LINKS WITH TRIVIAL ALEXANDER MODULE AND NONTRIVIAL MILNOR INVARIANTS

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Abstract. Cochran constructed many links with Alexander module that of the unlink and some nonvanishing Milnor invariants, using as input commutators in a free group and as an invariant the longitudes of the links. We present a different and conjecturally complete construction, that uses elementary properties of clasper surgery, and a different invariant, the tree-part of the LMO invariant. Our method also constructs links with trivial higher Alexander modules and nontrivial Milnor invariants.

1. Introduction

1.1. **History of the problem.** Two of the best studied topological invariants a link L in S^3 are its *Alexander module* $A(L)$ which measures the homology of the universal abelian cover of $S^3 - L$, and its collection of *Milnor invariants* $\bar{\mu}(L)$, which are concordance (and sometimes link homotopy) invariants, defined modulo a recursive indeterminacy. Let us say that L has *trivial* Alexander module (resp. Milnor invariants) if $A(L) = A(\mathcal{O})$ (resp. $\bar{\mu}(L) = \bar{\mu}(\mathcal{O}) = 0$) for an unlink $\mathcal{O}$. Despite the indeterminacy of the Milnor invariants, note that the vanishing of all Milnor invariants is a well-defined statement.

Using the language of *longitudes* λ_i of components of L, Milnor showed that a link L has vanishing Milnor invariants iff $\lambda_i(L) \subset \pi_\omega$ for all i, where $\pi = \pi_1(S^3 - L)$ and $\pi_\omega = \cap_{n=1}^\infty \pi_n$ is the intersection of the *lower central series* π_n of π, defined by $\pi_1 = \pi$ and $\pi_{n+1} = [\pi_n, \pi]$, see [Mil54]. L has trivial Alexander module iff there is a map $\pi \to F/[[F,F],[F,F]]$ which induces an isomorphism $\pi/[[\pi,\pi],[\pi,\pi]] \cong F/[[F,F],[F,F]]$.

It is natural to ask how independent are the conditions of trivial Alexander module and trivial Milnor invariants. In a sense, this question asks for a comparison between the lower central series and the commutator series of a link group.

In one direction, Levine showed that the vanishing of the Milnor invariants of a link L implies that a localization $A(L)_S$ of its Alexander module (although not the Alexander module itself) vanishes, where $S \subset \mathbb{Z}[t_1^{\pm 1}, \dots, t_r^{\pm 1}]$ is the multiplicative set of polynomials that evaluate to ± 1 at $t_1 = \dots = t_r = 1$; see [Lev83]. A boundary link has vanishing Milnor invariants, and its Alexander module splits as a direct sum of a trivial module and a torsion module. It was shown in [GL02] that all torsion modules with the appropriate symmetry can be realized.

In the opposite direction, if L has trivial Alexander module, then it is known that some low order Milnor invariants vanish, [Lev83, Tra84]. For example, all nonrepeated (link homotopy) invariants with at most 5 indices vanish. On the other hand, Cochran constructed a class of links with trivial Alexander module and nontrivial Milnor invariants; such links are not even be concordant to homology boundary links.

Cochran's construction used iteration, and used as a pattern certain elements in the lower central series of the free group. There is enough explicitness and control on the iteration that enabled Cochran to compute the longitudes directly and verify that these links have vanishing Alexander modules. Further, a geometric interpretation of Milnor invariants in terms of cycles on Seifert surfaces allowed Cochran to conclude that the constructed links have nontrivial Milnor invariants.

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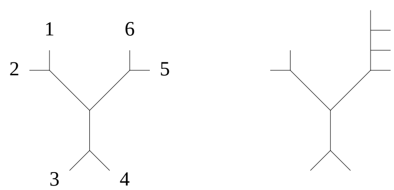
As an elementary application of the calculus of claspers, we will construct a plethora of links with vanishing Alexander module. For these links, we can compute the tree part of the LMO invariant (which can be identified with Milnor invariants, [HM00]), using formal Gaussian integration. As a result, we will construct many (and conjecturally all) links with trivial Alexander module and nontrivial Milnor invariants. The next definition explains the patterns that we will use in our construction.

Definition 1.1. Let $\mathcal{A}^{\mathrm{tr}}(r)$ (or simply, $\mathcal{A}^{\mathrm{tr}}$, in case r is clear) denote the vector space over $\mathbb{Q}$ generated by vertex-oriented unitrivalent trees, whose univalent vertices are labeled by r colors, modulo the AS and IHX relation. $\mathcal{A}^{\mathrm{tr}}(r)$ is a graded vector space, where the degree of a graph is half the number of vertices. We will call a tree of degree 1 (with two univalent vertices and no trivalent ones) a *strut*.

A *pattern* β is an element of $\mathcal{A}^{\mathrm{tr}}(r)$ which is represented by a tree which has a trivalent vertex v such that $\beta - v$ has no strut components.

The next figure gives some examples of nonvanishing patterns:



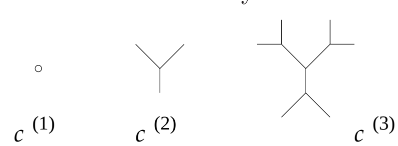
Theorem 1.1. *For every nonvanishing pattern $\beta \in \mathcal{A}^{\mathrm{tr}}_m(r)$ there exists a link $L(\beta)$ with r components such that $A(L(\beta)) = A(\mathcal{O})$, all Milnor invariants of degree less than m vanish and some Milnor invariant of degree m do not.*

Our construction adapts without change to the case of links with trivial *higher Alexander modules*. Although classical, these modules appeared only recently in work of Cochran-Orr-Teichner [COT03] and subsequent work of Cochran, [Coc04]. Given a group π, consider its *commutator series* defined by $\pi^{(0)} = \pi$ and $\pi^{(n+1)} = [\pi^{(n)}, \pi^{(n)}]$.

Definition 1.2. We will say that a link L in a homology sphere M has *trivial nth Alexander module* if it has a map $\pi \longrightarrow F/F^{(n+1)}$ which induces an isomorphism $\pi/\pi^{(n+1)} \cong F/F^{(n+1)}$, where $\pi = \pi_1(M - L)$.

The next definition explains the n-patterns which we will use.

Definition 1.3. Let $c^{(n)}$ be a unitrivalent tree defined by



In other words, we are adding two univalent vertices in $c^{(n+1)}$ to each of the univalent vertices of $c^{(n)}$. An *n-pattern* $\beta^{(n)}$ is an element of $\mathcal{A}^{\mathrm{tr}}(r)$ which is represented by a tree $\beta^{(n)}$ such that $c^{(n)} \subset \beta^{(n)}$ and $\beta^{(n)} - c^{(n)}$ has no strut components.

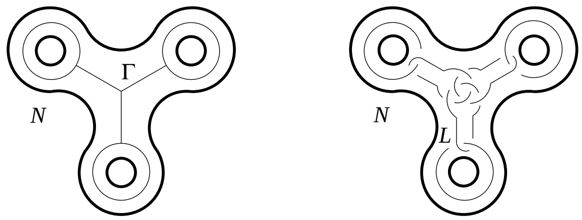
The proof of Theorem 1.1 generalizes without change to the following

Theorem 1.2. *For every nonvanishing n-pattern $\beta^{(n)} \in \mathcal{A}^{\mathrm{tr}}_m(r)$ there exists a link $L(\beta^{(n)})$ with r components with trivial nth Alexander module, such that all Milnor invariants of degree less than m vanish and some Milnor invariant of degree m do not.*

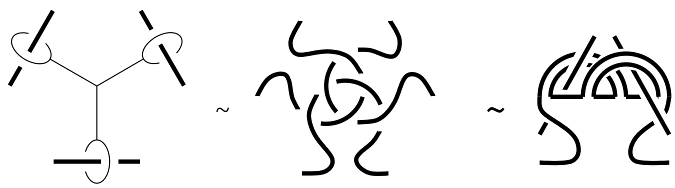
2. Constucting links by surgery on claspers

2.1. **What is surgery on a clasper?** As we mentioned in the introduction, we will construct links of Theorem 1.2 using *surgery on claspers*. Since claspers play a key role in geometric constructions, as well as in the theory of finite type invariants, we include a brief discussion here. For a reference on claspers and their associated surgery, we refer the reader to [Gus00, Hab00] and also to [GGP01, Section 2] (where claspers were called clovers instead). It suffices to say that a clasper is a thickening of a trivalent graph, and it has a preferred set of loops, called the leaves. The degree of a clasper is the number of trivalent vertices (excluding those at the leaves). With our conventions, the smallest clasper is a Y-clasper (which has degree one and three leaves), so we explicitly exclude struts (which would be of degree zero with two leaves).

A clasper G of degree 1 is an embedding $G : N \to M$ of a regular neighborhood of the graph Γ



in a 3-manifold M. Surgery on G can be described by cutting $G(N)$ from M (which is a genus 3 handlebody), twisting by a fixed diffeomorphism of its boundary (which acts trivially on the homology of the boundary) and gluing back. We will denote the result of surgery by M_G. Alternatively, we can describe surgery on G by surgery on a framed six component link (the image of L) in M. The six component link consists of a 0-framed Borromean ring and an arbitrarily framed three component link, the so-called *leaves* of G. If one of the leaves bounds a 0-framed disk disjoint from the rest of G, then surgery on G does not change the ambient 3-manifold M, although it can change an embedded link in M. In particular, surgery on a clasper of degree 1 is shown as follows:



In general, surgery on a clasper G of degree n can be described in terms of simultaneous surgery on n claspers $G_1, \dots, G_n$, which are obtained from G after breaking its edges and inserting Hopf links as follows:

2.2. **A basic principle.** Surgery on a clasper is described by twisting by a surface diffeomorphism that acts trivially on homology, thus we have the basic principle:

$$\boxed{\text{Clasper surgery preserves the homology}}$$

Surgery on claspers with leaves of a resticted type has already been studied and used successfully in [GR04] (where the leaves were assumed null homologous in a knot complement), [GL01b] (and where the leaves where null homotopic) and [GK04] (where the leaves where in the kernel of a map to a free group). It is important to study not only 3-manifolds but rather pairs of 3-manifolds together with a representation of their fundamental group into a fixed group. Claspers adapt well to this point of view, as we explain next.

Consider a pair (N, ρ) of a 3-manifold N (possibly noncompact) and a representation $\rho : \pi_1(N) \to \Gamma$ for some group Γ. Consider a clasper $G \subset N$ whose leaves are mapped to 1 under ρ. We will call such claspers *ρ-null*, or simply *null*, if ρ is clear. Surgery on G gives rise to a 4-manifold W whose boundary consists of one copy of N and one copy of N_G. We may think that W is obtained by attaching $6n$ 2-handles on $N \times I$, where $n = \text{degree}(G)$. Since the cores of these handles lie in the kernel of ρ, it follows that ρ extends over W, and in particular restricts to a representation ρ_G on the end N_G of W.

Lemma 2.1. We have $H_*(N, \rho) \cong H_*(N_G, \rho_G)$.

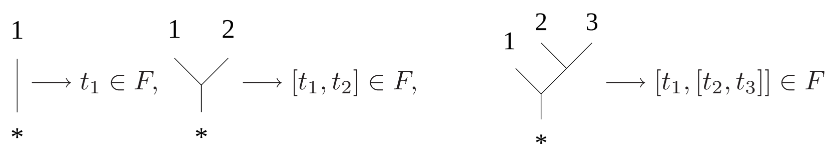
Proof. Let $\widetilde{N}$ (resp. $\widetilde{N_G}$) denote the cover of N (resp. N_G) corresponding to ρ (resp. ρ_G). Surgery on G is equivalent to surgery on a collection $\{G_1, \dots, G_k\}$ of degree 1 claspers, constructed by inserting Hopf links in the edges of G. Each G_i lifts to a collection $\widetilde{G}_i$ of claspers in $\widetilde{N}$; let $\widetilde{G} = \widetilde{G}_1 \cup \ldots \widetilde{G}_k$. Then, $\widetilde{N_G}$ can be identified with $(\widetilde{N})_{\widetilde{G}}$. Since clasper surgery preserves homology, the result follows. $\square$

We will adapt the above lemma in the following situation. Suppose that G is a clasper in the complement of an unlink $X_0 = S^3 - \mathcal{O}$ of r components whose leaves are null homologous in X_0, and let (M, L) denote the result of surgery along G on the pair $(S^3, \mathcal{O})$. It follows that G lifts to a family $\widetilde{G}$ of claspers in $\widetilde{X}_0$ (the universal abelian cover of X) and that $\widetilde{X}$ is obtained from $\widetilde{X}_0$, by surgery on $\widetilde{G}$, where $X = M - L$. Since $A(L) = H_1(\widetilde{X}, \widetilde{x})$, and clasper surgery preserves homology, it follows that $A(M, L) = A(\mathcal{O})$.

Remark 2.2. There are two known cases where surgery on a null clasper $G \subset X_0$ gives rise to a link (M, L) with vanishing Milnor invariants.
(a) If the leaves of G are null homotopic in X_0, then the constructed links would be boundary links, as was observed and used in [GK04]. Boundary links have vanishing Milnor invariants.
(b) If G is a connected clasper with at least one loop, then (M, L) is concordant to $(S^3, \mathcal{O})$, [GL01a] and also [CT04]. Concordance preserves Milnor invariants.

With a bit more effort, we can arrange that $M = S^3$. For this, it suffices to assume that each connected component G_i of G has a 0-framed leaf l_i, such that the union of the leaves $\{l_i\}$ is an unlink in S^3.

To finalize the construction of Theorem 1.1, consider a pattern β, and a vertex v of β such that $\beta - v = T_1 \cup T_2 \cup T_3$ where T_i are rooted trees which are not struts. Each rooted tree T corresponds to an element $\phi(T) \in F$ via a map defined in pictures by:



If T is not a strut, then $\phi(T) \in [F, F]$. Given β as above, we will choose a clasper $G(\beta)$ of degree 1 such that its three leaves l_i satisfy $l_i = \phi(T_i) \in [F, F]$, for $i = 1, 2, 3$. Then, $L(\beta)$ is obtained from the unlink by clasper surgery on $G(\beta)$.

Finally, let us modify the above discussion for the construction of Theorem 1.2. Given an n-pattern $\beta^{(n)}$, let $G(\beta^{(n)})$ be a tree clasper of degree n in X_0, which consists of $c^{(n+1)}$ and 2^{n+1} leaves l_i (one in each univalent vertex of $c^{(n+1)}$). There is a 1-1 correspondence between the connected components T_i of $\beta^{(n)} - c^{(n)}$ and the leaves l_i of $G(\beta^{(n)})$. We will choose these leaves so that $l_i = \phi(T_i) \in F$, and we will let $L(b^{(n)})$ be obtained from the unlink by clasper surgery on $G(\beta^{(n)})$.

We need to show that $L(b^{(n)})$ has trivial nth Alexander module. Indeed, using the figures above that describe clasper surgery, it follows that clasper surgery on $G(\beta^{(n)})$ is equivalent to surgery on a clasper $G'(\beta^{(n)})$ of degree 1 whose leaves lie in $F^{(n)}$. This implies that the nth Alexander module of $L(b^{(n)})$ is trivial.

We end this section with a comment on pictures. To get *pictures* of the constructed links, one may use various descriptions of surgery on a clasper that were discussed at length by Goussarov and Habiro at [Gus00, Hab00]. From our point of view though, these pictures are complicated and unnecessary, since not only claspers describe surgery adequately, but also the invariants which we will use behave well with respect to clasper surgery. This is the content of the next section.

3. Computing the tree part of the Aarhus integral

3.1. **The Aarhus integral in brief.** As was stated in the discussion of Theorem 1.1, we will not compute the Milnor invariants of the links $L(\beta)$ constructed via clasper surgery, but rather we will compute the tree-part of their Aarhus integral. The Aarhus integral is a graph version of stationary phase approximation that was introduced at [BNGRT02a, BNGRT02b, BNGRT04]. Despite its intimidating name, it is a rather harmless combinatorial object which we now describe.

Consider a framed link $C \subset S^3 - \mathcal{O}$ and let $(M, L) = (S^3, \mathcal{O})_G$ denote the result of surgery on C. That is, M is the 3-manifold obtained from S^3 by surgery on C and L is the image of $\mathcal{O}$ after surgery. Assuming that

M is a rational homology sphere (i.e., that the linking matrix of C has nonzero determinant) the Aarhus integral $Z(M,L)$ can be computed by the Kontsevich integral of the link $\mathcal{O}\cup C$ by integration as follows:

$$Z(M,L)=\int dX\, Z(S^3,\mathcal{O}\cup C)$$

(where X is a set of variables in 1-1 correspondence with the components of C). Let us briefly recall from [BNGRT02b] how this integration works. Consider an element

$$s=\exp\left(\frac{1}{2}\sum_{x,y\in X} \overset{x}{\underset{y}{|}} Q_{xy}\right)R,$$

with R a series of graphs that do not contain a strut whose legs are colored by X. Notice that Q and R, the X-*strutless part* of s, are uniquely determined by s. Then, the integration $\int dX(s)$ glues all the X-colored legs of R pairwise, using the negative inverse of the matrix Q. That is, when two legs x,y of R are glued, the resulting graph is multiplied by $-Q^{xy}$, the negative inverse of the matrix Q_{xy}.

It follows immediately that the *tree-part* $Z^{\mathrm{tr}}(M,L)$ of $Z(M,L)$ depends only on the tree-part $Z^{\mathrm{tr}}(S^3,\mathcal{O}\cup C)$ of $Z(S^3,\mathcal{O}\cup C)$.

3.2. **Claspers and the Aarhus integral.** Let us adapt the above discussion when the link C is one that describes clasper surgery. Consider a null clasper $G\subset S^3-\mathcal{O}$ of degree 1 constructed from a pattern β and let $(M,L)=(S^3,\mathcal{O})_G$. Let $Z^{\min}(M,L)$ denotes the *lowest degree nonvanishing tree part* of $Z^{\mathrm{tr}}(M,L)$. Assuming that the pattern is nonvanishing, and after we choose string-link representatives of $L\cup G$, we will show that

Proposition 3.1. *We have*

$$Z^{\min}(M,L)=\beta\in\mathcal{A}^{\mathrm{tr}}$$

It is clear that this concludes Theorem 1.1.

Proof. (of Proposition 3.1) Surgery on G is equivalent to surgery on a 6 component link $C=C^e\cup C^l$; see Section 2.1. C^e is a borromean link and C^l consists of the leaves of G. In the obvious basis, the linking matrix of C and its negative inverse are given as follows:

$$\begin{pmatrix} 0 & I\\ I & \mathrm{lk}(C^l_i,C^l_j)\end{pmatrix} \quad\text{and}\quad \begin{pmatrix} \mathrm{lk}(C^l_i,C^l_j) & -I\\ -I & 0\end{pmatrix}.$$

In particular, a univalent vertex labeled by a leaf has to be glued to a univalent vertex labeled by the corresponding edge. Let $A_i=\{C^e_i,C^l_i\}$ denote the arms of G for $i=1,2,3$. It is a key fact that surgery on any proper subcollection of the set $\{A_1,A_2,A_3\}$ of arms does not change the pair $(S^3,\mathcal{O})$. In other words, alternating with respect to the 8 subsets of the set of arms we have that

$$Z([(S^3,\mathcal{O}),G])=Z([(S^3,\mathcal{O}),\{A_1,A_2,A_3\}])$$

The nontrivial contributions to the left hand side come from the $(\mathcal{O}\cup C)$-strutless part of $Z(S^3,\mathcal{O}\cup C)$ that consists of graphs with legs on A_1 and on A_2 and on A_3.

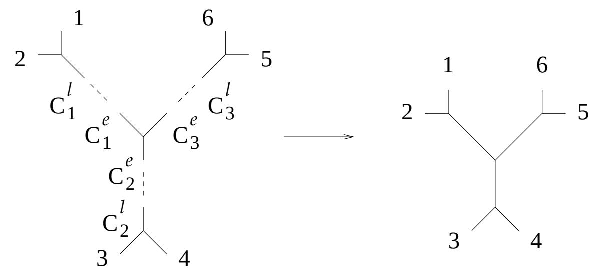
What kind of diagrams in $Z^{\mathrm{tr}}(S^3,\mathcal{O}\cup C)$ contribute to the above sum? Consider a disjoint union D of trees whose legs are labeled by $\mathcal{O}\cup C$. D must have a leg (i.e., univalent vertex) labeled by C^l_i or by C^e_i for each $i=1,2,3$. If D has a leg labeled by C^l_i, then due to the shape of the gluing matrix, D must have a C^e_i-labeled leg. Thus, in all cases, D must have legs labeled by all three edges C^e_i of G.

Consider a tree T labeled by $\mathcal{O}\cup C$. If T has a C^e_i-labeled leg, then it must either have legs labeled by all three edges of G, or else it must have a leg labeled by C^l_i. Indeed, C^e_i is an unknot in a ball disjoint from $\mathcal{O}\cup C-\{C^l_i\}$, thus the rest of the trees have vanishing coefficient in $Z^{\mathrm{tr}}(S^3,\mathcal{O}\cup C)$.

Consider further a *vortex* Y (that is, a unitrivalent graph of the shape Y with three univalent vertices and one trivalent one) whose legs are labeled by three leaves of G. Then, the coefficient of Y in $Z(S^3,\mathcal{O}\cup C)$ is 1.

Consider further a tree T with one univalent vertex labeled by a leaf C^l_i of G and all other vertices labeled by $\mathcal{O}$. Recall the corresponding rooted tree T_i which is a component of $\beta-v$. Then the coefficient of T

in $Z^{\rm tr}(S^3, \mathcal{O} \cup C)$ is zero if $\deg(T) < \deg(T_i)$ and equals to 1 if $T = T_i$. This, together with the above discussion and the gluing rules concludes the proof of Proposition 3.1. The argument is best illustrated by the following figure:



□

The above proposition and its proof generalize easily to the case of claspers G corresponding to nonvanishing n-patterns $\beta^{(n)}$. In that case, if (M, L) denote the corresponding link, we still have that

$$Z^{\rm min}(M, L) = \beta^{(n)} \in \mathcal{A}^{\rm tr}$$

which implies Theorem 1.2.

Remark 3.2. In the above discussion we have silently chosen dotted Morse link representatives (or equivalently, string-link representatives) and we ought to have normalized the Aarhus integral. But this does not affect the lowest degree nonvanishing tree part.

The links constructed by clasper surgery in Theorem 1.1 include the links that Cochran constructed via Seifert surfaces.

Question 3.3. Does Section 2 construct every link with trivial Alexander module?

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