**Math. 2403, Practice Test1, Solutions**

1. (a) Solve the initial value problem

$$\frac{dx}{dt} = \frac{2}{3x+1}, \quad x(0) = 3.$$

This is a separable equation. Therefore

$$\int (3x+1)dx = \int 2dt$$

and so

$$\frac{3x^2}{2} + x = 2t + c$$

which gives

$$x = \frac{-1 \pm \sqrt{1 + 6(2t+c)}}{3}.$$

Since we must have $x(0) = 3$ and this leads to

$$3 = \frac{-1 \pm \sqrt{1+6c}}{3},$$

which is equivalent to

$$10 = \pm\sqrt{1+6c}.$$

Therefore only the + solution makes sense and then $c = 33/2$. So finally

$$x = \frac{-1 + \sqrt{12t + 100}}{3}.$$

(b) Draw the associated slope field.
(c) Draw the phase portrait.
2. Solve the initial value problem

$$\frac{dx}{dt} - \frac{2x}{t} = t^2 \cos t, \quad x(\pi) = 1.$$

This is a linear first order equation. We calculate the integrating factor

$$\mu(t) = e^{-\int \frac{2}{t} dt} = e^{-2\ln t} = \frac{1}{t^2}.$$

Therefore, multiplying both sides by $\mu(t)$ and then integrating, we get

$$x \cdot \frac{1}{t^2} = \int \cos t \, dt = \sin t + c$$

and so

$$x = t^2(\sin t + c).$$

Now

$$1 = x(\pi) = \pi^2(0 + c) \quad \Longrightarrow \quad c = \frac{1}{\pi^2}.$$

Therefore

$$x = t^2(\sin t + \frac{1}{\pi^2}).$$

3. A falling object is subjected to air resistance that is proportional to the velocity of the object. Suppose that the proportionality constant is equal to $k > 0$, the object has mass $m$, and the acceleration due to gravity is equal to $g$.
(a) Derive an equation governing the velocity $v$ of the object.

Let $F_g$ denote the force acting on the object that is due to gravity and let $F_r$ denote the force due to air resistance. Denote by $v(t)$ the velocity of the object at time $t$. We know that $F_r = kv$. Then from Newton's Law

$$m \cdot \text{acceleration} = m\frac{dv}{dt} = F_g - F_r = mg - kv$$

which gives

$$\frac{dv}{dt} + \frac{k}{m}v = g.$$

(b) Solve the differential equation and determine the limiting (or terminal) velocity of the object.

The integrating factor is

$$\mu(t) = e^{\frac{k}{m}t}.$$

Therefore we obtain

$$ve^{\frac{k}{m}t} = \int e^{\frac{k}{m}t} g\, dt = \frac{mg}{k}e^{\frac{k}{m}t} + c$$

and finally

$$v(t) = \frac{mg}{k} + ce^{-\frac{k}{m}t}.$$

To find the limiting velocity of the object we have to find the limit of $v(t)$ as $t \to +\infty$. But

$$\lim_{t\to+\infty} v(t) = \lim_{t\to+\infty} \left(\frac{mg}{k} + ce^{-\frac{k}{m}t}\right) = \frac{mg}{k}.$$

4. Find the general solution of the system

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \mathbf{x}.$$

Draw a phase portrait for the system. The origin is a critical point. Describe its type and stability.

The characteristic equation is

$$det(A - \lambda I) = (-2 - \lambda)^2 - 1 = (\lambda + 1)(\lambda + 3) = 0$$

which gives the two eigenvalues $\lambda_1 = -1, \lambda_2 = -3$. The corresponding eigenvectors are

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Therefore the general solution is

$$\mathbf{x}(t) = c_1 e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

The origin is asymptotically stable. It is an improper node (nodal sink).

5. Solve the initial value problem

$$2y'' - 7y' + 3y = 0, \quad y(0) = 1, \; y'(0) = -2.$$

Describe the behavior of the solution for increasing $t$.

The characteristic equation is $2r^2 - 7r + 3 = 0$ which has roots $r_1 = 1/2, r_2 = 3$. Therefore the general solution is

$$y(t) = c_1 e^{\frac{t}{2}} + c_2 e^{3t}.$$

Solving for $c_1$ and $c_2$ we get $c_1 = 2, c_2 = -1$ and so the solution is

$$y(t) = 2e^{\frac{t}{2}} - e^{3t}.$$

The solution $y \to -\infty$ as $t \to \infty$.