Math. 2403, Practice Final Solutions

1. Solve the initial value problem

$$\frac{dx}{dt} = \frac{t^2}{x + t^3 x}, \quad x(0) = -2.$$

$$x(t) = -(\frac{2}{3}\ln(1+t^3) + 4)^{\frac{1}{2}}$$

2. A large tank is initially filled with 500 L of water containing 50 kg of salt. A brine containing 2 kg of salt per liter is pumped into the tank at a rate of 5 L/min. The well-mixed solution is pumped out at the same rate. After 10 min, a leak develops in the tank and an additional liter per minute of mixture flows out of the tank. What is the concentration, in kilograms per liter, of salf in the tank 20 min after the leak develops?

Let $x(t)$ be the amount of salt in the tank at time t. For $t \leq 10$ we have

$$\frac{dx}{dt} = 10 - \frac{x}{100}, \quad x(0) = 50,$$

which gives

$$x(t) = 1000 - 950e^{-\frac{t}{100}}.$$

The amount of salt in the tank at time $t = 10$ is then $1000 - 950e^{-\frac{1}{10}}$.

Let now $x(t)$ be the amount of salt in the tank t minutes after the leak has developed and let $V(t) = 500 - t$ be the volume of water t minutes after the leak has developed. The equation for x is now

$$\frac{dx}{dt} = 10 - 6\frac{x}{500 - t}, \quad x(0) = 1000 - 950e^{-\frac{1}{10}}.$$

The solution is now

$$2(500 - t) - \frac{950e^{-\frac{1}{10}}}{500^6}(500 - t)^6.$$

The concentration of salf in the tank 20 min after the leak develops is then

$$\frac{x(20)}{480} = 2 - \frac{950e^{-\frac{1}{10}}}{500^6}480^5$$

3. Solve the initial value problem

$$y'' - y = \sin t - e^{2t}, \quad y(0) = 1, y'(0) = -1.$$

$$y_h(t) = c_1 e^t + c_2 e^- t$$

We are therefore looking for a particular solution of the form

$$y_p(t) = A\cos t + B\sin t + Ce^{2t}.$$

Therefore we need

$$-2A\cos t - 2B\sin t + 3Ce^{2t} = \sin t - e^{2t}$$

which implies $A = 0, B = -1/2, C = -1/3$. Thus

$$x(t) = c_1 e^t + c_2 e^- t - \frac{1}{2}\sin t - \frac{1}{3}e^{2t}.$$

To match the initial conditions we must have

$$c_1 + c_2 - \frac{1}{3} = 1, \quad \text{and } c_1 - c_2 - \frac{1}{2} - \frac{2}{3} = -1.$$

Solving this system we get $c_1 = 3/4, c_2 = 7/12$.

4. Find a general solution of

$$\mathbf{x}'(t) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -5 & -4 \end{pmatrix} \mathbf{x}(t).$$

The eigenvalues are $\lambda = -1$ (with multiplicity 2) and $\lambda = -2$. $\lambda = -2$ has eigenvector

$$\mathbf{v_3} = \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}$$

and $\lambda = -1$ has two linearly independent generalized eigenvectors (solutions of $(A+I)^2\mathbf{v} = \mathbf{0}$)

$$\mathbf{v_1} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \quad \mathbf{v_2} = \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}.$$

The solution is

$$\mathbf{x}(t) = c_1 e^{-t}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + c_2 e^{-t}\left(\begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} + t\begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}\right) + c_3 e^{-2t}\begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}.$$

5. Let

$$A = \begin{pmatrix} 3 & -2 \\ 0 & 3 \end{pmatrix}.$$

(a) Find e^{At}.

$$e^{At} = \begin{pmatrix} e^{3t} & -2te^{3t} \\ 0 & e^{3t} \end{pmatrix}$$

(b) Find the solution of

$$\mathbf{x}'(t) = A\mathbf{x}(t), \quad \mathbf{x}(0) = \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

$$\mathbf{x}(t) = e^{At}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} e^{3t} & -2te^{3t} \\ 0 & e^{3t} \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

(c) Draw the phase portrait for $\mathbf{x}'(t) = A\mathbf{x}(t)$.

It is a source (improper node for the case of a repeated eigenvalue different from zero). All solution curves are tangent at 0 to the line spanned by the eigenvector

$$\mathbf{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

of A.

6. A mass-spring system contains two masses whose displacement functions $x(t), y(t)$ satisfy the differential equations

$$x'' = -4x + y, \quad y'' = x - 4y.$$

Find the two natural frequencies of the system, its two natural modes of oscillation, and write down a general solution of the system.

The two natural frequencies are $\omega_1 = \sqrt{3}$ and $\omega_2 = \sqrt{5}$ and the general solution is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = (a_1 \cos\sqrt{3}t + b_1 \sin\sqrt{3}t)\begin{pmatrix} 1 \\ 1 \end{pmatrix} + (a_2 \cos\sqrt{5}t + b_2 \sin\sqrt{5}t)\begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

and the two terms above are the natural modes of oscillation.

7. Find the critical points of

$$x' = 2x - x^2 - xy$$
$$y' = 3y - 2y^2 - xy$$

and classify their type and stability. Draw the phase portrait for the system. Indicate the directions of the vector field at various points.

The critical points are $(0,0), (0,3/2), (2,0), (1,1)$. The Jacobi matrix for the system is

$$D\mathbf{f}(x,y) = \begin{pmatrix} 2 - 2x - y & -x \\ -y & 3 - 4y - x \end{pmatrix}.$$

We have

$$D\mathbf{f}(0,0) = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$$

so $(0,0)$ is a source (unstable).

$$D\mathbf{f}(0,\frac{3}{2}) = \begin{pmatrix} \frac{1}{2} & 0 \\ -\frac{3}{2} & -3 \end{pmatrix}$$

so $(0,3/2)$ is a saddle (unstable).

$$D\mathbf{f}(2,0) = \begin{pmatrix} -2 & -2 \\ 0 & 1 \end{pmatrix}$$

so $(2,0)$ is also a saddle.

$$D\mathbf{f}(1,1) = \begin{pmatrix} -1 & -1 \\ -1 & -2 \end{pmatrix}$$

which has negative eigenvalues so $(1,1)$ is a sink (asymptotically stable).

8. Find $L\{f(t)\}$, where

$$f(t) = \begin{cases} e^{-t} & \text{for } 0 < t < 1, \\ 1 & \text{for } 1 < t < 2, \end{cases}$$

and $f(t)$ has period 2.

$$F(s) = \frac{\int_0^1 e^{-st}e^{-t}dt + \int_1^2 e^{-st}dt}{1-e^{-2s}} = \frac{1}{1-e^{-2s}}\left(\frac{1}{s+1}(1-e^{-(s+1)}) + \frac{1}{s}(e^{-s}-e^{-2s})\right).$$

9. Use the method of Laplace transforms to solve the initial value problem

$$\begin{cases} y'' + 5y' - 6y = 21e^t \\ y(0) = -1, y'(0) = 9. \end{cases}$$

Denote by $Y(s)$ the Laplace transform of $y(t)$. Taking the Laplace transform of the equation we get

$$s^2Y + s - 9 + 5(sY+1) - 6Y = \frac{21}{s-1}.$$

Therefore

$$Y(s) = \frac{21}{(s-1)^2(s+6)} - \frac{s-4}{(s-1)(s+6)} = \frac{3}{(s-1)^2} - \frac{1}{s+6}.$$

Taking the inverse transform we now obtain

$$y(t) = 3te^t - e^{-6t}.$$