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RESEARCH ARTICLE

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On the trace fields of hyperbolic Dehn fillings

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Abstract

Assuming Lehmer's conjecture, we estimate the degree of the trace field $K(M_{p/q})$ of a hyperbolic Dehn filling $M_{p/q}$ of a 1-cusped hyperbolic 3-manifold M by

$$\frac{1}{C}(\max\{|p|,|q|\}) \leqslant \deg K(M_{p/q}) \leqslant C(\max\{|p|,|q|\}),$$

where $C = C_M$ is a constant that depends on M.

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1 | INTRODUCTION

1.1 | Main result

An important arithmetic invariant of a complete hyperbolic 3-manifold M of finite volume is its trace field $K(M) = \mathbb{Q}(\mathrm{tr}\,\rho(g) \mid g \in \pi_1(M))$ generated by the traces $\rho(g)$ of the elements of the fundamental group $\pi_1(M)$ of the geometric representation $\rho : \pi_1(M) \to \mathrm{PSL}_2(\mathbb{C})$. Mostow rigidity implies that ρ is rigid; hence, it can be conjugated to lie in $\mathrm{PSL}_2(\overline{\mathbb{Q}})$, and as a result, it follows that $K(M)$ is a number field.

Given a 1-cusped hyperbolic 3-manifold M, Thurston showed that the manifolds $M_{p/q}$ obtained by p/q-Dehn filling on M are hyperbolic for all but finitely many pairs of coprime integers (p, q) [16]. Thus, it is natural to ask how the trace field $K(M_{p/q})$ depends on the Dehn-filling parameters. In [9], C. Hodgson proved the following.

Theorem 1.1 (Hodgson). *Let M be a 1-cusped hyperbolic 3-manifold. Then there are only finitely many hyperbolic Dehn fillings of M of bounded trace field degree.*

Additional proofs of this theorem were given by Long–Reid [12, Thm.3.2] and by the second author in his thesis [11]. A simple invariant of a number field is its degree. Hence, having the above theorem, the next question is to ask for the behavior of the degree of the trace field of a cusped hyperbolic 3-manifold under Dehn filling. In this paper, we give a partial and conditional answer to the question as follows.

Theorem 1.2. *Let M be a 1-cusped hyperbolic 3-manifold and $M_{p/q}$ be its p/q-Dehn filling. Assuming Lehmer's conjecture, there exists $C = C_M$ depending on M such that*

$$\frac{1}{C}(\max\{|p|, |q|\}) \leqslant \deg K(M_{p/q}) \leqslant C(\max\{|p|, |q|\}). \tag{1}$$

Note that the upper bound in (1) follows from a relatively easy observation (see Theorems 2.4–2.5) and so what really matters in (1) is its lower bound.

1.2 | Key observation

The proof of the above theorem uses the fact that the geometric representation of $M_{p/q}$ lies in the geometric component of the $\mathrm{PSL}_2(\mathbb{C})$-representation variety of M (known as Thurston's hyperbolic Dehn-surgery theorem [16]). The latter defines an algebraic curve in $\mathbb{C}^* \times \mathbb{C}^*$ (the so-called A-polynomial curve [1]) in meridian–longitude coordinates, and a suitable point in its intersection with $m^p \ell^q = 1$ determines the geometric representation of $M_{p/q}$. Thus, the determination of the degree of $K(M_{p/q})$ is reduced to a bound for the irreducible components of the Dehn-filling polynomial, a polynomial whose coefficients are independent of (p, q) for large $|p| + |q|$.

To explain the key idea further in detail, as a toy model, let us assume the A-polynomial of a 1-cusped hyperbolic 3-manifold M is simply given as

$$m\left(l - \frac{1}{l}\right) + 1 + \frac{1}{m}\left(\frac{1}{l} - l\right) = 0. \tag{2}$$

Then finding the intersection between (2) and $m^p\ell^q = 1$ is equivalent to solving

$$t^{-q}(t^p - t^{-p}) + 1 + t^q(t^{-p} - t^p) = 0,$$

which is normalized as

$$t^{2p+2q} - t^{2p} - t^{p+q} - t^{2q} + 1 = 0 \tag{3}$$

under $0 < q < p$. Since the Mahler measure of a polynomial concerns about the size of its roots (see Section 2.3), to apply it alongside Lehmer's conjecture (see Conjecture 2.7), we first bound the modulus of a root of (3) in terms of p and q as follows. Let t_0 is a root of (3) with $|t_0| > 1$. Then,

$$|t_0|^{2p+2q} < |t_0|^{2p} + |t_0|^{p+q} + |t_0|^{2q} + 1 < 4|t_0|^{2p} \Longrightarrow |t_0|^{2q} < 4$$

and, taking logarithms,

$$|t_0| < 1 + \frac{1}{q}.$$

If S is an arbitrary constant with $\frac{p}{q} < S$, clearly

$$|t_0| < 1 + \frac{S}{p}, \tag{4}$$

and this implies that the Mahler measure of the minimal polynomial of t_0 is bounded above by

$$\left(1 + \frac{S}{p}\right)^{\deg t_0}.$$

According to Lehmer's conjecture, the above number is at least 1.176280818 … , thus there exists some constant C depending only on S such that[†]

$$Cp < \deg\ t_0. \tag{5}$$

Remark that (5) already verifies Theorem 1.2 for the given example partially over the following restricted domain of p and q:

$$\left\{(p, q) \in \mathbb{Z}^2\ :\ 1 < \frac{p}{q} < S\right\}, \tag{6}$$

and C, as a function of S, goes to 0 as S approaches ∞.

Now suppose $\frac{p}{q} > S$. Let t_0 (with $|t_0| > 1$) be a root of

$$t^{2p+2q} - t^{2p} - t^{p+q} - t^{2q} + 1 = t^{2p}(t^{2q} - 1) - t^{p+q} - (t^{2q} - 1) = 0 \tag{7}$$

[†] In this case, one may further obtain (5) from (4) unconditionally, thanks to Dimitrov's recent proof of the Schinzel–Zassenhaus conjecture. See Theorem 2.8 and Theorem 4.1.

and $\delta_{p,q}$ be

$$\min\left\{ |t_0^{2q}-1| \mid t_0 \text{ is a root of } (7) \text{ with } |t_0|>1\right\}. \tag{8}$$

Then,

$$\begin{aligned} &\delta_{p,q}|t_0^{2p}|(<|t_0^{2p}||t_0^{2q}-1|) < |t_0^{p+q}| + |t_0^{2q}| + 1 < 3|t_0^{p+q}| \\ &\Longrightarrow |t_0^{p-q}| < \tfrac{3}{\delta_{p,q}} \Longrightarrow |t_0^{(1-\frac{1}{S})p}| < \tfrac{3}{\delta_{p,q}} \Longrightarrow |t_0| < 1 + \tfrac{D_{p,q}}{p} \end{aligned} \tag{9}$$

for some $D_{p,q}$ depending on S and $\delta_{p,q}$. Hence, combining with Lehmer's conjecture again, it follows that

$$C_{p,q}\,p < \deg\ t_0 \tag{10}$$

with $C_{p,q}$ depending on $D_{p,q}$. Note that, when S is fixed to be sufficiently large, $D_{p,q}$ depends only on $\delta_{p,q}$, and

$$\lim_{\delta_{p,q}\to 0} D_{p,q} = \infty, \qquad \lim_{D_{p,q}\to\infty} C_{p,q} = 0.$$

Consequently, if

$$\inf_{\substack{(p,q)\in\mathbb{Z}^2 \\ 0<q<p}} \delta_{p,q} > 0, \tag{11}$$

the statement of Theorem 1.2 holds over

$$\left\{(p,q)\in\mathbb{Z}^2 \ : \ 0<q<p\right\} \tag{12}$$

for the example provided.

However, (11) is not generally true; in fact, the following holds:

$$\inf_{\substack{(p,q)\in\mathbb{Z}^2 \\ 0<q<p}} \delta_{p,q} = 0.$$

Moreover, for any sufficiently large D, we always find (p,q) in (12) and roots of (7) whose absolute values are bigger than $1+\frac{D}{p}$, which means that the above heuristic argument is not enough to establish the claim of Theorem 1.2. This aspect constitutes a highly nontrivial point of the problem.

To resolve the issue, in Lemmas 3.2–3.3, we fix some sufficiently large D and count the number of roots of (7) whose absolute values are larger than $1+\frac{D}{p}$. We further get the lower and upper bounds of those roots and examine their distribution within the range. It is then verified that the product of the root moduli stays in manageable limits, making it small enough to deduce the desired outcome from Lehmer's conjecture.

The key technical heart of the paper lies in proving Lemmas 3.2–3.3, and this requires a thorough examination of the local geometry of an analytic curve. Employing solely elementary methods such as trigonometry and basic analysis, we investigate the desired properties of the curve in the proofs. We particularly encourage readers to contrast the content in this subsection with the assertions given in Lemma 3.3.

Finally, let us remark that once the claim of Theorem 1.2 is demonstrated over the domain in (12), the rest will follow naturally by the symmetric properties of the A-polynomial, along with a change of variables applied to it.

2 | PRELIMINARIES

2.1 | The A-polynomial

For a given 1-cusped hyperbolic 3-manifold M, the A-polynomial of M is a polynomial with 2-variables introduced by Cooper–Culler–Gillett–Long–Shalen in [1]. The subject has been studied in great detail, as it provides much topological information of M. We will not present all the technical details about the topic in this paper, but will instead provide a brief outline of its construction and necessary properties that will be utilized later in proving the main theorem. For a more comprehensive description of the A-polynomial, we suggest readers refer to, for instance, [3] or [1].

According to Thurston [16], a geometric ideal triangulation $\mathcal{T}$ on M induces a so-called *gluing variety* $G(\mathcal{T})$ of M. The variety represents the required conditions for how the tetrahedra in $\mathcal{T}$ are glued together along their edges to get a hyperbolic structure on M. Roughly, $G(\mathcal{T})$ is seen as the set of all the possible hyperbolic structures on M, or simply, the moduli space of M.

If T is a torus cross section of the cusp of M and μ, λ are the chosen meridian–longitude pair of T, then each point of $G(\mathcal{T})$ gives rise to a (Euclidean) similarity structure on T, thus inducing the following *holonomy map*

$$\pi_1(T) \longrightarrow \mathrm{Aff}(\mathbb{C}) := \{az + b \ : \ a \neq 0, b \in \mathbb{C}\}. \tag{13}$$

Consequently, the dilation components of the holonomies of μ, λ produce rational functions m, l, respectively, on $G(\mathcal{T})$. Now the *A-polynomial* of M is defined as the Zariski closure of the image under

$$G(\mathcal{T}) \longrightarrow (m, l).$$

The A-polynomial turns out to be independent of the triangulation $\mathcal{T}$, but depends only on the fundamental group of M as well as the chosen meridian–longitude pair of T.

The following properties are well known as follows.

Theorem 2.1 [1]. *Let $A(m, \ell) \in \mathbb{Z}[m, \ell]$ be the A-polynomial of a 1-cusped hyperbolic 3-manifold M.*

(1) $A(m, \ell) = \pm A(m^{-1}, \ell^{-1})$ *up to powers of m and l.*
(2) $(m, \ell) = (1, 1)$ *is a point on $A(m, \ell) = 0$, which gives rise to a discrete faithful representation of $\pi_1(M)$.*

Writing

$$A(m,\ell)=\sum_{i,j} c_{i,j} m^i \ell^j, \tag{14}$$

the Newton polygon $\mathcal{N}(A)$ of $A(m,\ell)$ is defined as the convex hull in the plane of the set $\{(i,j)\ :\ c_{i,j}\neq 0\}$.

Theorem 2.2 [1, 4]. *Let M, $A(m,\ell)=0$ and $\mathcal{N}(A)$ be the same as above. Suppose that $A(m,\ell)$ is normalized so that the greatest common divisor of the coefficients is 1.*

(1) *If (i,j) is a corner of $\mathcal{N}(A)$, then $c_{i,j}=\pm 1$.*
(2) *For each edge of $\mathcal{N}(A)$, the corresponding edge-polynomial is a product of cyclotomic polynomials.*

2.2 | The Dehn filling polynomial

Fix a 1-cusped hyperbolic 3-manifold M with a meridian and longitude μ and λ. Let $M_{p/q}$ denote its p/q-Dehn filling where (here and throughout) (p,q) represents a pair of coprime integers. Thurston's hyperbolic Dehn-surgery theorem [16] implies that $M_{p/q}$ is a hyperbolic manifold for all but finitely many pairs (p,q), or equivalently for almost all pairs (p,q), or yet equivalently, for all pairs (p,q) with $|p|+|q|$ large. It follows by the Seifert–Van Kampen theorem that $\pi_1(M_{p/q})=\pi_1(M)/(\mu^p\lambda^q=1)$. Hence, if the A-polynomial of M is given by $A(m,\ell)=0$, a discrete faithful representation $\phi\ :\ \pi_1(M_{p/q})\longrightarrow \mathrm{PSL}_2(\mathbb{C})$ is obtained by finding a point on

$$(A(m,\ell)=0)\cap(m^p\ell^q=1), \tag{15}$$

which is reduced to an equation $A_{p,q}(t)=0$ for a one-variable Dehn-filling polynomial $A_{p,q}(t)=A(t^{-q},t^p)$. More precisely, if (14) is normalized as

$$A(m,\ell)=\sum_{j=0}^{n}\left(\sum_{i=a_j}^{b_j} c_{i,j} m^i\right)\ell^j \tag{16}$$

where a_j, b_j and $c_{i,j}$ are integers with $c_{a_j,j}, c_{b_j,j}\neq 0$ for all j, then the Dehn-filling polynomial is given by

$$A_{p,q}(t)=\sum_{j=0}^{n}\left(\sum_{i=a_j}^{b_j} c_{i,j} t^{-qi}\right)t^{pj}. \tag{17}$$

We now discuss some elementary properties of the Dehn-filling polynomial. Let

$$S_A := \max_{0\leqslant j\leqslant n-1}\left\{\frac{a_n-a_j}{n-j}\right\} \tag{18}$$

denote the largest slope of the Newton polygon of the A-polynomial (see Figure 1).

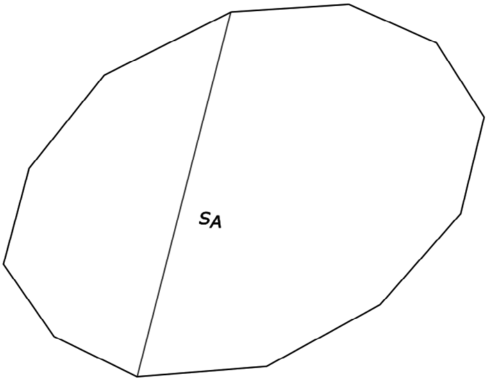

FIGURE 1 For instance, if the Newton polygon of the A-polynomial M appears as above, then S_A is the slope of the depicted edge.

Lemma 2.3. *When $|p| + |q|$ are large,*

(a) *no two terms in (17) are equal,*
(b) *further, if $q > 0$, the leading term of $A_{p,q}(t)$ is of the form $c_{a_k,k}t^{-a_kq+kp}$ for some $0 \leqslant k \leqslant n$. Moreover, $k = n$ for $p/q > S_A$ and $q > 0$.*

It follows that for almost all pairs (p, q), $A_{p,q}(t)$ has degree piece-wise linear in (p, q), and leading coefficient (due to Theorem 2.2) ± 1.

Proof. Observe that $t^{-qi+pj} = t^{-qi'+pj'}$ implies that $p/q = (i' - i)/(j' - j)$ that takes finitely many values according to (16). It follows that when $|p| + |q|$ is sufficiently large, then no two terms in (17) are equal, and hence the Newton polygon of $A_{p,q}(t)$ is the image of the Newton polygon of $A(m, \ell)$ under the linear map $\mathbb{R}^2 \to \mathbb{R}$ that sends (i, j) to $-qi + pj$. This proves that the leading term of $A_{p,q}(t)$ is the image under the above map of a corner of $A(m, \ell)$. When $p/q > S_A$ and $q > 0$, it is easy to see that this corner of $A(m, \ell)$ is (a_n, n). □

We now relate the trace field $K(M_{p/q})$ with (15).

Theorem 2.4. *Let $(m, \ell) = (t_0^{-q}, t_0^p)$ be the point on (15) that gives rise to the discrete faithful representation of $\pi_1(M_{p/q})$. (Of course, t_0 depends on p and q.) Then there exist constants $C = C(M)$ depending only on M such that*

$$\frac{1}{C} \deg t_0 \leqslant \deg K(M_{p/q}) \leqslant C \deg t_0. \tag{19}$$

Proof. First note that $K(M)$ is generated by the traces of products of at most three generators of $\pi_1(M)$ (see, e.g., [14]). By Theorem 3.1 in [6], the trace of any generator of $\pi_1(M)$ is represented as an algebraic function of the traces of μ and λ. As

$$\operatorname{tr} \mu = \pm\left(\sqrt{m} + \frac{1}{\sqrt{m}}\right) \quad \text{and} \quad \operatorname{tr} \lambda = \pm\left(\sqrt{l} + \frac{1}{\sqrt{l}}\right),$$

it follows that the trace of any $\pi_1(M)$ is an algebraic function of m and ℓ. Since $\pi_1(M)$ is finitely generated, the result follows. □

It is well known (e.g., see [8]) in number theory that $A_{p,q}(t)$ has a bounded number of cyclotomic factors for any p and q (although of unknown multiplicity). Therefore, thanks to Theorem 2.4, proving Theorem 1.2 is reduced to showing the degree of any noncyclotomic factor of $A_{p,q}(t)$ is bounded both above and below by some multiples of $\max\{|p|,|q|\}$ under Lehmer's conjecture. That is, Theorem 1.2 is derived from the following.

Theorem 2.5. *Fix M as in Theorem 1.2. Assuming Lehmer's conjecture, there exist $C = C(M)$ depending only on M such that, for any noncyclotomic integer irreducible factor $g(t)$ of $A_{p,q}(t)$,*

$$\frac{1}{C}\max\{|p|,|q|\} \leqslant \deg g(t) \leqslant C\max\{|p|,|q|\}. \tag{20}$$

The upper bound follows trivially from (17), and can even be replaced by a piecewise linear function of (p,q), as was commented after Lemma 2.3. The difficult part is the lower bound, and this requires an analysis of the roots of $A_{p,q}(t)$ near 1, along with careful estimates, as well as Lehmer's conjecture.

Remark 2.6. In some special circumstances, such as Dehn-filling on one cusp of the Whitehead link complement W, it is possible to prove unconditionally that the degree of $K(W_{1/q})$ is given by an explicit piece-wise linear function of q (see Hoste–Shanahan [10]). More generally, one can prove that the degree of $K(W_{p/q})$ for fixed p and large q is given by an explicit piecewise linear function of q [7].

2.3 | Mahler measure and Lehmer's conjecture

The Mahler measure $\mathscr{M}(f)$ and length $\mathscr{L}(f)$ of an integer polynomial

$$f(x) = a_n x^n + \cdots + a_1 x + a_0 = a_n(x-\alpha_1)\cdots(x-\alpha_n) \in \mathbb{Z}[x]$$

are defined by

$$\mathscr{M}(f) = |a_n|\prod_{i=1}^n \max(|\alpha_i|,1) \quad \text{and} \quad \mathscr{L}(f) = |a_0| + \cdots + |a_n|,$$

respectively. Then, the following properties are standard [13]:

(1) $\mathscr{M}(f_1 f_2) = \mathscr{M}(f_1)\mathscr{M}(f_2)$,
(2) $\mathscr{M}(f) \leqslant \mathscr{L}(f)$,

where $f_1, f_2 \in \mathbb{Z}[x]$. One of the renowned unsolved problems in number theory is the following conjecture proposed by D. Lehmer in 1930s.

Conjecture 2.7 (D. Lehmer). *The Mahler measure of any noncyclotomic irreducible integer polynomial f satisfies*

$$\mathscr{M}(f) \geqslant 1.176280818\ldots,$$

where 1.176280818 ... *is the Mahler measure of*

$$x^{10}+x^9-x^7-x^6-x^5-x^4-x^3+x+1$$

and the smallest known Salem number.

2.4 | Schinzel–Zassenhaus conjecture

Recently, there has been some progress toward Lehmer's conjecture. In [5], V. Dimitrov proved the following theorem.

Theorem 2.8 (Schinzel–Zassenhaus conjecture). *Let $f(t)$ be a monic integer irreducible polynomial of degree $n > 1$. If $f(t)$ is not cyclotomic, then*

$$\max_{t_0\,:\,f(t_0)=0} |t_0| \geqslant 2^{\frac{1}{4n}} = 1 + \frac{\log 2}{4n} + O\left(\frac{1}{n^2}\right).$$

As mentioned in Section 1.2, once we restrict our attention to a small domain such as (6), the conclusion of Theorem 1.2 follows unconditionally (i.e., without relying on Lehmer's conjecture) over the domain solely based on the above theorem (see Theorem 4.1). However, we do not believe that Theorem 2.8 in general provides an unconditional proof of Theorem 1.2 over the entire domain. Please also refer to Remark 3.5.

3 | BOUNDS FOR THE ROOTS OF THE DEHN-FILLING POLYNOMIAL

In this section, we study the roots of the Dehn-filling polynomial $A_{p,q}(t)$ for sufficiently large $|p|+|q|$. Recall the constant S_A from (18). The next lemma shows that when $p/q > S_A$ with p and q sufficiently large, the roots of $A_{p,q}(t)$ are near the unit circle in the complex plane.

Lemma 3.1.

(a) *Fix a positive constant S_1 with $S_1 > S_A$. There exists $D > 0$ that depends on S_1 such that for any coprime pair $(p,q) \in \mathbb{N}^2$ satisfying $\frac{p}{q} > S_1$, and for any root t_0 of $A_{p,q}(t)$, we have*

$$|t_0| < 1 + \frac{D}{q}.$$

(b) *If in addition* $S_2 > S_1$, $S_1 < \frac{p}{q} < S_2$, *and* t_0 *is any root of* $A_{p,q}(t)$, *then*

$$|t_0| < 1 + \frac{D}{p}$$

for some D *that depends on* S_1 *and* S_2 *only.*

Proof. For $S_1 > S_A$, the leading term of $A_{p,q}(t)$ is $c_{a_n,n}t^{-a_nq+np}$ and $c_{a_n,n} = \pm 1$ by Lemma 2.3. Thus,

$$|t_0^{-a_nq+np}| < Lc_{\max}|t_0^{-aq+bp}| \Longrightarrow |t_0^{(n-b)p+(a-a_n)q}| < Lc_{\max}, \tag{21}$$

where L is the number of terms of $A_{p,q}(t)$, $c_{\max}$ is the maximum among all the coefficients of $A_{p,q}(t)$, and $-aq + bp$ is the second largest exponent of $A_{p,q}(t)$. Note that L and $c_{\max}$ are independent of p and q.

We now consider two cases: $b = n$ and $b < n$. If $b = n$, then $|t_0^q| < Lc_{\max}$, implying

$$|t_0| < 1 + \frac{D}{q} \tag{22}$$

for some constant D depending on L and $c_{\max}$.

If $b < n$, since $p, q > 0$, Lemma 2.3 implies that $(a, b) = (a_j, j)$ for some $0 \leqslant j \leqslant n-1$. Thus,

$$Lc_{\max} > |t_0^{(n-j)p+(a_i-a_n)q}| = \left|t_0^{(n-j)\left(p-\frac{a_n-a_j}{n-j}q\right)}\right| > \left|t_0^{(n-j)\left(S_1-\frac{a_n-a_j}{n-j}\right)q}\right|$$

by (21). By the assumption, as S_1 is strictly larger than $\frac{a_n-a_j}{n-j}$ for every j, we get

$$|t_0^q| < D_1 \Longrightarrow |t_0| < 1 + \frac{D}{q}$$

for some constant D_1 and D that depend only on S_1. Part (b) follows easily from part (a). □

The next lemma is a model for the roots of the Dehn-filling polynomial outside the unit circle. Indeed, as we will see later (in the proof of Lemma 3.3), after a change of variables, we will bring the equation $A_{p,q}(t) = 0$ into the form $z^q\phi(z)^p = 1$ where ϕ is an analytic function at $z = 0$ with $\phi(0) = 1$. Hence, the lemma below is the key technical tool used to bound the roots of the Dehn-filling polynomial. In a simplified form, note that the equation $z^q(1+z)^p = 1$ has $p + q$ solutions in the complex plane for $p, q > 0$. On the other hand, only a fraction of them are near zero, whereas at the same time $1 + z$ is outside the unit circle. Moreover, we have a bound for the size of such solutions.

Lemma 3.2. *Let* $w = \phi(z)$ *be an analytic function defined near* $(z, w) = (0, 1)$ *and* ϵ *be a sufficiently small number. Then, there exists* $\gamma(\epsilon) > 0$ *such that, for every coprime pair* $(p, q) \in \mathbb{N}^2$ *with* $p/q > \frac{1}{\epsilon}$, *the number of* (z, w) *satisfying*

$$z^q w^p = 1, \quad |w| > 1, \quad |w - 1| < \epsilon \tag{23}$$

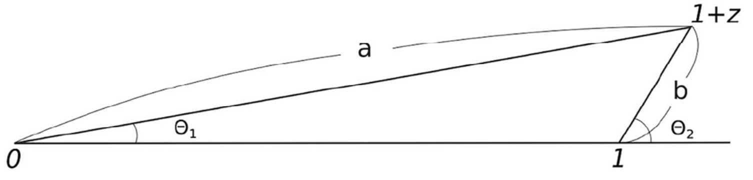

FIGURE 2 $\triangle(z)$.

is at most $2\left(\left\lceil \frac{\gamma(\epsilon)p}{2\pi q}\right\rceil + 1\right) q$. *Moreover,* $\gamma(\epsilon) \to 0$ *as* $\epsilon \to 0$ *and, for each* h $(1 \leqslant h \leqslant \left\lceil \frac{\gamma(\epsilon)p}{2\pi q}\right\rceil + 1)$*, the product of the moduli of the first* $2hq$ *largest* w *satisfying (23) is bounded above by*

$$\prod_{l=1}^{h}\left(1 + \frac{d \log \frac{p/q}{l}}{p/q}\right)^{2q}, \tag{24}$$

where d *is some constant depending only on* ϕ.

Proof.

(1) We first prove the lemma in the special case of $w = \phi(z) = 1 + z$. The first equation in (23) is equivalent to

$$(1+z)^{p/q} = \frac{e^{2\pi i k/q}}{z}, \tag{25}$$

where $0 \leqslant k \leqslant q - 1$.

Considering $0, 1, 1+z$ as three vertices of a triangle $\triangle(z)$ in the complex plane, we denote $1 + z$ and z by

$$ae^{i\theta_1} := 1 + z, \quad be^{i\theta_2} := z, \tag{26}$$

respectively, where $a, b > 0$ and $-\pi < \theta_1, \theta_2 < \pi$ (Figure 2).† Then, (25) is equivalent to

$$(ae^{i\theta_1})^{p/q} = \frac{e^{2\pi i k/q}}{be^{i\theta_2}} \implies a^{p/q} = b^{-1}, \ \frac{\theta_1 p}{q} \equiv \frac{2\pi k}{q} - \theta_2 \pmod{2\pi}. \tag{27}$$

By trigonometry, we have:

$$b^2 = a^2 + 1 - 2a\cos\theta_1, \quad a^2 = b^2 + 1 + 2b\cos\theta_2.$$

Since $b = a^{-p/q}$ by (27), it follows that

$$\begin{aligned}\cos\theta_1 &= \tfrac{1}{2a}(a^2 + 1 - a^{-\frac{2p}{q}}) = \tfrac{1}{2}\left(a + \tfrac{1}{a} - \tfrac{1}{a^{\frac{2p}{q}+1}}\right),\\ \cos\theta_2 &= \tfrac{1}{2b}(a^2 - b^2 - 1) = \tfrac{a^{\frac{p}{q}}}{2}(a^2 - a^{\frac{-2p}{q}} - 1) = \tfrac{1}{2}\left(a^{\frac{p}{q}+2} - a^{\frac{-p}{q}} - a^{\frac{p}{q}}\right).\end{aligned} \tag{28}$$

† Note that if $\theta_1 > 0$ (resp. $\theta_1 < 0$), then $\theta_2 > 0$ (resp. $\theta_2 < 0$).

To simplify notation, we let[†]

$$r := p/q \quad \text{and} \quad x := r(a-1), \tag{29}$$

and rewrite the equations in (28) as

$$\begin{aligned} \cos\theta_1 &= \frac{1}{2}\left(1+\frac{x}{r}+\frac{1}{1+\frac{x}{r}}-\frac{1}{\left(1+\frac{x}{r}\right)^{2r+1}}\right) = \frac{1}{2}\left(2+\frac{x}{r}-\frac{\frac{x}{r}}{1+\frac{x}{r}}-\frac{1}{\left(1+\frac{x}{r}\right)^{2r+1}}\right), \\ \cos\theta_2 &= \frac{1}{2}\left(\left(1+\frac{x}{r}\right)^{r+2}-\frac{1}{\left(1+\frac{x}{r}\right)^{r}}-\left(1+\frac{x}{r}\right)^{r}\right), \end{aligned} \tag{30}$$

which implies

$$\begin{aligned} \sin^2\theta_1 &= 1-\cos^2\theta_1 = 1-\left(1+\frac{1}{2}\left(\frac{x}{r}-\frac{\frac{x}{r}}{1+\frac{x}{r}}-\frac{1}{\left(1+\frac{x}{r}\right)^{2r+1}}\right)\right)^2 \\ &= -\left(\frac{x}{r}-\frac{\frac{x}{r}}{1+\frac{x}{r}}-\frac{1}{\left(1+\frac{x}{r}\right)^{2r+1}}\right)-\frac{1}{4}\left(\frac{x}{r}-\frac{\frac{x}{r}}{1+\frac{x}{r}}-\frac{1}{\left(1+\frac{x}{r}\right)^{2r+1}}\right)^2 \\ \Longrightarrow \sin\theta_1 &= \pm\sqrt{\left(\frac{1}{\left(1+\frac{x}{r}\right)^{2r+1}}+\frac{\frac{x}{r}}{1+\frac{x}{r}}-\frac{x}{r}\right)-\frac{1}{4}\left(\frac{1}{\left(1+\frac{x}{r}\right)^{2r+1}}+\frac{\frac{x}{r}}{1+\frac{x}{r}}-\frac{x}{r}\right)^2}. \end{aligned}$$

Considering $\theta_1 = \theta_1(x)$ as a function of x and assuming $\theta_1(x) \geqslant 0$ for the sake of simplicity, we briefly go over some basic behavior of the above function, set up the domain of x satisfying $|ae^{i\theta_1(x)} - 1| = |z| < \epsilon$, and then list all the solutions to the equation $z^q(1+z)^p = 1$ over the domain found.

Note that the following

$$\sqrt{\frac{1}{\left(1+\frac{x}{r}\right)^{2r+1}}+\frac{\frac{x}{r}}{1+\frac{x}{r}}-\frac{x}{r}} = \sqrt{\frac{1}{\left(1+\frac{x}{r}\right)^{2r+1}}-\frac{1}{\frac{r}{x}\left(\frac{r}{x}+1\right)}} = \sqrt{\frac{1}{\left(1+\frac{x}{r}\right)^{2r+1}}-\frac{x^2}{r(x+r)}}$$

is a decreasing function of x and is 1 when $x = 0$. Further, if

$$\frac{1}{(1+\frac{x}{r})^{2r+1}} = \frac{x^2}{r(x+r)},$$

then

$$\left(\frac{r}{x+r}\right)^{2r+1} = \frac{x^2}{r(x+r)} \Rightarrow x^2 = \frac{r^{2r+2}}{(x+r)^{2r}} \Rightarrow x = \frac{r^{r+1}}{(x+r)^r} \Rightarrow r = x\left(1+\frac{x}{r}\right)^r. \tag{31}$$

And as $\left(1+\frac{x}{r}\right)^{\frac{r}{x}}$ is approximately equal to e (simply say $\left(1+\frac{x}{r}\right)^{\frac{r}{x}} \approx e$) for $\frac{x}{r}$ sufficiently small, the root of (31) is very close to the root of the Lambert equation $r = xe^x$, studied in

[†] Since $|w| = a > 1$ and $|z| = |ae^{i\theta_1} - 1| < \epsilon$ by the assumption, we suppose $x > 0$ and $\frac{x}{r}$ is sufficiently small.

great detail in [2]. Hence, for $x = \phi(r)$ satisfying (31), it follows that

$$\log r - \log\log r < \phi(r) < \log r - \log\log r + \log\log\log r. \tag{32}$$

Remark that if $x = 0$, then $\theta_1(0) = \frac{\pi}{3}, \theta_2(0) = \frac{2\pi}{3}$, and thus, $\triangle\left(z = e^{\frac{2\pi}{3}i}\right)$ is the unit equilateral triangle. As x increases, both $\theta_1(x)$ and $\theta_2(x)$ decrease, and finally, when $x = \phi(r)$, $\theta_1(\phi(r)) = \theta_2(\phi(r)) = 0$ and $\triangle\left(z = \frac{1}{\left(1+\frac{\phi(r)}{r}\right)^r}\right)$ becomes a flat triangle.

(a) By the assumption,

$$|z| = |ae^{i\theta_1(x)} - 1| < \epsilon$$

and, as

$$|ae^{i\theta_1(x)} - 1| = \left|\left(1 + \frac{x}{r}\right)e^{i\theta_1(x)} - 1\right| = \left(1 + \frac{x}{r}\right)^2 (2 - 2\cos\theta_1(x))$$

(with $\cos\theta_1(x)$ in (30)) is a decreasing function of x, there exists $\gamma = \gamma(\epsilon)$ depending on ϵ such that

$$\left|\left(1 + \frac{x}{r}\right)e^{i\theta_1(x)} - 1\right| < \epsilon \Longleftrightarrow \gamma < x \leqslant \phi(r).$$

In conclusion, $(\gamma, \phi(r)]$ is a desired domain for $\theta_1(x)$ (and $\theta_2(x)$) with the required property.

(b) Now we list the solutions of $z^q(1+z)^p = 1$ over the above domain. First, the second equation in (27) is reduced to

$$r\theta_1(x) + \theta_2(x) - \frac{2\pi k}{q} \in 2\pi\mathbb{Z} \tag{33}$$

and

$$-\frac{2\pi k}{q} \leqslant r\theta_1(x) + \theta_2(x) - \frac{2\pi k}{q} \leqslant r\theta_1(\gamma) + \theta_2(\gamma) - \frac{2\pi k}{q}$$

for $\gamma \leqslant x \leqslant \phi(r)$. Since $r\theta_1(x) + \theta_2(x)$ is a decreasing function of x and $\theta_2(\gamma) \leqslant \frac{2\pi}{3}$, for each k $(0 \leqslant k \leqslant q-1)$, the number of x satisfying (33) is at most

$$\left\lfloor \frac{r\theta_1(\gamma) + \theta_2(\gamma)}{2\pi} - \frac{2\pi k}{q} \right\rfloor + 1 = \left\lceil \frac{r\theta_1(\gamma)}{2\pi} \right\rceil + 1.$$

Let $x_l^k \in (\gamma, \phi(r)]$ be a number satisfying

$$\frac{r\theta_1(x_l^k) + \theta_2(x_l^k) - \frac{2\pi k}{q}}{2\pi} = l,$$

where $0 \leqslant l \leqslant \left\lceil \frac{r\theta_1(\gamma)}{2\pi} \right\rceil$ and $l \in \mathbb{Z}$. Then,

$$\theta_1(x_l^k) = \frac{2\pi l - \theta_2(x_l^k) + \frac{2\pi k}{q}}{r} \geqslant \frac{2\pi(l - \frac{1}{3})}{r} \geqslant \frac{l+1}{r}$$

for every k $(0 \leqslant k \leqslant q-1)$ and $l \geqslant 1$. Using the fact that $\theta_1(x)$ is a decreasing function, one further gets

$$x_l^k \leqslant \log\frac{r}{l+1} \quad \text{and so} \quad 1 + \frac{x_l^k}{r} \leqslant 1 + \frac{\log\frac{r}{l+1}}{r} \quad (l \geqslant 1).$$

Clearly, $x_0^k \leqslant \phi(r) \leqslant \log r$ (by (32)) for any $0 \leqslant k \leqslant q-1$. Consequently, for each h $(1 \leqslant h \leqslant \left\lceil \frac{r\theta_1(\gamma)}{2\pi} \right\rceil + 1)$, the product of the moduli of the first hq largest w satisfying $\theta_1(x), \theta_2(x) \geqslant 0$ and (33) is bounded above by[†]

$$\prod_{k=0}^{q-1}\prod_{l=0}^{h-1}\left(1 + \frac{x_l^k}{r}\right) \leqslant \prod_{l=0}^{h-1}\left(1 + \frac{\log\frac{p/q}{l+1}}{p/q}\right)^q = \prod_{l=1}^{h}\left(1 + \frac{\log\frac{p/q}{l}}{p/q}\right)^q.$$

Similarly, counting x with $\theta_1(x), \theta_2(x) \leqslant 0$, one finally attains the desired result. This concludes the proof of the lemma for the special case of $w = 1 + z$.

(2) To prove the lemma for the general case, suppose $w = \phi(z)$ is given as $\phi(z) = 1 + \sum_{i=1}^{\infty} c_i z^i$ and let

$$ae^{i\theta_1} := 1 + \sum_{i=1}^{\infty} c_i z^i, \quad be^{i\theta_2} := \sum_{i=1}^{\infty} c_i z^i, \quad ce^{i\theta_3} := z.$$

We consider b (resp. θ_2) as an analytic function of c (resp. θ_3). Thus, for z sufficiently small, b (resp. θ_1) is approximately very close to $|c_m|c^m$ (resp. $\arg c_m + m\theta_3$) where c_m is the coefficient of the first nonzero (and nonconstant) term of ϕ. Now (23) is

$$(1 + \phi(z))^{p/q} = \frac{e^{2\pi i k/q}}{z} \quad (0 \leqslant k \leqslant q-1),$$

which is equivalent to

$$c = a^{-r}, \quad r\theta_1 + \theta_3 \equiv \frac{2\pi i k}{q} \quad (\mod 2\pi i)$$

where $r = p/q$. Similar to the previous case, provided $x := r(a-1)$, we get $\frac{1}{c} = a^r = \left(1 + \frac{x}{r}\right)^r$ and

[†] Recall from (26) and (29) that $|w| = |1 + z| = |a| = 1 + \frac{x}{r}$.

$$\sin\theta_1 = \pm\sqrt{-\left(\frac{x}{r} - \frac{\frac{x}{r}}{1+\frac{x}{r}} - \frac{b^2}{a}\right) - \frac{1}{4}\left(\frac{x}{r} - \frac{\frac{x}{r}}{1+\frac{x}{r}} - \frac{b^2}{a}\right)^2}$$
$$= \pm\sqrt{\left(\frac{b^2}{a} - \frac{x^2}{r(x+r)}\right) - \frac{1}{4}\left(\frac{b^2}{a} - \frac{x^2}{r(x+r)}\right)^2}.$$

Since $c = a^{-r}$ and $b \approx |c_m|c^m$ for z small, $b \approx |c_m|a^{-rm}$ and so

$$\sin\theta_1 \approx \pm\sqrt{\left(\frac{|c_m|^2}{a^{2rm+1}} - \frac{x^2}{r(x+r)}\right) - \frac{1}{4}\left(\frac{|c_m|^2}{a^{2rm+1}} - \frac{x^2}{r(x+r)}\right)^2}$$
$$= \pm\sqrt{\left(\frac{|c_m|^2}{\left(1+\frac{x}{r}\right)^{2rm+1}} - \frac{x^2}{r(x+r)}\right) - \frac{1}{4}\left(\frac{|c_m|^2}{\left(1+\frac{x}{r}\right)^{2rm+1}} - \frac{x^2}{r(x+r)}\right)^2} \tag{34}$$

for z sufficiently small. Let $\phi(r)$ be the number satisfying

$$\frac{|c_m|}{\left(1+\frac{x}{r}\right)^{rm}} = \frac{x}{r}.$$

As $\left(1+\frac{x}{r}\right)^r \approx e^x$ for $\frac{x}{r}$ small, $\phi(r)$ is very close to the root of $|c_m|r = xe^{xm}$ and

$$d_1 \log r < \phi(r) < d_2 \log r$$

for some $d_1, d_2 \in \mathbb{Q}$. Further, one can check (34) is a decreasing function of x over $0 \leqslant x \leqslant \phi(r)$. Applying similar methods used in the proof of the previous special case, we obtain the desired result.

□

In the next lemma, consider $M, M_{p/q}$ and the Dehn-filling polynomial $A_{p,q}(t)$ as usual. As remarked earlier, the motivation for each statement of Lemma 3.3 was already explained in Section 1.2 using a toy model.

Lemma 3.3. *For every $\epsilon > 0$ sufficiently small, there exist $D(\epsilon)$ and $\gamma(\epsilon)$ such that, for any coprime pair $(p, q) \in \mathbb{N}^2$ with $p/q > \frac{1}{\epsilon}$, the following hold.*

(1) *If t_0 is a root of $A_{p,q}(t)$ such that $1 < |t_0|^p < \frac{1}{\epsilon}$, then $|t_0| < 1 + \frac{D(\epsilon)}{p}$.*

(2) *If t_0 is a root of $A_{p,q}(t)$ such that $|t_0| > 1$ and $|t_0^q - \zeta| > \epsilon$ for every ζ satisfying $\sum_{i=a_n}^{b_n} c_{i,n}\zeta^{-i} = 0$, then $|t_0| < 1 + \frac{D(\epsilon)}{p}$.*

(3) *There are at most $2\lceil\gamma(\epsilon)p/q\rceil q$ roots of $A_{p,q}(t) = 0$ whose moduli are bigger than $1 + \frac{D(\epsilon)}{p}$. Further, for each h ($1 \leqslant h \leqslant \left\lceil\frac{\gamma(\epsilon)p}{q}\right\rceil + 1$), the product of the moduli of the first $2hq$ largest roots of*

$A_{p,q}(t)$ is bounded above by[†]

$$\prod_{l=1}^{h}\left(1+\frac{d\log\frac{p/q}{l}}{p/q}\right)^2, \tag{35}$$

where d is some constant depending only on M.

(4) *$D(\epsilon)\to\infty$ and $\gamma(\epsilon)\to 0$ as $\epsilon\to 0$.*

Proof. For (1), if $|t_0|^p<\frac{1}{\epsilon}$ with p sufficiently large, taking logarithms, there exists $D(\epsilon)$ such that $|t_0|<1+\frac{D(\epsilon)}{p}$.

For (2), if $|t_0^q-\zeta|>\epsilon$ for every ζ satisfying $\sum_{i=a_n}^{b_n} c_{i,n}\zeta^{-i}=0$, then there exists $\delta(\epsilon)>0$ such that $\left|\sum_{i=a_n}^{b_n} c_{i,n}t_0^{-iq}\right|>\delta(\epsilon)$, and so,

$$\delta(\epsilon)|t_0^{np}|<\left|\sum_{i=a_n}^{b_n} c_{i,n}t_0^{-iq}\right||t_0^{np}|<nL|t_0^{(n-1)p}|\Longrightarrow|t_0^p|<\frac{nL}{\delta(\epsilon)}, \tag{36}$$

where L is some number bigger than $\max_{0\leqslant j\leqslant n-1}\left|\sum_{i=a_j}^{b_j} c_{i,j}t_0^{-iq}\right|$. By (36), there exists some constant $D(\epsilon)$ depending on ϵ such that $|t_0|<1+\frac{D(\epsilon)}{p}$.

For (3), suppose that t_0 is a root of $A_{p,q}(t)=0$ such that $|t_0^q-\zeta|<\epsilon$ where $\sum_{i=a_n}^{b_n} c_{i,n}\zeta^{-i}=0$ and $|t_0|^p>\frac{1}{\epsilon}$ for some sufficiently small ϵ. Note that ζ is a root of unity by Theorem 2.2 and, without loss of generality, we further assume $\zeta=1$. If $A(m,\ell)=0$ is the A-polynomial of M as given in (16), one can view $(m,\ell)=(t_0^{-q},t_0^p)$ as an intersection point between $A(m,\ell)=0$ and $m^p\ell^q=1$. If we let $m':=\frac{1}{m}$ and $f(m',\ell):=A(\frac{1}{m'},\ell)$, then $(m',\ell)=(t_0^q,t_0^p)$ is a point lying over $f(m',\ell)=0$ and $\ell^q=(m')^p$. For the sake of simplicity, we consider the projective closure of $f(m',\ell)=0$ and work with a different affine chart of it. More precisely, let $h(x',y',z')=0$ be the homogeneous polynomial representing the projective closure of $f(m',\ell)=0$ with $m'=\frac{x'}{z'},\ell=\frac{y'}{z'}$. That is, $h(x',y',z')=0$ is obtained from $f\left(\frac{x'}{z'},\frac{y'}{z'}\right)=0$ by multiplying by a power of z' if necessary. Further, if $x:=\frac{x'}{y'},z:=\frac{z'}{y'}$ and $k(x,z):=h(x,1,z)$, since $x=\frac{m'}{\ell},z=\frac{1}{\ell}$ and

$$\ell^q=(m')^p\Rightarrow\left(\frac{y'}{z'}\right)^q=\left(\frac{x'}{z'}\right)^p\Rightarrow z^{p-q}=x^p,$$

$(x,z)=(t_0^{-p+q},t_0^{-p})$ is an intersection point between $k(x,z)$ and $z^{p-q}=x^p$. As (t_0^{-p+q},t_0^{-p}) is sufficiently close to (0,0), it follows that $k(0,0)=0$, and thus, x is represented as an analytic function $\varphi(z)$ of z near (0,0) (i.e., $k(\varphi(z),z)=0$) with $(t_0^{-p+q},t_0^{-p})\in(\varphi(z),z)$. Moreover, since $\frac{t_0^{-p+q}}{t_0^{-p}}=t_0^q$

[†] Note that the exponent 2 in (35) is differed from the exponent $2q$ given in (24).

is close enough to 1, $\varphi(z)$ is of the following form:

$$x = \varphi(z) = z\left(1 + \sum_{i=1}^{\infty} c_i z^i\right).$$

Let $\phi(z) := 1 + \sum_{i=1}^{\infty} c_i z^i$. Then,

$$z^{p-q} = x^p \Rightarrow z^{p-q} = z^p \phi(z)^p \Rightarrow z^q \phi(z)^p = 1. \tag{37}$$

In conclusion, (t_0^q, t_0^{-p}) is a point on $(\frac{x}{z}, z) = (\phi(z), z)$ satisfying (37). Let $w := \phi(z)$. As $|t_0| > 1$ and $|t_0^q - 1| < \epsilon$ by the assumption, we get the desired result by Lemma 3.2.

Finally, (4) is clear by the construction of $D(\epsilon)$ and Lemma 3.2. □

Remark 3.4. Switching p and q, one obtains an analogous result for a coprime pair (p, q) with q/p is sufficiently large.

Remark 3.5. Note that if t_0 (where $|t_0| > 1$) is a root of $A_{p,q}(t)$ giving rise to the discrete faithful representation of $M_{p/q}$, for $|p| + |q|$ sufficiently large, $|t_0|$ is asymptotic to $1 + \frac{2\pi \mathrm{Im}\, \tau}{|p+\tau q|^2}$ where τ is a complex number depending only on M satisfying $\mathrm{Im}\, \tau \neq 0$ (τ is called the cusp shape of M) [15]. This, combining with the above remark, implies that t_0 is not the root of its minimal polynomial that appears in Theorem 2.8 for $|p| + |q|$ sufficiently large.

4 | PROOF OF THE MAIN THEOREM

Now we prove the main theorem of the paper. Recall that Theorem 1.2 follows from Theorem 2.5. We first prove a special case of Theorem 2.5 over a restricted domain of p and q. In particular, on this restricted domain, the statement of Theorem 2.5 holds unconditionally, without relying on Lehmer's conjecture, thanks to Theorem 2.8. We then expand the domain further and prove Theorem 2.5 over the extended range. We shall use Lemma 3.3 and assume Lehmer's conjecture from Theorem 4.2. Once Theorem 4.2 is proven, the rest of the proof of Theorem 2.5 will follow by symmetric properties of the A-polynomial as well as a monomial change of variables on it.

Finally, let us remark again that as the upper bound in Theorem 2.5 follows as a corollary of Bezóut's theorem, only the lower bound in (20) is proved.

Theorem 4.1 (Without Lehmer). *Let $M, M_{p/q}$ be as usual and $A_{p,q}(t), S_1, S_2$ be the same as in Lemma 3.1. Then there exists C depending on S_1 and S_2 such that, for any coprime $(p, q) \in \mathbb{N}^2$ satisfying $S_1 < \frac{p}{q} < S_2$ and any noncyclotomic irreducible integer factor $g(t)$ of $A_{p,q}(t)$,*

$$C \max\{p, q\} \leqslant \deg g(t).$$

Proof. By Lemma 3.1(b), there exists D depending on S_1 and S_2 such that $|t_0| < 1 + \frac{D}{p}$ for any root t_0 of $A_{p,q}(t) = 0$. By Theorem 2.8, the result follows. □

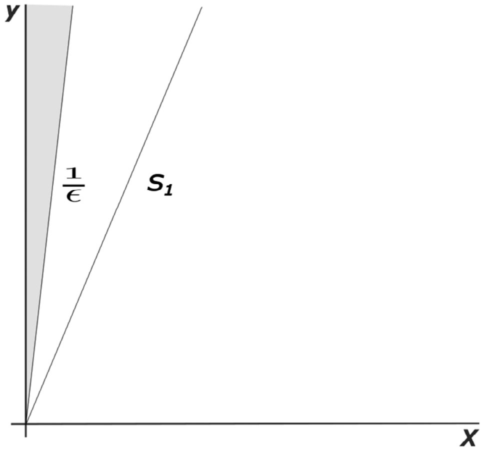

FIGURE 3 In the proof of Theorem 4.2, to verify the claim for (p, q) contained in the shaded region above, we need the estimations carried out in Lemmas 3.2–3.3.

Theorem 4.2 (With Lehmer). *Let $M, M_{p/q}$ be as usual and $A_{p,q}(t), S_1$ be the same as in Lemma 3.1. Assuming Lehmer's conjecture, there exists C depending on S_1 such that, for any coprime $(p, q) \in \mathbb{N}^2$ satisfying $\frac{p}{q} > S_1$ and any noncyclotomic irreducible integer factor $g(t)$ of $A_{p,q}(t)$,*

$$C \max\{p, q\} \leqslant \deg g(t). \tag{38}$$

Proof. Let ϵ be some sufficiently small number such that $S_1 < \frac{1}{\epsilon}$ (Figure 3). By Theorem 4.1, we find C' depending on S_1 and ϵ satisfying

$$C' \max\{p, q\} \leqslant \deg g(t)$$

for any coprime $(p, q) \in \mathbb{N}^2$ with $S_1 < \frac{p}{q} < \frac{1}{\epsilon}$ and any noncyclotomic irreducible integer factor $g(t)$ of $A_{p,q}(t)$.

Now suppose that there are sequences of positive real numbers $\{c_i\}_{i\in\mathbb{N}}$ and coprime pairs $\{(p_i, q_i)\}_{i\in\mathbb{N}}$ satisfying $\lim_{i\to\infty} c_i = 0$ and $\frac{p_i}{q_i} > \frac{1}{\epsilon}$. We further assume that, for each i, there exists a noncyclotomic irreducible integer factor $g_i(t)$ of $A_{p_i,q_i}(t)$ such that

$$\deg g_i(t) \leqslant c_i \max\{p_i, q_i\}. \tag{39}$$

We prove

Claim 4.3.

$$\overline{\lim_{i\to\infty}}\, \mathscr{M}(g_i(t)) < 1.176280818 \dots . \tag{40}$$

Once the above claim is established, the conclusion of Theorem 4.2 can be readily deduced as follows. Since the Mahler measure of any noncyclotomic polynomial is strictly bigger than 1.176280818 ... , by Lehmer's conjecture, this is a contradiction, and thus, there exists a constant

C'' depending on ϵ such that

$$C'' \max\{p,q\} \leqslant \deg g(t)$$

for any coprime $(p,q) \in \mathbb{N}^2$ satisfying $\frac{p}{q} > \frac{1}{\epsilon}$ and any noncyclotomic irreducible integer factor $g(t)$ of $A_{p,q}(t)$. As ϵ is arbitrarily, we may choose C satisfying the statement of Theorem 4.2.

Proof of Claim 4.3. We now give the proof of (40). By Theorem 2.2, Lemma 2.3, and multiplying by a power of t if necessary, we assume that both $A_{p_i,q_i}(t)$ and its irreducible factor $g_i(t)$ are monic integer polynomials.

As $c_i \to 0$, for i sufficiently large, the product of the moduli of the first $\lfloor c_i p_i \rfloor$ largest roots of $A(t^{-q_i}, t^{p_i}) = 0$ (and so $\mathscr{M}(g_i(t))$ as well) is bounded above by

$$\prod_{l=1}^{\lceil \frac{c_i p_i}{q_i} \rceil} \left(1 + \frac{d \log \frac{p_i/q_i}{l}}{p_i/q_i}\right)^2, \tag{41}$$

where d is some constant depending only on M by Lemma 3.3. We show (41) is strictly less than $1.176280818\ldots$ as $i \to \infty$. To simplify notation, let $r_i := p_i/q_i$. Taking logarithms to $\mathscr{M}(g_i(t))$ and (41),

$$\log \mathscr{M}(g_i(t)) \leqslant 2 \sum_{l=1}^{\lceil c_i r_i \rceil} \log \left(1 + \frac{d \log \frac{r_i}{l}}{r_i}\right).$$

Since

$$2 \textstyle\sum_{l=1}^{\lceil c_i r_i \rceil} \log \left(1 + \frac{d \log \frac{r_i}{l}}{r_i}\right) \leqslant 2 \textstyle\sum_{l=1}^{\lceil c_i r_i \rceil} \frac{d \log \frac{r_i}{l}}{r_i}$$

and

$$\lceil c_i r_i \rceil \log \lceil c_i r_i \rceil - \lceil c_i r_i \rceil < \log \lceil c_i r_i \rceil !,$$

it follows that

$$\log \mathscr{M}(g_i(t)) \leqslant 2 \sum_{l=1}^{\lceil c_i r_i \rceil} \frac{d \log \frac{r_i}{l}}{r_i} = \frac{2d \log \frac{r_i^{\lceil c_i r_i \rceil}}{\lceil c_i r_i \rceil !}}{r_i} = \frac{2d \lceil c_i r_i \rceil \log r_i - 2d \log \lceil c_i r_i \rceil !}{r_i}$$

$$< \frac{2d \lceil c_i r_i \rceil \log r_i - 2d(\lceil c_i r_i \rceil \log \lceil c_i r_i \rceil - \lceil c_i r_i \rceil)}{r_i} < \frac{2d \lceil c_i r_i \rceil \log \frac{r_i}{\lceil c_i r_i \rceil} + 2d \lceil c_i r_i \rceil}{r_i}. \tag{42}$$

We now consider two cases.

Case 1. If $c_i r_i \geqslant 1$, then (42) is bounded above by

$$\frac{4dc_i r_i \log \frac{1}{c_i} + 4dc_i r_i}{r_i} = -4dc_i \log c_i + 4dc_i. \tag{43}$$

Since $\lim_{i\to\infty} c_i = 0$, (43) (resp. (41)) converges to 0 (resp. 1) as $i \to \infty$.

Case 2. If $c_i r_i < 1$, then (42) is bounded above by

$$\frac{4d \log r_i + 4d}{r_i}. \tag{44}$$

As $r_i (= p_i/q_i) > \frac{1}{\epsilon}$ and d depends only on M, (44) is strictly less than $\log 1.176280818\dots$, provided that ϵ is sufficiently small. This completes the proof of (40) as well as the proof of Theorem 4.2. □

Remark 4.4. For $p > 0$ and $q < 0$, one analogously gets the following result. Suppose that $A_{p,q}(t)$ is given as in (17). Let S_1 be some positive constant such that $S_1 > \frac{b_j - b_n}{n-j}$ for every j ($0 \leqslant j \leqslant n-1$). Assuming Lehmer's conjecture, there exists C depending on S_1 such that, for any coprime pair (p,q) with $p > 0, q < 0, \frac{p}{|q|} > S_1$ and any noncyclic irreducible factor $g(t)$ of $A_{p,q}(t)$,

$$C \max\{p, |q|\} \leqslant \deg g(t).$$

We now use the $\mathrm{SL}_2(\mathbb{Z})$ action on the lattice $\mathbb{Z}^2$ that amounts to a monomial change of variables on $A(m,\ell)$ and hence on (p,q) and on $A_{p,q}(t)$. Combining this action with Theorem 4.2 and Remark 4.4, we obtain the following.

Theorem 4.5 (With Lehmer). *Let $M, M_{p/q}$, and $A_{p,q}(t)$ be as in Theorem 4.2. Assuming Lehmer's conjecture, the following statements hold.*

(1) *If $\frac{a_n - a_j}{n-j} < 0$ for all $0 \leqslant j \leqslant n-1$, then there exists C depending on M such that, for any coprime pair $(p,q) \in \mathbb{N}^2$ and any noncyclic irreducible factor $g(t)$ of $A_{p,q}(t)$,*

$$C \max\{p, q\} \leqslant \deg g(t).$$

(2) *If $\frac{a_n - a_j}{n-j} > 0$ for some $0 \leqslant j \leqslant n-1$, then there exists C depending on M such that, for any coprime pair $(p,q) \in \mathbb{N}^2$ with $\frac{p}{q} > S_A$ where S_A is the one given in (18) and any noncyclic irreducible factor $g(t)$ of $A_{p,q}(t)$,*

$$C \max\{p, q\} \leqslant \deg g(t).$$

Proof. For (1), if $\frac{a_n - a_j}{n-j} < 0$ for all $0 \leqslant j \leqslant n-1$, equivalently, it means that $c_{a_n,n} t^{-a_n q + np}$ is the leading term of $A_{p,q}(t)$ for every coprime pair $(p,q) \in \mathbb{N}^2$ with $p+q$ sufficiently large. Provided

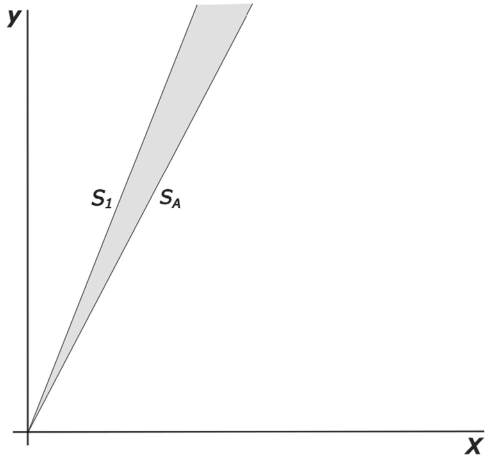

FIGURE 4 We have established the claim for (p, q) in the region bounded by the y-axis and the line with slope S_1, as proven in Theorem 4.2. To extend the claim further to the shaded region above, we apply a change of variables and reduce it to the previous case. See the proof of Theorem 4.5.

S_1 is chosen to be some sufficiently small ϵ, the claim is true for any coprime pair $(p, q) \in \mathbb{N}^2$ with $\frac{p}{q} > \epsilon$ by Theorem 4.2. If $\frac{p}{q} < \epsilon$ (or, equivalently, $\frac{q}{p} > \frac{1}{\epsilon}$), by switching p and q, one gets the desired result by an analog of Lemma 3.3 (see Remark 3.4) and similar arguments given in the proof of Theorem 4.2.

For (2), suppose $\frac{a_n - a_j}{n-j} > 0$ for some j $(0 \leqslant j \leqslant n-1)$ and let S_A be as in (18). Let $S_1 := S_A + \epsilon$ where ϵ some sufficiently small number. If $(p, q) \in \mathbb{N}^2$ with $\frac{p}{q} > S_1$, the result follows by Theorem 4.2. For $(p, q) \in \mathbb{N}^2$ satisfying $S_A < \frac{p}{q} < S_1$ (see Figure 4), first let (a, b) (resp. (r, s)) be a coprime pair such that $\frac{a}{b} = S_A$ (resp. $bs + ar = 1$). We further assume $a, b > 0$. Since $S_A (= \frac{a}{b}) < \frac{p}{q} < S_1 (= S_A + \epsilon)$,

$$\left|\frac{p}{q} - \frac{a}{b}\right| < \epsilon \Longrightarrow \left|\frac{bq}{aq - bp}\right| > \frac{1}{\epsilon},$$

and so,

$$\frac{rp + sq}{aq - bp} = \frac{-\frac{r}{b}(aq - bp) + \frac{bs+ar}{b}q}{aq - bp} = -\frac{r}{b} + \frac{1}{b}\frac{q}{aq - bp}, \tag{45}$$

implying

$$\left|\frac{rp + sq}{aq - bp}\right| > \frac{1}{b^2\epsilon} - \frac{r}{b}. \tag{46}$$

As given in Section 2.1, let μ, λ be a chosen meridian–longitude pair of T, a torus cross section of the cusp of M. By setting $\mu' := \mu^a \lambda^b, \lambda' := \mu^{-s}\lambda^r$, we change basis of T from μ, λ to μ', λ'. Also let $A'(m', \ell') = 0$ be the A-polynomial of M obtained from the new basis. Since $\mu = (\mu')^r(\lambda')^{-b}$,

$\lambda = (\mu')^s(\lambda')^a$ and $\mu^p\lambda^q = (\mu')^{rp+sq}(\lambda')^{-bp+aq}$, $\frac{p}{q}$-Dehn filling of M under the original basis corresponds to $\left(\frac{rp+sq}{-bp+aq}\right)$-Dehn filling of M under the new basis. Let $p' := rp + sq, q' := -bp + aq$. Note that

$$p' > 0, \quad q' < 0, \quad |p'/q'| > \frac{1}{b^2\epsilon} - \frac{r}{b} \tag{47}$$

by (45)–(46). Since $\frac{1}{b^2\epsilon} - \frac{r}{b}$ is sufficiently big, by Remark 4.4, there is C' such that, for any (p', q') satisfying (47) and any noncyclotomic irreducible factor of $A'(t^{-q'}, t^{p'}) = 0$,

$$C'\, p' \leqslant \deg g(t).$$

Equivalently, it means that the degree of any noncyclotomic irreducible factor of $A_{p,q}(t) = 0$ is bounded above and below by constant multiples of $rp + sq$. This completes the proof (recall that r and s are independent of p and q). □

Now we are ready to conclude the proof of Theorem 2.5.

Proof of Theorem 2.5. We show the theorem only for $p, q > 0$. The rest of the cases can be treated similarly. Let $A_{p,q}(t)$ be normalized as (17). For $\frac{a_n - a_j}{n-j} < 0$ for every j $(0 \leqslant j \leqslant n-1)$, the result follows by Theorem 4.5 (1), and so, it is assumed $\frac{a_n - a_j}{n-j} > 0$ for some j $(0 \leqslant j \leqslant n-1)$. Let S_A be given as in (18). For a coprime pair (p, q) satisfying $p/q > S_A$, since the claim was also proved in Theorem 4.5 (2), we further suppose $p/q < S_A$ and let

$$n_2 := \min_{0\leqslant j\leqslant n-1} \left\{ j \mid S_A = \frac{a_n - a_j}{n - j} \right\}.$$

Note that (a_{n_2}, n_2) is a corner of $\mathcal{N}(A)$ and S_A is the slope of the edge E_{S_A} connecting two corners (a_n, n) and (a_{n_2}, n_2) of $\mathcal{N}(A)$.

We distinguish two cases.

Case 1. If $n_2 = 0$ or $\frac{a_{n_2} - a_j}{n_2 - j} < 0$ for every $0 \leqslant j \leqslant n_2 - 1$, then $c_{a_{n_2}, n_2} t^{-a_{n_2}q + n_2 p}$ is the leading term of $f_{p,q}(t)$ for any $(p, q) \in \mathbb{N}^2$ with $\frac{p}{q} < S_A$ and $p + q$ sufficiently large. By interchanging p and q, one gets the desired result following similar steps shown in the proof of Theorem 4.5 (2).

Case 2. Now suppose $n_2 \neq 0$ and $\frac{a_{n_2} - a_j}{n_2 - j} > 0$ for some $0 \leqslant j \leqslant n_2 - 1$. Let

$$S_{A_2} := \max_{0\leqslant j\leqslant n_2-1} \left\{ \frac{a_{n_2} - a_j}{n_2 - j} \right\}.$$

Then, $S_A > S_{A_2}$ and S_{A_2} is the slope of the edge $E_{S_{A_2}}$ of $\mathcal{N}(f)$ adjacent to E_{S_A} (Figure 5). To show the claim for $(p, q) \in \mathbb{N}^2$ satisfying $S_{A_2} < \frac{p}{q} < S_A$, let $T, \mu, \lambda, \mu'(= \mu^a\lambda^b), \lambda'(= \mu^{-s}\lambda^r)$ be the same as in the proof of Theorem 4.5 (2). Also, we denote the A-polynomial of M obtained from μ', λ' by

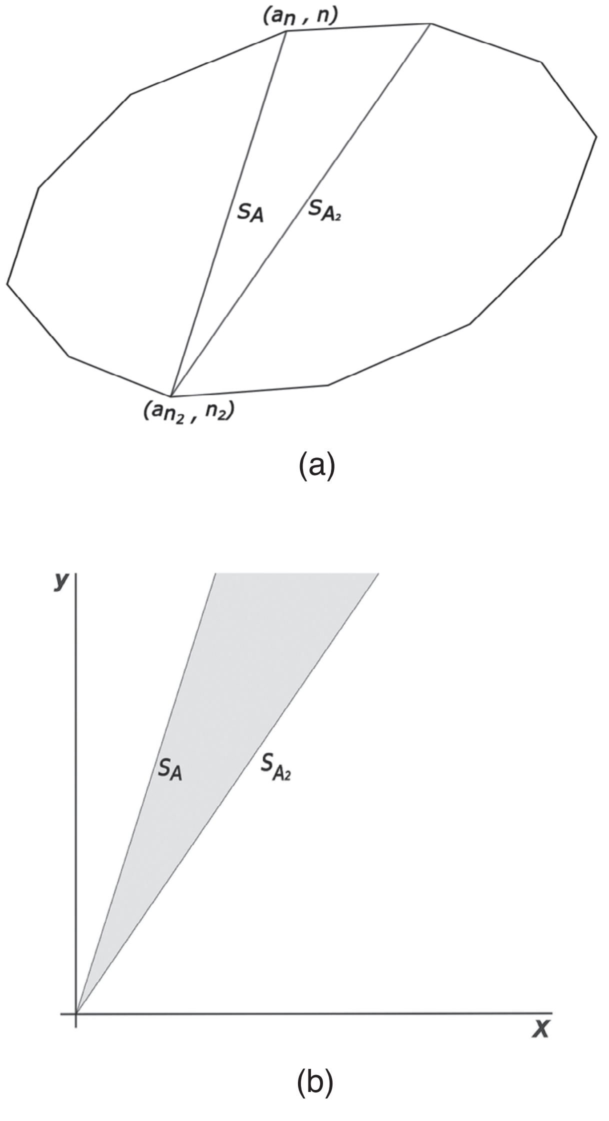

FIGURE 5 (a) If the Newton polygon of the A-polynomial M is the same as the one given in Figure 1, then the corners (a_n, n), (a_{n_2}, n_2) are shown as above in the picture, and S_A, S_{A_2} are the slopes of the depicted edges. (b) Once the claim is proven for (p, q) over the sector between the y-axis and the edge of slope S_A (as established in Theorem 4.5), the proof of it over the the shaded region is obtained through a change of variables.

$A'(m', \ell') = 0$ and assume its p'/q'-Dehn filling equation $A'_{p',q'}(t) := A'(t^{-q'}, t^{p'})$ is given as

$$\sum_{j=0}^{n'} \left(\sum_{i=a'_j}^{b'_j} c'_{i,j} t^{-q'i} \right) t^{p'j},$$

where $a'_j, b'_j \in \mathbb{Z}$. Under $p' = rp + sq$ and $q' = -bp + aq$, the set of coprime pairs (p, q) satisfying $S_{A_2} < \frac{p}{q} < S_A$ is transformed to the set of (p', q') satisfying $\frac{p'}{q'} > S_{A'}$ where

$$S_{A'} := \max_{0\leqslant j\leqslant n'-1} \frac{a'_{n'} - a'_j}{n' - j}.$$

Since the conclusion holds for any $A'_{p',q'}(t)$ with $(p',q') \in \mathbb{N}^2$, $\frac{p'}{q'} > S_{A'}$ by Theorem 4.5, equivalently, it holds for $A_{p,q}(t)$ with $(p,q) \in \mathbb{N}^2, S_{A_2} < \frac{p}{q} < S_A$ as well.

Analogously, using the same ideas, one can also prove the statement for $(p,q) \in \mathbb{N}^2$ satisfying $S_{A_3} < \frac{p}{q} < S_{A_2}$ where S_{A_3} is the slope of the edge $E_{S_{A_3}} (\neq E_A)$ of $\mathcal{N}(A)$ adjacent to $E_{S_{A_2}}$. Since $\mathcal{N}(A)$ has only finitely many edges, the desired result will follow eventually. □

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JOURNAL INFORMATION

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