and $s = (s_1,\dots,s_6) \in \mathbb{Z}^6$. Keep in mind the convention that $\begin{bmatrix} r \\ s \end{bmatrix} = 0$ if $s < 0$.

Let us end this example with some observations. The above formula has the form

$$\tilde{J}_L^{\mathfrak{sl}_n}(e_a) = \sum_{s\in\mathbb{Z}^6} F(a,s), \tag{40}$$

where $F(a,s)$ is a finite product of factors of the following shapes:

(i) $(\pm q)^{A(a,s)}$,

(ii) $\mathrm{He}(A(a,s))$,

(iii) quantum binomial $\begin{bmatrix} A(a,s) \\ B(a,s) \end{bmatrix}$,

(iv) quantum binomial $\begin{bmatrix} n+A(a,s) \\ B(a,s) \end{bmatrix} = \begin{bmatrix} q^n;A(a,s) \\ B(a,s) \end{bmatrix}$, where for $s,l \in \mathbb{Z}$ we define

$$\begin{bmatrix} x;s \\ l \end{bmatrix} = \begin{cases} 0 & \text{if } l < 0, \\ \prod_{j=1}^{l} \frac{xq^{s-j+1}-x^{-1}q^{-s+j-1}}{q^j-q^{-j}} & \text{if } l \geq 0. \end{cases} \tag{41}$$

Here $A(a, s)$ and $B(a, s)$ are $\mathbb{Z}$-linear functions. Moreover, for each *integer* value of a and n, the sum on the right-hand side of (40) is terminating in the sense that only a finite number of terms are nonzero. The number of terms are bounded by a polynomial function of a.

We will show that a similar formula exists for any framed, oriented link colored with e_a. But before we do so, let us recall q-holonomic functions.

4. q-Holonomic functions

q-Holonomic functions of one variable were introduced in the seminal paper of Zeilberger [58]. The class of q-holonomic functions resembles in several ways the class of holonomic D-modules, as acknowledged by conversations of Zeilberger and Bernstein prior to the introduction of holonomic functions (see [58]). An extension of the definition to q-holonomic functions with several variables was given by Sabbah [52], using the language of homological algebra. In this section we will review the definition of q-holonomic functions of several variables, give examples, and list the closure properties of this class under some operations. Our exposition follows Zeilberger, Sabbah, and the survey paper of the first and third authors [20].

We should point out, however, that the precise definition of q-holonomic functions is not used in the proof of Theorem 1.1. If the reader wishes to take as a black box the examples of q-holonomic functions given below and their closure properties, then they can skip this section altogether and still deduce the proof of Theorem 1.1.

4.1. The quantum Weyl algebra

Let V denote a fixed (not necessarily finitely generated) $\mathcal{A}$-module, where $\mathcal{A} = \mathbb{Z}[q^{\pm 1}]$. For a natural number r, let $S_r(V)$ be the set of all functions $f : \mathbb{Z}^r \to V$ and $S_{r,+}(V)$ the set of functions $f : \mathbb{N}^r \to V$. For $i = 1, \ldots, r$ consider the operators L_i and M_i which act on functions $f \in S_r(V)$ by

$$(\mathsf{L}_i f)(n_1, \ldots, n_i, \ldots, n_r) = f(n_1, \ldots, n_i + 1, \ldots, n_r), \tag{42}$$

$$(\mathsf{M}_i f)(n_1, \ldots, n_r) = q^{n_i} f(n_1, \ldots, n_r). \tag{43}$$

It is clear that L_i and M_j are invertible operators that satisfy the q-commutation relations

$$\mathsf{M}_i \mathsf{M}_j = \mathsf{M}_j \mathsf{M}_i, \tag{44a}$$

$$\mathsf{L}_i \mathsf{L}_j = \mathsf{L}_j \mathsf{L}_i, \tag{44b}$$

$$\mathsf{L}_i \mathsf{M}_j = q^{\delta_{i,j}} \mathsf{M}_j \mathsf{L}_i, \tag{44c}$$

for all $i, j = 1, \ldots, r$.

Definition 4.1
The r-dimensional quantum Weyl algebra $\mathbb{W}_r$ is the $\mathcal{A}$-algebra generated by $\mathsf{L}_1^{\pm 1}, \dots, \mathsf{L}_r^{\pm 1}, \mathsf{M}_1^{\pm 1}, \dots, \mathsf{M}_r^{\pm 1}$ subject to the relations (44a)–(44c). Let $\mathbb{W}_{r,+}$ be the subalgebra of $\mathbb{W}_r$ generated by the nonnegative powers of $\mathsf{M}_j, \mathsf{L}_j$.

Given $f \in S_r(V)$, the annihilator $\mathrm{Ann}(f)$ (a left $\mathbb{W}_r$-module) is defined by

$$\mathrm{Ann}(f) = \{P \in \mathbb{W}_r \,|\, Pf = 0\}. \tag{45}$$

This gives rise to a cyclic $\mathbb{W}_r$-module M_f, defined by $M_f = \mathbb{W}_r f \subset S_r(V)$, and isomorphic to $\mathbb{W}_r / \mathrm{Ann}(f)$.

4.2. *Definition of q-holonomic functions*

In this section we follow closely the work of Sabbah [52]. Let N be a finitely generated $\mathbb{W}_{r,+}$-module. Consider the increasing filtration $\mathcal{F}$ on $\mathcal{W}_{r,+}$ given

$$\begin{aligned} \mathcal{F}_n \mathcal{W}_{r,+} = \{ & \mathcal{A}\text{-span of all monomials } \mathsf{M}^\alpha \mathsf{L}^\beta \text{ with } \alpha, \beta \in \mathbb{N}^r \\ & \text{with total degree at most } n\}. \end{aligned} \tag{46}$$

The filtration $\mathcal{F}$ on $\mathbb{W}_{r,+}$ induces an increasing filtration on N, defined by $\mathcal{F}_n N = \mathcal{F}_n \mathbb{W}_{r,+} \cdot N$. Note that $\mathcal{F}_n \mathbb{W}_{r,+}$ and, consequently, $\mathcal{F}_n N$ are finitely generated $\mathcal{A}$-modules for all natural numbers n. An analogue of Hilbert's theorem for this noncommutative setting holds: the dimension of the $\mathbb{Q}(q)$-vector space $\mathbb{Q}(q) \otimes_{\mathcal{A}} \mathcal{F}_n N$ is a polynomial in n, for big enough n. The degree of this polynomial is called the *dimension of* N and is denoted by $d(N)$.

In [52, Theorem 1.5.3] Sabbah proved that $d(N) = 2r - \mathrm{codim}(N)$, where

$$\mathrm{codim}(N) = \min\bigl\{ j \in \mathbb{N} \mid \mathrm{Ext}^j_{\mathbb{W}_{r,+}}(N, \mathbb{W}_{r,+}) \neq 0 \bigr\}.$$

Sabbah also proved that $d(N) \geq r$ if N is nonzero and does not have monomial torsion. Here a monomial torsion is a monomial P in $\mathbb{W}_{r,+}$ such that $Px = 0$ for a certain nonzero $x \in N$. It is easy to see that N embeds in the $\mathbb{W}_r$-module $\mathbb{W}_r \otimes_{\mathbb{W}_{r,+}} N$ if and only if N has no monomial torsion. Throughout the article, we will assume that all $\mathbb{W}_{r,+}$-modules do not have monomial torsion.

Definition 4.2
(a) A $\mathbb{W}_{r,+}$-module N is q-holonomic if $N = 0$ or N is finitely generated, does not have monomial torsion, and $d(N) = r$.
(b) An element $f \in N$, where N is a $\mathbb{W}_{r,+}$-module, is q-holonomic over $\mathbb{W}_{r,+}$ if $\mathbb{W}_{r,+} \cdot f$ is a q-holonomic $\mathbb{W}_{r,+}$-module.

The above definition defines q-holonomic $\mathbb{W}_{r,+}$-modules, and our next task is to define q-holonomic $\mathbb{W}_r$-modules. Let M be a nonzero finitely generated left $\mathcal{W}_r$-module. Following [52, Section 2.1], the codimension and dimension of M are defined in terms of homological algebra by

$$\mathrm{codim}(M) = \min\{j \in \mathbb{N} \mid \mathrm{Ext}^j_{\mathbb{W}_r}(M, \mathcal{W}_r) \neq 0\}, \qquad \dim(M) = 2r - \mathrm{codim}(M).$$

The key Bernstein inequality (proved by Sabbah [52, Theorem 2.1.1] in the q-case) asserts that if $M \neq 0$ is a finitely generated $\mathbb{W}_r$-module, then $\dim(M) \geq r$.

Definition 4.3

(a) A $\mathbb{W}_r$-module M is q-holonomic if either $M = 0$ or M is finitely generated and $\dim(M) = r$.

(b) An element $f \in M$, where M is a $\mathbb{W}_r$-module, is q-holonomic over $\mathbb{W}_r$ if $\mathbb{W}_r \cdot f$ is a q-holonomic $\mathbb{W}_r$-module.

Next we compare q-holonomic modules over $\mathbb{W}_r$ versus over $\mathbb{W}_{r,+}$. The following proposition was proven in [20, Section 3]. Next we compare q-holonomic modules over $\mathbb{W}_r$ versus over $\mathbb{W}_{r,+}$, using Sabbah [52, Corollary 2.1.4].

PROPOSITION 4.4

Suppose that $f \in M$, where M is a $\mathbb{W}_r$-module. Then f is q-holonomic over $\mathbb{W}_r$ if and only if it is q-holonomic over $\mathbb{W}_{r,+}$.

The next corollary is taken from [20, Section 3].

COROLLARY 4.5

If $f \in S_r(V)$ is q-holonomic and $g \in S_{r,+}(V)$ is its restriction to $\mathbb{N}^r$, then g is q-holonomic.

Remark 4.6

The definition of q-holonomic $\mathcal{A}$-modules can be extended to q-holonomic $\mathcal{R}$-modules, where $\mathcal{R}$ is the ring (and also an $\mathcal{A}$-module)

$$\mathcal{R} = \mathbb{Q}(q)[x^{\pm 1}]. \tag{47}$$

Proposition 4.4 and Theorems 4.7 and 4.8 below hold after replacing $\mathcal{A}$ by $\mathcal{R}$.

4.3. *Properties of q-holonomic functions*

In this section we summarize some closure properties of q-holonomic functions, whose proofs can be found in [20, Section 5].

THEOREM 4.7
Suppose that $f, g \in S_r(V)$ are q-holonomic functions. Then, the following hold.
(a) *$f + g$ is q-holonomic.*
(b) *fg is q-holonomic.*
(c) *Restriction. For a fixed $a \in \mathbb{Z}$, the function $g \in S_{r-1}(V)$ defined by*

$$g(n_1, \ldots, n_{r-1}) = f(n_1, \ldots, n_{r-1}, a)$$

is q-holonomic.
(d) *Extension. The function $h \in S_{r+1}(V)$ defined by*

$$h(n_1, \ldots, n_{r+1}) = f(n_1, \ldots, n_r)$$

is q-holonomic.
(e) *Linear substitution. If $A \in \mathrm{GL}(r, \mathbb{Z})$ and $f \in S_r(V)$ is q-holonomic, so is the composition $f \circ A \in S_r(V)$.*

Let $S_{r-1,1}(V)$ denote the set of all functions $f : \mathbb{Z}^r \to V$ such that, for every $(n_1, \ldots, n_{r-1}) \in \mathbb{Z}^{r-1}$, $f(n_1, \ldots, n_r) = 0$ for all but a finite number of n_r's.

THEOREM 4.8
(a) *Suppose that $f \in S_{r-1,1}(V)$ is q-holonomic. Then, $g \in S_{r-1}(V)$, defined by*

$$g(n_1, \ldots, n_{r-1}) = \sum_{n_r \in \mathbb{Z}} f(n_1, \ldots, n_r),$$

is q-holonomic.
(b) *Suppose that $f \in S_r(V)$ is q-holonomic. Then $h \in S_{r+1}(V)$, defined by*

$$h(n_1, \ldots, n_{r-1}, a, b) = \sum_{n_r = a}^{b} f(n_1, n_2, \ldots, n_r), \tag{48}$$

is q-holonomic.

4.4. Elementary q-holonomic functions

A function $g : \mathbb{Z}^s \to \mathbb{Z}^r$ is *affine* if there is an $(r \times s)$-matrix A with integer entries and $b \in \mathbb{Z}^r$ such that $g(a) = Aa + b$. If $b = 0$, then such a function is called *linear*.

A function $f : \mathbb{Z}^r \to \mathbb{Q}(q)[x^{\pm 1}]$ is called an *elementary block* if f is a finite product of compositions of a linear function $\mathbb{Z}^r \to \mathbb{Z}^s$ (for $s = 1, 2$) with one of the following functions:
(i) $\mathbb{Z} \to \mathbb{Z}[q^{\pm 1}], k \to (-1)^k$, or $k \to q^k$, or $k \to \mathrm{He}(k)$,

(ii) $\mathbb{Z}^2 \to \mathbb{Z}[q^{\pm 1}]$, $(k,l) \to (-1)^{kl}$, or $(k,l) \to \delta_{k,l}$, or $(k,l) \to \left[\begin{smallmatrix} k \\ l \end{smallmatrix}\right]$, or $(k,l) \to \left[\begin{smallmatrix} x;k \\ l \end{smallmatrix}\right]$.

Observe that functions of the form (i) or (ii) above are q-holonomic (see [20]). Consider the function $f(n_1, n_2) = (-1)^{n_1 n_2}$. Its annihilator ideal contains the monic operators $\mathsf{L}_1^2 - 1$ and $\mathsf{L}_2^2 - 1$, which generate a q-holonomic ideal (see [20, Theorem 7.2(a)]); hence, f is q-holonomic.

A function $f : \mathbb{Z}^r \to \mathbb{Q}(q)[x^{\pm 1}]$ is called *elementary* if it can be presented by a terminating sum

$$f(a) = \sum_{b \in \mathbb{Z}^l} g(a,b),$$

where $g : \mathbb{Z}^{k+l} \to \mathbb{Q}(q)[x^{\pm 1}]$ is an elementary block. Here "the sum is terminating" means that for each a there are only a finite number of b's such that $g(a,b) \neq 0$. Theorems 4.7 and 4.8 imply the following.

COROLLARY 4.9
Every elementary block and every elementary function is q-holonomic.

5. Proof of Theorem 1.1

5.1. *Evaluation of monomials is q-holonomic*

For $n \in \mathbb{Z}$, let $\mathrm{eval}_n : \mathbb{Q}(q)[x^{\pm 1}] \to \mathbb{Q}(q)$ be the $\mathbb{Q}(q)$-algebra homomorphism defined by

$$\mathrm{eval}_n(f) = f|_{x=q^n}. \tag{49}$$

The next lemma recovers an element of $\mathbb{Q}(q)[x^{\pm 1}]$ from its evaluations.

LEMMA 5.1
Suppose that $f, g \in \mathbb{Q}(q)[x^{\pm 1}]$ satisfy $\mathrm{eval}_n(f) = \mathrm{eval}_n(g)$ for infinitely many n. Then $f = g$.

Proof
This follows from the fact that a Laurent polynomial in x has at most k roots, where k is the difference between the highest order and the lowest order in x. □

Let $X = (X_1, \dots, X_k)$ be a sequence of elements of the set $\mathcal{X}_{2m}$ from (21). Recall that, for $b = (b_1, \dots, b_k) \in \mathbb{Z}^k$, the monomial $X^{(b)} \in \mathfrak{A}_{2m}$ and its weight are defined in Section 3.1. By convention, $X^{(b)} = 0$ if one of the b_i's is negative. The goal of this section is to calculate

$$\mathrm{ev}_n(X^{(b)}) = \mathrm{ev}(1_\vartheta X^{(b)} 1_\vartheta),$$

where $\vartheta = (n^m, 0^m) \in \mathbb{Z}^{2m}$.

PROPOSITION 5.2

Suppose that $X = (X_1, \dots, X_k)$ is a sequence of elements of the set $\mathcal{X}_{2m}$. There exists a unique function

$$Q_X : \mathbb{Z}^k \to \mathbb{Q}(q)[x^{\pm 1}]$$

such that, for all $b \in \mathbb{Z}^k$, $n \in \mathbb{N}$,

$$\mathrm{ev}_n(X^{(b)}) = \mathrm{eval}_n\big(Q_X(b)\big). \tag{50}$$

Moreover, Q_X is an elementary function given by

$$Q_X(b) = \sum_{j \in \mathbb{Z}^l} F_X(b, j) \tag{51}$$

for a certain $l \in \mathbb{N}$ and elementary block $F_X : \mathbb{Z}^{k+l} \to \mathbb{Q}(q)[x^{\pm 1}]$. In addition,

(i) $F_X(b, j) = 0$ *if* $\|j\|_\infty > \|b\|_\infty$ *(which implies that the sum (51) is terminating), and*

(ii) $F_X(b, j) = 0$ *if* (X, b) *is tail-negative or if one of the components of b is negative.*

Proof

The uniqueness follows from Lemma 5.1. Let us prove the existence. The idea is to move the E_i's to the right of the F_j's by using (22) and (23) (this creates a sum of a product of q-binomials) and then use (17), which creates a product of δ-functions. Besides, we insert Heaviside functions to make the sum terminating. The result is an elementary function. Now we give the details of the proof.

Let $l \le k$ be the maximal index such that $X_l \in \{E_1, \dots, E_{2m-1}\}$. We use induction on k and then induction on l. If $k = 0$, then the statement is obvious.

For fixed k, we use induction on l, beginning with $l = k$ and going down.

(a) Suppose that $l = k$. Recall that $\vartheta = (n^m, 0^m)$. Using (17), we have

$$X^{(b)} 1_\vartheta = \delta_{b_k, 0} Y^{(b')} 1_\vartheta,$$

where $Y = (X_1, \dots, X_{k-1})$ and $b' = (b_1, \dots, b_{k-1})$. For Y the statement holds. Define

$$F_X(b, j) := F_Y(b', j)\delta_{b_k, 0}, \qquad Q_X(b) = \sum_{j \in \mathbb{Z}^l} F_X(b, j).$$

Then $F_X(b, j)$ is an elementary summand. Both statements (i) and (ii) for $F_X(b, j)$ follow immediately from those for $F_Y(b', j)$. Then Q_X is an elementary q-holonomic function, and (50) holds.

(b) Suppose that $l < k$. Assume that $X_l = E_r$ and $X_{l+1} = F_s$. Let $Y = (Y_1, \dots, Y_k)$ be the sequence defined by $Y_i = X_i$ for all i's except for those such that $Y_l = X_{l+1}$ and $Y_{l+1} = X_l$. By induction, the statement holds for Y, and we can define an elementary summand $F_Y(b, j)$ for $(b, j) \in \mathbb{Z}^{k+l}$. Consider two cases.

Case 1: $r \neq s$. Because $E_r F_s = F_s E_r$, we have $X^{(b)} = Y^{(b')}$ where b' is obtained from b by swapping the lth and $(l + 1)$th components. This case is reduced to the case of Y by defining $F_X(b, j) = F_Y(b', j)$.

Case 2: $r = s$. We have

$$X^{(b)} = X_{\text{left}}(E_r^{(b_l)} F_r^{(b_{l+1})}) X_{\text{right}},$$

where

$$X_{\text{left}} = \overrightarrow{\prod_{j\in[1,l-1]}} X_j^{(b_j)}, \qquad X_{\text{right}} = \overrightarrow{\prod_{j\in[l+2,k]}} X_j^{(b_j)}.$$

We have $X_{\text{right}} 1_\vartheta \in \mathrm{Hom}_{n\mathbf{Lad}_{2m}}(\vartheta, \mu)$, where

$$\mu = \vartheta + \mathrm{w}(X_{\text{right}}) = \vartheta - \sum_{j=l+2}^{k} b_j \alpha_{i_j}.$$

Here the index i_j is defined so that $X_j = F_{i_j}$ for $j > l$. Using (14a), we have

$$\begin{aligned} X^{(b)} 1_\vartheta &= \sum_{t\in\mathbb{Z}} \begin{bmatrix} \langle \mu, \alpha_r \rangle + b_l - b_{l+1} \\ t \end{bmatrix} X_{\text{left}} F_r^{(b_{l+1}-t)} E_r^{(b_l-t)} X_{\text{right}} 1_\vartheta \\ &= \sum_{t\in\mathbb{Z}} \begin{bmatrix} \langle \mu, \alpha_r \rangle + b_l - b_{l+1} \\ t \end{bmatrix} Y^{(b')} 1_\vartheta, \end{aligned} \tag{52}$$

where $b' = (b'_1, \dots, b'_k)$ such that $b'_i = b_i$ for all i's except for $i = l, l + 1$, with $b'_l = b_{l+1} - t, b'_{l+1} = b_l - t$. Clearly b' is a linear function of (b, t).

Note that $\langle \vartheta, \alpha_r \rangle = n\delta(r, m)$. From the definition of μ,

$$\langle \mu, \alpha_r \rangle + b_l - b_{l+1} = n\delta(r, m) + \mathrm{Lin}(b),$$

where $\mathrm{Lin}(b) = \langle \mathrm{w}(X_{\text{right}}), \alpha_r \rangle + b_l - b_{l+1}$ is a $\mathbb{Z}$-linear form of b. For $j \in \mathbb{Z}^{l+1}$, we write $j = (j', t)$; that is, t is the last component of j. For $b \in \mathbb{Z}^k$ and $j \in \mathbb{Z}^{l+1}$, define

$$F_X(b, j) = \begin{cases} \left[\begin{smallmatrix} x;\mathrm{Lin}(b) \\ t \end{smallmatrix}\right] F_Y(b', j') \mathcal{H}(X, b) & \text{if } r = m, \\ \left[\begin{smallmatrix} \mathrm{Lin}(b) \\ t \end{smallmatrix}\right] F_Y(b', j') \mathcal{H}(X, b) & \text{if } r \neq m, \end{cases} \tag{53}$$

where $\mathcal{H}(X,b)$, defined by (26), is an elementary function of b.

Then $F_X(b,j)$ is an elementary function. Let us prove (i) and (ii), which claim $F_X(b,j)=0$ under certain conditions. If $t<0$, then the first factor on the right-hand side of (53) is 0. Hence, we will assume $t\geq 0$ in what follows.

(i) Suppose that $\|j\|_\infty > \|b\|_\infty$. Then either $\|j'\|_\infty > \|b\|_\infty$ or $|t| > \|b\|_\infty$. In the first case, $\|j'\|_\infty > \|b\|_\infty \geq \|b'\|_\infty$, and $F_Y(b',j')=0$. In the latter case, the lth component of b' is negative. By (ii) we have $F_Y(b',j')=0$. Hence, $F_Y(b,j)=0$.

(ii) First assume that one of the components of b is negative. Then one of the components of b' is negative. Hence, $F_Y(b',j')=0$, implying $F_X(b,j)=0$.

Now assume that (X,b) is tail-negative. Then the third factor on the right-hand side of (53) is 0. Hence, $F_X(b,j)=0$.

Let us prove (50). If (X,b) is tail-negative, then both sides of (50) are 0, by Lemma 3.1 and the property of $\mathcal{H}(X,b)$. Assume now that (X,b) is not tail-negative. Then $\mathcal{H}(X,b)=1$, and (50) follows from (52), (53), and the identity (50) applicable to Y. This completes the proof of the proposition. □

5.2. *Coloring with partitions with one column*

THEOREM 5.3

Suppose L is an oriented, framed link with r ordered components. There exists a unique function

$$Q_L : \mathbb{N}^r \to \mathbb{Q}(q)[x^{\pm 1}]$$

such that, for any integer $n\geq 2$ and $a=(a_1,\dots,a_r)\in\mathbb{N}^r\cap[0,n-1]^r$,

$$\tilde{J}_L^{\mathfrak{sl}_n}(e_{a_1},\dots,e_{a_r}) = \mathrm{eval}_n\big(Q_L(a)\big). \tag{54}$$

Moreover, Q_L is elementary and, hence, a q-holonomic function.

Proof

The uniqueness follows from Lemma 5.1. Let us prove the existence. Suppose that L is the closure of a braid β on m strands, as in Section 3.5. By Proposition 3.3, there exists a sequence $X=(X_1,\dots,X_k)$ of elements from E_i, F_i with $i=1,\dots,2m-1$ and linear functions $g_1,g_2:\mathbb{Z}^{r+t}\to\mathbb{Z}$ and $f:\mathbb{Z}^{r+t}\to\mathbb{Z}^k$ such that

$$\tilde{J}_L^{\mathfrak{sl}_n}(e_{a_1},\dots,e_{a_r}) = \sum_{s\in\mathbb{Z}^t}(-1)^{g_1(a,s)}q^{g_2(a,s)}\mathrm{ev}_n(X^{(f(a,s))}).$$

By Proposition 5.2, there exists an elementary summand function $F_X:\mathbb{Z}^{k+l}\to\mathbb{Q}(q)[x^{\pm 1}]$ such that

$$\tilde{J}_L^{\mathfrak{sl}_n}(e_{a_1},\dots,e_{a_r}) = \sum_{s\in\mathbb{Z}^t}(-1)^{g_1(a,s)}q^{g_2(a,s)}\mathrm{eval}_n\Big(\sum_{j\in\mathbb{Z}^l} F_X\big(f(a,s),j\big)\Big).$$

By Proposition 5.2(i),

$$F_X\big(f(a,s),j\big) = 0 \quad \text{if } \|j\|_\infty > \big\|f(a,s)\big\|_\infty. \tag{55}$$

When $\|s\|_\infty > \|a\|_\infty$, $(X, f(a,s))$ is tail-negative (see Proposition 3.3). Hence, by Proposition 5.2(ii),

$$F_X\big(f(a,s),j\big) = 0 \quad \text{if } \|s\|_\infty > \|a\|_\infty. \tag{56}$$

Then (55) and (56) imply that the sum

$$Q_L(a) := \sum_{s\in\mathbb{Z}^t}\sum_{j\in\mathbb{Z}^l}(-1)^{g_1(a,s)}q^{g_2(a,s)}F\big(f(a,s),j\big)$$

is terminating for each $a\in\mathbb{Z}^r$. Then Q_L is elementary q-holonomic, and (54) holds. □

Remark 5.4
By our construction, Q_L vanishes in $\mathbb{Z}^r\setminus\mathbb{N}^r$.

Remark 5.5
Theorem 5.3 gives an alternative construction of the colored HOMFLYPT polynomial W_L of a framed, oriented link colored by partitions with one column. By the uniqueness,

$$Q_L(a_1,\dots,a_r) = W_L(e_{a_1},\dots,e_{a_r}).$$

5.3. *The Jacobi–Trudi formula*

In this section we explain how to extend the q-holonomicity of the HOMFLYPT polynomial of a link colored by partitions with one row to the case of partitions with a fixed number of rows. The key idea is the Jacobi–Trudi formula, which expresses the Schur function s_λ of a partition $\lambda\in\mathcal{P}_\ell$, considered as an element of the algebra Λ, as a determinant of a matrix whose entries are partitions with one row. Observe that for partitions with one row (resp., one column) we have $s_{(a)} = h_a$ (resp., $s_{(1^a)} = e_a$).

The Jacobi–Trudi formula (see [39]) states that if $\lambda = (\lambda_1,\dots,\lambda_\ell)\in\mathcal{P}_\ell$, then in Λ,

$$s_\lambda = \det\big((e_{\lambda_i^\dagger+j-i})_{i,j=1}^{\ell}\big),$$

where the right-hand side is an $(\ell \times \ell)$-determinant, with the convention $e_0 = 1$ and $e_n = 0$ for $n < 0$. For example, if λ is a partition with three rows with λ_1, λ_2, and λ_3 boxes, then we have

$$s_{\lambda_1,\lambda_2,\lambda_3} = -e_{\lambda_1+2}e_{\lambda_2}e_{\lambda_3-2} + e_{\lambda_1+1}e_{\lambda_2+1}e_{\lambda_3-2} + e_{\lambda_1+2}e_{\lambda_2-1}e_{\lambda_3-1} - e_{\lambda_1}e_{\lambda_2+1}e_{\lambda_3-1} - e_{\lambda_1+1}e_{\lambda_2-1}e_{\lambda_3} + h_{\lambda_1}e_{\lambda_2}e_{\lambda_3}.$$

Let L denote a framed, oriented link L with r ordered components, and choose a partition $\lambda \in \mathcal{P}_\ell$ and partitions $\mu_2, \dots, \mu_r$. Then, Proposition 2.4(c) implies that

$$W_L(\lambda, \mu_1, \dots, \mu_r) = \sum_{\sigma \in \mathrm{Sym}_\ell} \mathrm{sgn}(\sigma) W_{L'}(e_{\lambda_1+\sigma(1)-1}, \dots, e_{\lambda_\ell+\sigma(\ell)-\ell}, \mu_1, \dots, \mu_r), \tag{57}$$

where L' is the link obtained from L by replacing the first framed component of L by ℓ of its parallels.

5.4. Proof of Theorem 1.1

Fix a framed, oriented link L with r ordered components. By using the symmetry of the HOMFLYPT polynomial from Proposition 2.4(c), it suffices to show that the colored HOMFLYPT polynomial of L, colored by partitions with at most ℓ rows, is q-holonomic. Said differently, it suffices to show that the function $W_L \circ (\iota_\ell^\dagger)^r : \mathbb{N}^{r\ell} \to \mathbb{Q}(q)[x^{\pm 1}]$ is q-holonomic. Let $\lambda = (\lambda_1, \dots, \lambda_{r\ell}) \in \mathbb{N}^{r\ell}$. Using (57), we have

$$\big(W_L \circ (\iota_\ell^\dagger)^r\big)(\lambda) = \sum_\sigma \mathrm{sgn}(\sigma) W_{\Delta L}(e_{f_{\sigma,1}(\lambda)}, \dots, e_{f_{\sigma,r\ell}(\lambda)}),$$

where the sum is over $\sigma = (\sigma_1, \dots, \sigma_r) \in (\mathrm{Sym}_\ell)^r$, $\mathrm{sgn}(\sigma) = \mathrm{sgn}(\sigma_1)\cdots\mathrm{sgn}(\sigma_r)$, ΔL is the link with $r\ell$ components obtained from L by replacing each component with its ℓth parallel, and $f_{\sigma,i} : \mathbb{Z}^{r\ell} \to \mathbb{Z}$ are affine. Theorem 4.7(a) and 4.7(e) together with Theorem 5.3 imply that $W_L \circ (\iota_\ell^\dagger)^r$ is a sum of q-holonomic functions and, thus, is q-holonomic. This concludes the proof of Theorem 1.1. □

Appendices

A. The formula for the invariant of the trefoil

In this section we give the omitted details of how (38) implies (39). We start with (38) and observe that

$$\sum_{s_1,s_2,s_3 \in \mathbb{Z}} (-q)^{-(s_1+s_2+s_3)} E_2^{(a)} E_1^{(a)} E_3^{(a)} E_2^{(a)} E_3^{(s_1)} F_3^{(s_1)} E_3^{(s_2)} F_3^{(s_2)} E_3^{(s_3)} F_3^{(s_3)} \times F_2^{(a)} F_3^{(a)} F_1^{(a)} F_2^{(a)} 1_{(n,n,0,0)}$$

$$= \sum_{s_1,s_2,s_3\in\mathbb{Z}} (-q)^{-(s_1+s_2+s_3)} E_2^{(a)} E_1^{(a)} E_3^{(a)} E_2^{(a)} (E_3^{(s_1)} F_3^{(s_1)} E_3^{(s_2)} F_3^{(s_2)} E_3^{(s_3)} \times F_3^{(s_3)} 1_{(n-a,n-a,a,a)}) F_2^{(a)} F_3^{(a)} F_1^{(a)} F_2^{(a)} 1_{(n,n,0,0)}, \tag{58}$$

where we used (13) to include the idempotent in the middle term (and the fact that $(n,n,0,0) - a\alpha_1 - 2a\alpha_2 - a\alpha_3 = (n-a,n-a,a,a)$). The term in parentheses can be simplified as follows:

$$\begin{aligned}
&E_3^{(s_1)} F_3^{(s_1)} E_3^{(s_2)} F_3^{(s_2)} (E_3^{(s_3)} F_3^{(s_3)} 1_{(n-a,n-a,a,a)}) \\
&\overset{(14a)}{=} E_3^{(s_1)} F_3^{(s_1)} F_3^{(s_2)} E_3^{(s_2)} F_3^{(s_3)} E_3^{(s_3)} 1_{(n-a,n-a,a,a)} \\
&\overset{(13)}{=} (F_3^{(s_1)} E_3^{(s_1)} 1_{(n-a,n-a,a,a)}) F_3^{(s_2)} E_3^{(s_2)} F_3^{(s_3)} E_3^{(s_3)} \\
&\overset{(14a)}{=} F_3^{(s_1)} E_3^{(s_1)} F_3^{(s_2)} E_3^{(s_2)} F_3^{(s_3)} E_3^{(s_3)} 1_{(n-a,n-a,a,a)} \\
&\overset{(13)}{=} F_3^{(s_1)} 1_{(n-a,n-a,a+s_1,a-s_1)} E_3^{(s_1)} F_3^{(s_2)} 1_{(n-a,n-a,a+s_2,a-s_2)} E_3^{(s_2)} \\
&\quad \times F_3^{(s_3)} 1_{(n-a,n-a,a+s_3,a-s_3)} E_3^{(s_3)} \\
&\overset{(3.1)}{=} \mathrm{He}(a-s_1)\mathrm{He}(a-s_2)\mathrm{He}(a-s_3) F_3^{(s_1)} (E_3^{(s_1)} F_3^{(s_2)} 1_{(n-a,n-a,a+s_2,a-s_2)}) \\
&\quad \times (E_3^{(s_2)} F_3^{(s_3)} 1_{(n-a,n-a,a+s_3,a-s_3)}) E_3^{(s_3)} \\
&\overset{(14a)}{=} \mathrm{He}(a-s_1)\mathrm{He}(a-s_2)\mathrm{He}(a-s_3) \sum_{s_4,s_5} \begin{bmatrix} s_2+s_1 \\ s_4 \end{bmatrix} \begin{bmatrix} s_2+s_3 \\ s_5 \end{bmatrix} \\
&\quad \times F_3^{(s_1)} (F_3^{(s_2-s_4)} E_3^{(s_1-s_4)} 1_{(n-a,n-a,a+s_2,a-s_2)}) \\
&\quad \times (F_3^{(s_3-s_5)} E_3^{(s_2-s_5)} 1_{(n-a,n-a,a+s_3,a-s_3)}) E_3^{(s_3)} 1_{(n-a,n-a,a,a)} \\
&\overset{(13)}{=} \mathrm{He}(a-s_1)\mathrm{He}(a-s_2)\mathrm{He}(a-s_3) \sum_{s_4,s_5} \begin{bmatrix} s_2+s_1 \\ s_4 \end{bmatrix} \begin{bmatrix} s_2+s_3 \\ s_5 \end{bmatrix} F_3^{(s_1)} \\
&\quad \times F_3^{(s_2-s_4)} 1_{(n-a,n-a,a+s_1+s_2-s_4,a-s_1-s_2+s_4)} \\
&\quad \times (E_3^{(s_1-s_4)} F_3^{(s_3-s_5)} 1_{(n-a,n-a,a+s_2+s_3-s_5,a-s_2-s_3+s_5)}) \\
&\quad \times E_3^{(s_2-s_5)} E_3^{(s_3)} 1_{(n-a,n-a,a,a)} \\
&\overset{(3.1)}{=} \mathrm{He}(a-s_1)\mathrm{He}(a-s_2)\mathrm{He}(a-s_3)\mathrm{He}(a+s_1+s_2-s_4)\mathrm{He}(a+s_2+s_3-s_5) \\
&\quad \times \sum_{s_4,s_5} \begin{bmatrix} s_2+s_1 \\ s_4 \end{bmatrix} \begin{bmatrix} s_2+s_3 \\ s_5 \end{bmatrix} F_3^{(s_1)} F_3^{(s_2-s_4)} (E_3^{(s_1-s_4)}
\end{aligned}$$

$$\times F_3^{(s_3-s_5)} 1_{(n-a,n-a,a+s_3+s_2-s_5,a-s_3-s_2+s_5)})$$
$$\times E_3^{(s_2-s_5)} E_3^{(s_3)} 1_{(n-a,n-a,a,a)}$$
$$\overset{(14a)}{=} \mathrm{He}(a-s_1)\mathrm{He}(a-s_2)\mathrm{He}(a-s_3)\mathrm{He}(a+s_1+s_2-s_4)\mathrm{He}(a+s_2+s_3-s_5)$$
$$\times \sum_{s_4,s_5,s_6} \begin{bmatrix} s_2+s_1 \\ s_4 \end{bmatrix} \begin{bmatrix} s_2+s_3 \\ s_5 \end{bmatrix} \begin{bmatrix} s_1+2s_2+s_3-s_4-s_5 \\ s_6 \end{bmatrix} F_3^{(s_1)}$$
$$\times F_3^{(s_2-s_4)} (F_3^{(s_3-s_5-s_6)} E_3^{(s_1-s_4-s_6)}) E_3^{(s_2-s_5)} E_3^{(s_3)} 1_{(n-a,n-a,a,a)}$$
$$\overset{(3.1)}{=} \mathrm{He}(a-s_1)\mathrm{He}(a-s_2)\mathrm{He}(a-s_3)\mathrm{He}(a+s_1+s_2-s_4)$$
$$\times \mathrm{He}(a+s_2+s_3-s_5)\mathrm{He}(a+s_1+s_2+s_3-s_4-s_5-s_6)$$
$$\times \sum_{s_4,s_5,s_6} \begin{bmatrix} s_2+s_1 \\ s_4 \end{bmatrix} \begin{bmatrix} s_2+s_3 \\ s_5 \end{bmatrix} \begin{bmatrix} s_1+2s_2+s_3-s_4-s_5 \\ s_6 \end{bmatrix}$$
$$\times (F_3^{(s_1)} F_3^{(s_2-s_4)} F_3^{(s_3-s_5-s_6)})(E_3^{(s_1-s_4-s_6)} E_3^{(s_2-s_5)} E_3^{(s_3)}) 1_{(n-a,n-a,a,a)}$$
$$\overset{(14d)}{=} \mathrm{He}(a-s_1)\mathrm{He}(a-s_2)\mathrm{He}(a-s_3)\mathrm{He}(a+s_1+s_2-s_4)$$
$$\times \mathrm{He}(a+s_2+s_3-s_5)\mathrm{He}(a+s_1+s_2+s_3-s_4-s_5-s_6)$$
$$\times \sum_{s_4,s_5,s_6} \begin{bmatrix} s_2+s_1 \\ s_4 \end{bmatrix} \begin{bmatrix} s_2+s_3 \\ s_5 \end{bmatrix} \begin{bmatrix} s_1+2s_2+s_3-s_4-s_5 \\ s_6 \end{bmatrix}$$
$$\times \begin{bmatrix} s_1+s_2-s_4 \\ s_1 \end{bmatrix} \begin{bmatrix} s_1+s_2+s_3-s_4-s_5-s_6 \\ s_1+s_2-s_4 \end{bmatrix}$$
$$\times \begin{bmatrix} s_2+s_3-s_5 \\ s_3 \end{bmatrix} \begin{bmatrix} s_1+s_2+s_3-s_4-s_5-s_6 \\ s_2+s_3-s_5 \end{bmatrix}$$
$$\times F_3^{(s_1+s_2+s_3-s_4-s_5-s_6)} E_3^{(s_1+s_2+s_3-s_4-s_5-s_6)} 1_{(n-a,n-a,a,a)}.$$

Then to complete the computation of $\Upsilon_{\mathcal{E}}(a)1_{\vartheta}$ from (58), set $\tau = s_1+s_2+s_3-s_4-s_5-s_6$ for simplicity, and use the above computation to simplify each term in the sum from (58)

$$1_{(n,n,0,0)} E_2^{(a)} E_1^{(a)} E_3^{(a)} E_2^{(a)} F_3^{(s_1+s_2+s_3-s_4-s_5-s_6)} E_3^{(s_1+s_2+s_3-s_4-s_5-s_6)}$$
$$\times F_2^{(a)} F_3^{(a)} F_1^{(a)} F_2^{(a)} 1_{(n,n,0,0)}$$
$$= E_2^{(a)} E_1^{(a)} E_3^{(a)} (E_2^{(a)} F_3^{(\tau)})(E_3^{(\tau)} F_2^{(a)}) F_3^{(a)} F_1^{(a)} F_2^{(a)} 1_{(n,n,0,0)}$$
$$\overset{(14b)}{=} E_2^{(a)} E_1^{(a)} E_3^{(a)} (F_3^{(\tau)} E_2^{(a)})(F_2^{(a)} E_3^{(\tau)}) F_3^{(a)} F_1^{(a)} F_2^{(a)} 1_{(n,n,0,0)}$$

$$\overset{(13)}{=} E_2^{(a)} E_1^{(a)} (E_3^{(a)} F_3^{(\tau)} 1_{(n-a,n,\tau,a-\tau)}) E_2^{(a)} F_2^{(a)} (E_3^{(\tau)} F_3^{(a)} 1_{(n-a,n,a,0)}) F_1^{(a)} F_2^{(a)}$$

$$\overset{(14a)}{=} \sum_{p_1,p_2} \begin{bmatrix} \tau \\ p_2 \end{bmatrix} \begin{bmatrix} \tau \\ p_1 \end{bmatrix} E_2^{(a)} E_1^{(a)} (F_3^{(\tau-p_2)} E_3^{(a-p_2)} 1_{(n-a,n,\tau,a-\tau)})$$

$$\times E_2^{(a)} F_2^{(a)} (F_3^{(a-p_1)} E_3^{(\tau-p_1)} 1_{(n-a,n,a,0)}) F_1^{(a)} F_2^{(a)}$$

$$= \sum_{p_1,p_2} \begin{bmatrix} \tau \\ p_2 \end{bmatrix} \begin{bmatrix} \tau \\ p_1 \end{bmatrix} 1_\vartheta (E_2^{(a)} E_1^{(a)} F_3^{(\tau-p_2)}) E_3^{(a-p_2)} E_2^{(a)} F_2^{(a)} F_3^{(a-p_1)}$$

$$\times (E_3^{(\tau-p_1)} F_1^{(a)} F_2^{(a)}) 1_\vartheta$$

$$\overset{(14b)}{=} \sum_{p_1,p_2} \begin{bmatrix} \tau \\ p_2 \end{bmatrix} \begin{bmatrix} \tau \\ p_1 \end{bmatrix} 1_\vartheta (F_3^{(\tau-p_2)} E_2^{(a)} E_1^{(a)}) E_3^{(a-p_2)} E_2^{(a)} F_2^{(a)} F_3^{(a-p_1)}$$

$$\times (F_1^{(a)} F_2^{(a)} E_3^{(\tau-p_1)}) 1_\vartheta$$

$$= \sum_{p_1,p_2} \begin{bmatrix} \tau \\ p_2 \end{bmatrix} \begin{bmatrix} \tau \\ p_1 \end{bmatrix} (1_\vartheta F_3^{(\tau-p_2)}) E_2^{(a)} E_1^{(a)} E_3^{(a-p_2)} E_2^{(a)} F_2^{(a)} F_3^{(a-p_1)} F_1^{(a)} F_2^{(a)}$$

$$\times (E_3^{(\tau-p_1)} 1_\vartheta)$$

$$\overset{(17)}{=} \sum_{p_1,p_2} \begin{bmatrix} \tau \\ p_2 \end{bmatrix} \begin{bmatrix} \tau \\ p_1 \end{bmatrix} (1_\vartheta \delta_{\tau,p_2}) E_2^{(a)} E_1^{(a)} E_3^{(a-p_2)} E_2^{(a)} F_2^{(a)} F_3^{(a-p_1)} F_1^{(a)}$$

$$\times F_2^{(a)} (\delta_{\tau,p_1} 1_\vartheta)$$

$$\overset{(13)}{=} E_2^{(a)} E_1^{(a)} E_3^{(a-\tau)} (E_2^{(a)} F_2^{(a)} 1_{\maltese(n-a,n,\tau,a-\tau)}) F_3^{(a-\tau)} F_1^{(a)} F_2^{(a)} 1_\vartheta$$

$$\overset{(14a)}{=} \sum_{s_7} \begin{bmatrix} n-\tau \\ s_7 \end{bmatrix} E_2^{(a)} E_1^{(a)} E_3^{(a-\tau)} (F_2^{(a-s_7)} E_2^{(a-s_7)} 1_{(n-a,n,\tau,a-\tau)})$$

$$\times F_3^{(a-\tau)} F_1^{(a)} F_2^{(a)} 1_\vartheta$$

$$= \sum_{s_7} \begin{bmatrix} n-\tau \\ s_7 \end{bmatrix} E_2^{(a)} (E_1^{(a)} E_3^{(a-\tau)} F_2^{(a-s_7)}) (E_2^{(a-s_7)} F_3^{(a-\tau)} F_1^{(a)}) F_2^{(a)} 1_\vartheta$$

$$\overset{(14b)}{=} \sum_{s_7} \begin{bmatrix} n-\tau \\ s_7 \end{bmatrix} E_2^{(a)} (F_2^{(a-s_7)} E_1^{(a)} E_3^{(a-\tau)}) (F_3^{(a-\tau)}$$

$$\times F_1^{(a)} E_2^{(a-s_7)}) F_2^{(a)} 1_\vartheta$$

$$\overset{(13)}{=}\sum_{s_7}\begin{bmatrix} n-\tau \\ s_7 \end{bmatrix}(E_2^{(a)}F_2^{(a-s_7)}1_{(n,n-s_7,s_7,0)})E_1^{(a)}E_3^{(a-\tau)}$$

$$\times F_3^{(a-\tau)}F_1^{(a)}(E_2^{(a-s_7)}F_2^{(a)}1_\vartheta)$$

$$\overset{(14a)}{=}\sum_{s_7,v_1,v_2}\begin{bmatrix} n-\tau \\ s_7 \end{bmatrix}\begin{bmatrix} n-s_7 \\ v_1 \end{bmatrix}\begin{bmatrix} n-s_7 \\ v_2 \end{bmatrix}(1_\vartheta F_2^{(a-s_7-v_2)}E_2^{(a-v_2)})E_1^{(a)}E_3^{(a-\tau)}$$

$$\times F_3^{(a-\tau)}F_1^{(a)}(F_2^{(a-v_1)}E_2^{(a-s_7-v_1)}1_\vartheta)$$

$$=\sum_{s_7,v_1,v_2}\begin{bmatrix} n-\tau \\ s_7 \end{bmatrix}\begin{bmatrix} n-s_7 \\ v_1 \end{bmatrix}\begin{bmatrix} n-s_7 \\ v_2 \end{bmatrix}(1_\vartheta F_2^{(a-s_7-v_2)})E_2^{(a-v_2)}E_1^{(a)}E_3^{(a-\tau)}F_3^{(a-\tau)}$$

$$\times F_1^{(a)}F_2^{(a-v_1)}(E_2^{(a-s_7-v_1)}1_\vartheta)$$

$$\overset{(17)}{=}\sum_{s_7,v_1,v_2}\begin{bmatrix} n-\tau \\ s_7 \end{bmatrix}\begin{bmatrix} n-s_7 \\ v_1 \end{bmatrix}\begin{bmatrix} n-s_7 \\ v_2 \end{bmatrix}(1_\vartheta\delta_{v_2,a-s_7})E_2^{(a-v_2)}E_1^{(a)}E_3^{(a-\tau)}F_3^{(a-\tau)}$$

$$\times F_1^{(a)}F_2^{(a-v_1)}(\delta_{v_1,a-s_7}1_\vartheta)$$

$$=\sum_{s_7}\begin{bmatrix} n-\tau \\ s_7 \end{bmatrix}\begin{bmatrix} n-s_7 \\ a-s_7 \end{bmatrix}\begin{bmatrix} n-s_7 \\ a-s_7 \end{bmatrix}E_2^{(s_7)}E_1^{(a)}E_3^{(a-\tau)}F_3^{(a-\tau)}F_1^{(a)}F_2^{(s_7)}1_\vartheta$$

$$\overset{(13)}{=}\sum_{s_7}\begin{bmatrix} n-\tau \\ s_7 \end{bmatrix}\begin{bmatrix} n-s_7 \\ a-s_7 \end{bmatrix}\begin{bmatrix} n-s_7 \\ a-s_7 \end{bmatrix}E_2^{(s_7)}E_1^{(a)}(E_3^{(a-\tau)}F_3^{(a-\tau)}1_{(n-a,n+a-s_7,s_7,0)})$$

$$\times F_1^{(a)}F_2^{(s_7)}1_\vartheta$$

$$\overset{(14a)}{=}\sum_{s_7,v_3}\begin{bmatrix} n-\tau \\ s_7 \end{bmatrix}\begin{bmatrix} n-s_7 \\ a-s_7 \end{bmatrix}\begin{bmatrix} n-s_7 \\ a-s_7 \end{bmatrix}\begin{bmatrix} s_7 \\ v_3 \end{bmatrix}E_2^{(s_7)}E_1^{(a)}$$

$$\times (F_3^{(a-\tau-v_3)}E_3^{(a-\tau-v_3)}1_{(n-a,n+a-s_7,s_7,0)})F_1^{(a)}F_2^{(s_7)}1_\vartheta$$

$$\overset{(14b)}{=}\sum_{s_7,v_3}\begin{bmatrix} n-\tau \\ s_7 \end{bmatrix}\begin{bmatrix} n-s_7 \\ a-s_7 \end{bmatrix}^2\begin{bmatrix} s_7 \\ v_3 \end{bmatrix}(1_\vartheta F_3^{(a-\tau-v_3)})E_2^{(s_7)}E_1^{(a)}F_1^{(a)}F_2^{(s_7)}$$

$$\times (E_3^{(a-\tau-v_3)}1_\vartheta)$$

$$\overset{(17)}{=}\sum_{s_7,v_3}\begin{bmatrix} n-\tau \\ s_7 \end{bmatrix}\begin{bmatrix} n-s_7 \\ a-s_7 \end{bmatrix}^2\begin{bmatrix} s_7 \\ v_3 \end{bmatrix}(1_\vartheta\delta_{v_3,a-\tau})E_2^{(s_7)}E_1^{(a)}F_1^{(a)}F_2^{(s_7)}(\delta_{v_3,a-\tau}1_\vartheta)$$

$$
\begin{aligned}
&\overset{(13)}{=} \sum_{s_7} \begin{bmatrix} n-\tau \\ s_7 \end{bmatrix} \begin{bmatrix} n-s_7 \\ a-s_7 \end{bmatrix}^2 \begin{bmatrix} s_7 \\ a-\tau \end{bmatrix} E_2^{(s_7)} (E_1^{(a)} F_1^{(a)} 1_{(n,n-s_7,s_7,0)}) F_2^{(s_7)} \\
&\overset{(14a)}{=} \sum_{s_7,v_4} \begin{bmatrix} n-\tau \\ s_7 \end{bmatrix} \begin{bmatrix} n-s_7 \\ a-s_7 \end{bmatrix}^2 \begin{bmatrix} s_7 \\ a-\tau \end{bmatrix} \begin{bmatrix} s_7 \\ v_4 \end{bmatrix} E_2^{(s_7)} \\
&\quad \times (F_1^{(a-v_4)} E_1^{(a-v_4)} 1_{(n,n-s_7,s_7,0)}) F_2^{(s_7)} 1_\vartheta \\
&\overset{(14b)}{=} \sum_{s_7,v_4} \begin{bmatrix} n-\tau \\ s_7 \end{bmatrix} \begin{bmatrix} n-s_7 \\ a-s_7 \end{bmatrix}^2 \begin{bmatrix} s_7 \\ a-\tau \end{bmatrix} \begin{bmatrix} s_7 \\ v_4 \end{bmatrix} (1_\vartheta F_1^{(a-v_4)}) E_2^{(s_7)} F_2^{(s_7)} (E_1^{(a-v_4)} 1_\vartheta) \\
&\overset{(17)}{=} \sum_{s_7} \begin{bmatrix} n-\tau \\ s_7 \end{bmatrix} \begin{bmatrix} n-s_7 \\ a-s_7 \end{bmatrix}^2 \begin{bmatrix} s_7 \\ a-\tau \end{bmatrix} \maltese \begin{bmatrix} s_7 \\ a \end{bmatrix} 1_\vartheta E_2^{(s_7)} F_2^{(s_7)} 1_\vartheta . \qquad (59)
\end{aligned}
$$

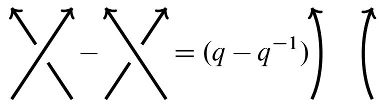
Tracing through this computation we have placed the symbol ✠ to indicate places where we must introduce Heaviside functions, so the end result should be multiplied by $\mathrm{He}(\tau)$. The Heaviside functions $\mathrm{He}(a-\tau)\mathrm{He}(a-s_7)\mathrm{He}(s_7-a)$ are implied by the definition of quantum binomial coefficients from (15c). Thus, $s_7 = a$ and the sum simplifies to

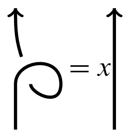
$$
\begin{aligned}
&\mathrm{He}(\tau) \begin{bmatrix} n-\tau \\ a \end{bmatrix} \begin{bmatrix} n-a \\ 0 \end{bmatrix} \begin{bmatrix} n-a \\ 0 \end{bmatrix} \begin{bmatrix} a \\ a-\tau \end{bmatrix} \begin{bmatrix} a \\ a \end{bmatrix} 1_{(n,n,0,0)} E_2^{(a)} F_2^{(a)} 1_{(n,n,0,0)} \\
&\quad = \mathrm{He}(\tau) \begin{bmatrix} n-\tau \\ a \end{bmatrix} \begin{bmatrix} a \\ a-\tau \end{bmatrix} 1_{(n,n,0,0)} E_2^{(a)} F_2^{(a)} 1_{(n,n,0,0)} \\
&\quad \overset{(14a)}{=} \mathrm{He}(\tau) \sum_{v_6} \begin{bmatrix} n-\tau \\ a \end{bmatrix} \begin{bmatrix} a \\ a-\tau \end{bmatrix} \begin{bmatrix} n \\ v_6 \end{bmatrix} 1_{(n,n,0,0)} F_2^{(a-v_6)} E_2^{(a-v_6)} 1_{(n,n,0,0)} \\
&\quad \overset{(17)}{=} \mathrm{He}(\tau) \begin{bmatrix} n-\tau \\ a \end{bmatrix} \begin{bmatrix} a \\ a-\tau \end{bmatrix} \begin{bmatrix} n \\ a \end{bmatrix} 1_\vartheta \\
&\quad \overset{(17)}{=} \mathrm{He}(\tau) \begin{bmatrix} n-\tau \\ a \end{bmatrix} \begin{bmatrix} n \\ a-\tau \end{bmatrix} 1_\vartheta . \qquad (60)
\end{aligned}
$$

By putting it all together, the a-colored trefoil evaluates to (39).

B. Proof of Propositions 2.4(b) and 2.4(c)

For a compact oriented surface (possibly with boundary) Σ, let $\mathcal{S}(\Sigma)$ be the HOMFLYPT skein algebra of Σ, as defined in [3] and [42]. Recall that, as a $\mathbb{Q}(x,q)$-module, $\mathcal{S}(\Sigma)$ is generated by oriented links diagrams on Σ modulo the regular isotopy, the two relations





and the relation that a disjoint trivial knot can be removed from a diagram at the expense of multiplication with $\frac{x-x^{-1}}{q-q^{-1}}$. The product $L_1 L_2$ of two links diagrams is obtained by placing L_1 atop L_2. When Σ is a disk, $\mathcal{S}(\Sigma) \cong \mathbb{Q}(q,x)$ via a map $L \to \langle L \rangle$, where $\langle L \rangle$ is a framed version of the HOMFLYPT polynomial.

The HOMFLYPT skein algebra of the annulus contains the subalgebra $\mathcal{C}^+$ generated by the closure of all braids. It is known that $\mathcal{C}^+$ is isomorphic to the algebra of symmetric functions (with ground ring $\mathbb{Q}(q,x)$). Under this isomorphism, the Schur function s_λ of a partition λ corresponds to a certain skein element Q_λ, which will be recalled later. The relation with the colored HOMFLYPT polynomial is as follows. For an oriented link diagram L on the disk with r ordered components and for partitions λ_i for $i = 1, \ldots, r$, we have

$$W_L(\lambda_1, \ldots, \lambda_r) = \big\langle L * (Q_{\lambda_1}, \ldots, Q_{\lambda_r}) \big\rangle. \tag{61}$$

Here, $L * (Q_{\lambda_1}, \ldots, Q_{\lambda_r})$ is the $\mathbb{Q}(q,x)$-linear combination of link diagrams on the disk obtained by replacing the ith component of L by Q_{λ_i}. The above equality implies Proposition 2.4(b).

Let $\sigma : \mathbb{Q}(x,q) \to \mathbb{Q}(x,q)$ denote the $\mathbb{Q}$-algebra automorphism given by $\sigma(x) = x$, $\sigma(q) = -q^{-1}$. One can easily check that σ extends to a $\mathbb{Q}$-linear automorphism of $\mathcal{S}(\Sigma)$ for any Σ by setting $\sigma(L) := L$ for any link diagram L on Σ. It is easy to see that if y is an element of the HOMFLYPT skein algebra of the disk, then

$$\sigma\big(\langle y \rangle\big) = \big\langle \sigma(y) \big\rangle. \tag{62}$$

LEMMA B.1
For any partition λ one has

$$\sigma(Q_\lambda) = Q_{\lambda^\dagger}. \tag{63}$$

Proof
Aiston and Morton [3] gave a geometric description of Q_λ in terms of closures of

braids. Let us recall this formula for partitions with one row $h_a = (a)$ or one column $e_a = (1^a)$ from [3, p. 11]:

$$Q_{(a)} = \frac{1}{\alpha_{(a)}} \sum_{\pi \in \mathrm{Sym}_a} q^{l(\pi)} \widehat{\omega}_\pi, \qquad Q_{(1^a)} = \frac{1}{\alpha_{(1^a)}} \sum_{\pi \in \mathrm{Sym}_a} (-q^{-1})^{l(\pi)} \widehat{\omega}_\pi. \tag{64}$$

Here, for a permutation π of Sym_n, ω_π denotes the positive braid corresponding to π, and $\widehat{\omega}_\pi \in \mathcal{C}$ denotes the closure of ω_π. Moreover, α_λ is given by [3, p. 14] as

$$\alpha_\lambda = \prod_{(i,j)\in\lambda} q^{j-i} \big[\mathrm{hook}(ij)\big], \tag{65}$$

where $\mathrm{hook}(ij)$ is the hook length of the cell (i, j) of the partition λ.

From (64) and (65) one can readily check that $\sigma(Q_{(a)}) = Q_{(1^a)}$, proving the lemma for the case $\lambda = h_a = (a)$. The case of general λ can be proved similarly, using explicit formulas of Q_λ as described in [3]. Alternatively, one can reduce the general case to the case of one row as follows. The two Jacobi–Trudi formulas

$$s_\lambda = \det\big((h_{\lambda_i+j-i})_{i,j=1}^{\ell}\big), \qquad s_{\lambda^\dagger} = \det\big((e_{\lambda_i+j-i})_{i,j=1}^{\ell}\big),$$

together with the case $\lambda = h_a$, imply the lemma for general partitions. □

Suppose that L is an oriented link diagram L on the disk with r ordered components, and suppose that λ_i for $i = 1, \dots, r$ are partitions. We have

$$\begin{aligned}
\sigma\big(W_L(\lambda_1,\dots,\lambda_r)\big) &= \sigma\big(\big\langle L * (Q_{\lambda_1},\dots,Q_{\lambda_r})\big\rangle\big) \quad \text{by (61)}\\
&= \big\langle \sigma\big(L * (Q_{\lambda_1},\dots,Q_{\lambda_r})\big)\big\rangle \quad \text{by (62)}\\
&= \big\langle L * \big(\sigma(Q_{\lambda_1}),\dots,\sigma(Q_{\lambda_r})\big)\big\rangle\\
&= \big\langle L * (Q_{\lambda_1^\dagger},\dots,Q_{\lambda_r^\dagger})\big\rangle \quad \text{by (63)}\\
&= W_L(\lambda_1^\dagger,\dots,\lambda_r^\dagger).
\end{aligned}$$

This concludes the proof of Proposition 2.4(c). □

C. The recursion for the colored HOMFLYPT of the trefoil

Let $\lambda \in \mathcal{P}_{n-1}$ be a partition of length at most $n-1$. We also use λ to denote the corresponding $\mathrm{U}_q(\mathfrak{sl}_n)$-module. For every positive integer k, the theory of ribbon categories gives a representation $J : \mathfrak{B}_k \to \mathrm{Aut}(\lambda^{\otimes k})$, where $\mathfrak{B}_k$ is the braid group on k strands and $\mathrm{Aut}(\lambda^{\otimes k})$ is the group of $\mathrm{U}_q(\mathfrak{sl}_n)$-linear automorphisms of $\lambda^{\otimes k}$.

Suppose that $\beta \in B_m$ is a braid on m strands, and suppose that $L = \mathrm{cl}(\beta)$ is the oriented, framed link obtained by closing β in the standard way, with blackboard framing. Then

$$J_L(\lambda,\lambda,\dots) = \mathrm{tr}_q^{\lambda^{\otimes m}}\big(J(\beta)\big), \tag{66}$$

where for a $U_q(\mathfrak{sl}_n)$-linear map $f : V \to V$,

$$\mathrm{tr}_q^V(f) = \mathrm{tr}(f\mathbf{g}, V).$$

Here the right-hand side is the usual trace of $f\mathbf{g}$ acting on V, and $\mathbf{g} \in U_q(\mathfrak{sl}_n)$ is the so-called *charm element* whose exact formula is not needed here. In particular, for a finite-dimensional weight $U_q(\mathfrak{sl}_n)$-module V, the quantum dimension $\dim_q(V) := J_U(V)$ (where U is the unknot) is

$$\dim_q(V) := \mathrm{tr}_q^V(\mathrm{id}) = \mathrm{tr}(\mathfrak{g}, V).$$

Let σ be the standard generator of $\mathfrak{B}_2$ (see $X_{a,b}$ of Figure 1). Then $J(\sigma)$ is defined by the universal R-matrix, and the action of $J(\sigma)$ on $h_m^{\otimes 2}$ can be calculated as follows. The decomposition of $h_m^{\otimes 2}$ into irreducible $U_q(\mathfrak{sl}_n)$-modules has the form

$$h_m^{\otimes 2} = \bigoplus_{k=0}^{m} \mu_{m,k},$$

where $\mu_{m,k}$ is the partition $(2m-k,k)$. Since $J(\sigma)$ is $U_q(\mathfrak{sl}_n)$-linear, the Schur lemma shows that there are scalars $c_{m,k} \in \mathbb{Q}(q^{1/n})$ such that on $h_m^{\otimes 2}$,

$$J(\sigma)|_{h_m^{\otimes 2}} = \bigoplus_{k=0}^{m} c_{m,k}\,\mathrm{id}_{\mu_{m,k}}. \tag{67}$$

One of the axioms of the ribbon structure of $U_q(\mathfrak{sl}_n)$ is that

$$J(\sigma^2)|_{V\otimes W} = (\mathbf{r}_V^{-1} \otimes \mathbf{r}_W^{-1})\mathbf{r}_{V\otimes W}, \tag{68}$$

where $\mathbf{r}$ is the ribbon element, which belongs to the center of a certain completion of $U_q(\mathfrak{sl}_n)$ and acts on any finite-dimensional weight $U_q(\mathfrak{sl}_n)$-module (see [46], [54]). Geometrically, $\mathbf{r} = J_T$, where T is the trivial 1–1 tangle with framing 1, and its action on λ is known (see, e.g., [35, (1.7)]):

$$\mathbf{r}|_{V_\lambda} = \mathbf{r}(\lambda)\mathrm{id}_\lambda, \quad \text{where } \mathbf{r}(\lambda) = q^{\langle\lambda,\lambda+2\rho\rangle}. \tag{69}$$

Here $\langle\cdot,\cdot\rangle$ is the inner product on the weight space of $U_q(\mathfrak{sl}_n)$ normalized such that each root has square length 2, and 2ρ is the sum of all positive roots.

Using (68) in the square of (67), we get

$$(c_{m,k})^2 = \mathbf{r}(\mu_{m,k})\mathbf{r}(h_m)^{-2}.$$

Taking the square root and using (69), one gets the value of $(c_{m,k})$, up to sign ± 1. The sign can be determined by noting that, when $q = 1$, $J(\sigma)$ is the permutation, $J(\sigma)(x_1 \otimes x_2) = x_2 \otimes x_1$. Eventually, we get

$$c_{m,k} = (-1)^k q^{-m^2/n} q^{m^2-2mk+k^2-k}. \tag{70}$$

Suppose that T_s is the link closure of σ^s, which is a torus link of type $(2, s)$. By (66) and the decomposition (67),

$$\begin{aligned}\tilde{J}_{T_s}(h_m) &= q^{sm^2/n} J_{T_s}(h_m) = q^{sm^2/n} \sum_{k=0}^{m} (c_{m,k})^s \dim_q(\mu_{m,k}) \\ &= \sum_{k=0}^{m} (-1)^{sk} q^{s(m^2-2mk+k^2-k)} \dim_q(\mu_{m,k}) \\ &= \sum_{k=0}^{m} (-1)^{sk} q^{s(m^2-2mk+k^2-k)} \begin{bmatrix} x; k-2 \\ k \end{bmatrix} \begin{bmatrix} x; 2m-k-1 \\ 2m-k \end{bmatrix} \\ &\quad \times \frac{[2m-2k+1]}{[2m-k+1]},\end{aligned} \tag{71}$$

where $x = q^n$. In the last equality we use the well-known formula for the quantum dimension (see, e.g., [41, (11)]), which was first established by Reshetikhin. The right-hand side of (71) gives a formula for $W_{T_r}(h_m)$. When $s = 3$, we get another formula of W_{T_3} for the trefoil, which is simpler than the one given in Section 3.7, since it is a 1-dimensional sum.

For odd s, let $\mathring{T}_s$ be the torus knot T_s with 0 framing. Then, adjusting the framing, from (71) we get

$$\begin{aligned}W_{\mathring{T}_s}(h_m) = x^{-m} \sum_{k=0}^{m} (-1)^k q^{s(m-2mk+k^2-k)} \begin{bmatrix} x; k-2 \\ k \end{bmatrix} \begin{bmatrix} x; 2m-k-1 \\ 2m-k \end{bmatrix} \\ \times \frac{[2m-2k+1]}{[2m-k+1]}.\end{aligned} \tag{72}$$

Using the Zeilberger algorithm (see [47]), we get the recurrence relation for $W_{\mathring{T}_3}(h_m)$ as described in Section 1.4.

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References

[1] M. AGANAGIC, T. EKHOLM, L. NG, and C. VAFA, *Topological strings, D-model, and knot contact homology*, Adv. Theor. Math. Phys. **18** (2014), 827–956. MR 3277674. *(401)*

[2] M. AGANAGIC and C. VAFA, *Large N duality, mirror symmetry, and a Q-deformed A-polynomial for knots*, preprint, arXiv:1204.4709v4 [hep-th]. *(401)*

[3] A. K. AISTON and H. R. MORTON, *Idempotents of Hecke algebras of type A*, J. Knot Theory Ramifications **7** (1998), 463–487. MR 1633027. DOI 10.1142/S0218216598000243. *(439, 440, 441)*

[4] A. BELIAKOVA, Q. CHEN, and T. T. Q. LÊ, *On the integrality of the Witten-Reshetikhin-Turaev 3-manifold invariants*, Quantum Topol. **5** (2014), 99–141. MR 3176311. DOI 10.4171/QT/48. *(406)*

[5] S. CAUTIS, J. KAMNITZER, and S. MORRISON, *Webs and quantum skew Howe duality*, Math. Ann. **360** (2014), 351–390. MR 3263166. DOI 10.1007/s00208-013-0984-4. *(403, 407, 408, 409, 410, 413, 414, 415)*

[6] V. CHARI and A. PRESSLEY, *A Guide to Quantum Groups*, Cambridge Univ. Press, Cambridge, 1994. MR 1300632. *(404, 405)*

[7] I. CHEREDNIK, *DAHA-Jones polynomials of torus knots*, Selecta Math. (N.S.) **22** (2016), 1013–1053. MR 3477341. DOI 10.1007/s00029-015-0210-1. *(400)*

[8] J. CHUANG and R. ROUQUIER, *Derived equivalences for symmetric groups and $\mathfrak{sl}_2$-categorification*, Ann. of Math. (2) **167** (2008), 245–298. MR 2373155. DOI 10.4007/annals.2008.167.245. *(414)*

[9] T. DIMOFTE and S. GUKOV, *Refined, motivic, and quantum*, Lett. Math. Phys. **91** (2010), 1–27. MR 2577296. DOI 10.1007/s11005-009-0357-9. *(401)*

[10] N. M. DUNFIELD, S. GUKOV, and J. RASMUSSEN, *The superpolynomial for knot homologies*, Experiment. Math. **15** (2006), 129–159. MR 2253002. *(401)*

[11] P. ETINGOF, S. GELAKI, D. NIKSHYCH, and V. OSTRIK, *Tensor Categories*, Math. Surveys Monogr. **205**, Amer. Math. Soc., Providence, 2015. MR 3242743. DOI 10.1090/surv/205. *(407)*

[12] P. FREYD, D. YETTER, J. HOSTE, W. B. R. LICKORISH, K. MILLETT, and A. OCNEANU, *A new polynomial invariant of knots and links*, Bull. Amer. Math. Soc. (N.S.) **12** (1985), 239–246. MR 0776477. DOI 10.1090/S0273-0979-1985-15361-3. *(399)*

[13] H. FUJI, S. GUKOV, and P. SUŁKOWSKI, *Super-A-polynomial for knots and BPS states*, Nuclear Phys. B **867** (2013), 506–546. MR 2992793. DOI 10.1016/j.nuclphysb.2012.10.005. *(401, 402)*

[14] H. FUJI, S. GUKOV, P. SUŁKOWSKI, and H. AWATA, *Volume conjecture: Refined and categorified*, Adv. Theor. Math. Phys. **16** (2012), 1669–1777. MR 3065082. *(401)*

[15] W. FULTON and J. HARRIS, *Representation Theory*, Grad. Texts in Math. **129**, Springer, New York, 1991. MR 1153249. DOI 10.1007/978-1-4612-0979-9. *(399)*

[16] S. GAROUFALIDIS, “On the characteristic and deformation varieties of a knot” in *Proceedings of the Casson Fest*, Geom. Topol. Monogr. **7**, Geom. Topol. Publ., Coventry, R.I., 2004, 291–309. MR 2172488. DOI 10.2140/gtm.2004.7.291. *(398, 401)*

[17] ———, *Quantum knot invariants*, preprint, arXiv:1201.3314v3 [math.GT]. *(399)*

[18] ———, *The colored HOMFLY polynomial is q-holonomic*, preprint, arXiv:1211.6388v1 [math.GT]. *(403)*

[19] S. GAROUFALIDIS and T. T. Q. LÊ, *The colored Jones function is q-holonomic*, Geom. Topol. **9** (2005), 1253–1293. MR 2174266. DOI 10.2140/gt.2005.9.1253. *(398, 401)*

[20] ———, *A survey of q-holonomic functions*, Enseign. Math. **62** (2016), 501–525. MR 3692896. *(399, 425, 427, 429)*

[21] S. GAROUFALIDIS and R. VAN DER VEEN, *A generating series for Murakami-Ohtsuki-Yamada graph evaluations*, Acta Math. Vietnam. **39** (2014), 529–539. MR 3292581. DOI 10.1007/s40306-014-0081-0. *(401)*

[22] S. GAROUFALIDIS and D. ZAGIER, *Quantum modularity of the Kashaev invariant*, preprint, 2013. *(399)*

[23] S. GUKOV and I. SABERI, “Lectures on knot homology and quantum curves” in *Topology and Field Theories*, Contemp. Math. **613**, Amer. Math. Soc., Providence, 2014, 41–78. MR 3221290. DOI 10.1090/conm/613/12235. *(401)*

[24] S. GUKOV, A. SCHWARZ, and C. VAFA, *Khovanov-Rozansky homology and topological strings*, Lett. Math. Phys. **74** (2005), 53–74. MR 2193547. DOI 10.1007/s11005-005-0008-8. *(401)*

[25] S. GUKOV and M. STOŠIĆ, “Homological algebra of knots and BPS states” in *Proceedings of the Freedman Fest*, Geom. Topol. Monogr. **18**, Geom. Topol. Publ., Coventry, R.I., 2012, 309–367. MR 3084243. DOI 10.2140/gtm.2012.18.309. *(401)*

[26] H. ITOYAMA, A. MIRONOV, A. MOROZOV, and A. MOROZOV, *Character expansion for HOMFLY polynomials, III: All 3-strand braids in the first symmetric representation*, Internat. J. Modern Phys. A **27** (2012), no. 1250099. MR 2958629. DOI 10.1142/S0217751X12500996. *(403)*

[27] ———, *HOMFLY and superpolynomials for figure eight knot in all symmetric and antisymmetric representations*, J. High Energy Phys. **2012**, no. 131. MR 2967690. *(403)*

[28] J. C. JANTZEN, *Lectures on Quantum Groups*, Grad. Stud. in Math. **6**, Amer. Math. Soc., Providence, 1996. MR 1359532. *(404, 405)*

[29] V. F. R. JONES, *Hecke algebra representations of braid groups and link polynomials*, Ann. of Math. (2) **126** (1987), 335–388. MR 0908150. DOI 10.2307/1971403. *(398)*

[30] ———, *Planar algebras, I*, preprint, arXiv:math/9909027v1 [math.QA]. *(407)*

[31] C. KASSEL, M. ROSSO, and V. TURAEV, *Quantum Groups and Knot Invariants*, Panor. Synthèses **5**, Soc. Math. France, Paris, 1997. MR 1470954. *(407)*

[32] K. KAWAGOE, *On the formulae for the colored HOMFLY polynomials*, J. Geom. Phys. **106** (2016), 143–154. MR 3508910. DOI 10.1016/j.geomphys.2016.02.012. *(400, 403)*

[33] G. KUPERBERG, *Spiders for rank* 2 *Lie algebras*, Comm. Math. Phys. **180** (1996), 109–151. MR 1403861. *(407)*

[34] J. LABASTIDA, M. MARIÑO, and C. VAFA, *Knots, links and branes at large* N, J. High Energy Phys. **2000**, no. 7. MR 1806596. DOI 10.1088/1126-6708/2000/11/007. *(403, 407)*

[35] T. T. Q. LÊ, *Integrality and symmetry of quantum link invariants*, Duke Math. J. **102** (2000), 273–306. MR 1749439. DOI 10.1215/S0012-7094-00-10224-4. *(398, 405, 442)*

[36] ———, *The colored Jones polynomial and the* A*-polynomial of knots*, Adv. Math. **207** (2006), 782–804. MR 2271986. DOI 10.1016/j.aim.2006.01.006. *(398, 401)*

[37] S. G. LUKAC, *Homfly skeins and the Hopf link*, Ph.D. dissertation, University of Liverpool, Liverpool, UK, 2001. *(406)*

[38] G. LUSZTIG, *Introduction to Quantum Groups*, reprint of the 1994 edition, Mod. Birkhäuser Class., Birkhäuser/Springer, New York, 2010. MR 2759715. DOI 10.1007/978-0-8176-4717-9. *(404, 413, 414)*

[39] I. G. MACDONALD, *Symmetric functions and Hall polynomials*, with contributions by A. Zelevinsky, 2nd ed., Oxford Math. Monogr., Oxford Univ. Press, New York, 1995. MR 1354144. *(405, 406, 433)*

[40] G. MASBAUM and H. WENZL, *Integral modular categories and integrality of quantum invariants at roots of unity of prime order*, J. Reine Angew. Math. **505** (1998), 209–235. MR 1662260. DOI 10.1515/crll.1998.505.209. *(405)*

[41] H. R. MORTON and S. G. LUKAC, *The Homfly polynomial of the decorated Hopf link*, J. Knot Theory Ramifications **12** (2003), 395–416. MR 1983094. DOI 10.1142/S0218216503002536. *(400, 443)*

[42] H. R. MORTON and P. MANCHÓN, *Geometrical relations and plethysms in the Homfly skein of the annulus*, J. Lond. Math. Soc. (2) **78** (2008), 305–328. MR 2439627. DOI 10.1112/jlms/jdn026. *(400, 406, 439)*

[43] H. MURAKAMI, T. OHTSUKI, and S. YAMADA, *Homfly polynomial via an invariant of colored plane graphs*, Enseign. Math. (2) **44** (1998), 325–360. MR 1659228. *(407, 409)*

[44] S. NAWATA, P. RAMADEVI, ZODINMAWIA, and X. SUN, *Super-A-polynomials for twist knots*, J. High Energy Phys. **2012**, no. 157. MR 3036447. *(403)*

[45] L. NG, *Framed knot contact homology*, Duke Math. J. **141** (2008), 365–406. MR 2376818. DOI 10.1215/S0012-7094-08-14125-0. *(401)*

[46] T. OHTSUKI, *Quantum Invariants*, Ser. Knots Everything **29**, World Sci., River Edge, N.J., 2002. MR 1881401. *(398, 442)*

[47] M. PETKOVŠEK, H. S. WILF, and D. ZEILBERGER, $A = B$, A K Peters, Wellesley, Mass., 1996. MR 1379802. *(402, 443)*

[48] J. H. PRZYTYCKI and P. TRACZYK, *Conway algebras and skein equivalence of links*, Proc. Amer. Math. Soc. **100** (1987), 744–748. MR 0894448. DOI 10.2307/2046716. *(399)*

[49] H. QUEFFELEC and A. SARTORI, *HOMFLY-PT and Alexander polynomials from a doubled Schur algebra*, to appear in Quantum Topol., preprint, arXiv:1412.3824v1 [math.QA]. *(423)*

[50] ———, *Mixed quantum skew Howe duality and link invariants of type A*, preprint, arXiv:1504.01225v2 [math.RT]. *(423)*

[51] N. Y. RESHETIKHIN and V. G. TURAEV, *Ribbon graphs and their invariants derived from quantum groups*, Comm. Math. Phys. **127** (1990), 1–26. MR 1036112. *(398, 413)*

[52] C. SABBAH, "Systèmes holonomes d'équations aux q-différences" in *D-Modules and Microlocal Geometry (Lisbon, 1990)*, de Gruyter, Berlin, 1993, 125–147. MR 1206016. *(399, 425, 426, 427)*

[53] V. G. TURAEV, *The Yang-Baxter equation and invariants of links*, Invent. Math. **92** (1988), 527–553. MR 0939474. DOI 10.1007/BF01393746. *(399, 406)*

[54] ———, *Quantum Invariants of Knots and 3-Manifolds*, de Gruyter Stud. Math. **18**, de Gruyter, Berlin, 1994. MR 1292673. *(398, 405, 407, 413, 442)*

[55] P. WEDRICH, *q-holonomic formulas for colored HOMFLY polynomials of 2-bridge links*, preprint, arXiv:1410.3769v1 [math.GT]. *(400)*

[56] H. WENZL, *Representations of braid groups and the quantum Yang-Baxter equation*, Pacific J. Math. **145** (1990), 153–180. MR 1066402. *(400, 405)*

[57] H. S. WILF and D. ZEILBERGER, *An algorithmic proof theory for hypergeometric (ordinary and "q") multisum/integral identities*, Invent. Math. **108** (1992), 575–633. MR 1163239. DOI 10.1007/BF02100618. *(399)*

[58] D. ZEILBERGER, *A holonomic systems approach to special functions identities*, J. Comput. Appl. Math. **32** (1990), 321–368. MR 1090884. DOI 10.1016/0377-0427(90)90042-X. *(399, 425)*

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