Math. 2403, Test1, 2/7/2008 **Name**: SOLUTIONS

Please work carefully, show your work and justify your answers. Each problem is worth 6 points.

1. (a) Draw the slope field of

$$\frac{dy}{dx} = y(1-y)(2-y).$$

(b) Draw the phase portrait and determine the stability of equilibria.

—-<—-0—>—1—<—2—->—-

Equilibria are at $0, 1$, and 2. 1 is stable and $0, 2$ are unstable.

2. Find the general solution of the differential equation

$$\frac{dy}{dt} + \frac{t}{t^2+1} y = \frac{t}{t^2+1}.$$

This is a linear equation. The integrating factor is

$$\mu(t) = e^{\int \frac{t}{t^2+1} dt} = e^{\frac{1}{2}\ln(t^2+1)} = \sqrt{t^2+1}.$$

Therefore we get

$$\sqrt{t^2+1}\, y = \int \sqrt{t^2+1} \frac{t}{t^2+1}\, dt = \int \frac{t}{\sqrt{t^2+1}}\, dt = \sqrt{t^2+1} + c$$

so finally

$$y(t) = 1 + \frac{c}{\sqrt{t^2+1}}.$$

3. Verify that the functions $\cos(\ln t), \sin(\ln t)$ form a fundamental set of solutions of the differential equation

$$t^2 y'' + t y' + y = 0$$

on the interval $(0, +\infty)$. Write down the general solution of the equation.

First, by plugging the functions into the equation we check that they are solutions. For instance for $y_1(t) = \cos(\ln t)$ we have

$$t^2 y_1''(t) + t y_1'(t) + y_1(t) = t^2(-\frac{1}{t^2}\cos(\ln t) + \frac{1}{t^2}\sin(\ln t)) + t(-\frac{1}{t}\sin(\ln t)) + \cos(\ln t) = 0.$$

Next we check that they are linearly independent.

$$W[\cos(\ln t), \sin(\ln t)](t) = \frac{1}{t} \neq 0 \quad \text{on } (0, +\infty)$$

so the functions are linearly independent on $(0, +\infty)$ and therefore they form a fundamental set of solutions. The general solution is

$$y(t) = c_1 \cos(\ln t) + c_2 \sin(\ln t).$$

4. Find the general solution of the system

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 5 & 1 \\ -2 & 3 \end{pmatrix} \mathbf{x}.$$

Draw a phase portrait for the system. Discuss the stability of the origin.

The eigenvalues of the matrix are $\lambda = 4 \pm i$. The eigenvector corresponding to the eigenvalue $4 + i$ is

$$\begin{bmatrix} 1 \\ i-1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} + i \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Therefore the two linearly independent solutions are

$$\mathbf{x}_1(t) = e^{4t} \left(\begin{bmatrix} 1 \\ -1 \end{bmatrix} \cos t - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \sin t \right),$$

$$\mathbf{x}_2(t) = e^{4t} \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \cos t + \begin{bmatrix} 1 \\ -1 \end{bmatrix} \sin t \right).$$

The general solution is thus

$$\mathbf{x}(t) = e^{4t} \left(c_1 \begin{bmatrix} \cos t \\ -\sin t - \cos t \end{bmatrix} + c_2 \begin{bmatrix} \sin t \\ \cos t - \sin t \end{bmatrix} \right).$$

Since the real part of the eigenvalues is positive, the origin is an unstable equilibrium. Trajectories are unstable spirals going clockwise.

5. The rate at which a drug disseminates into the bloodstream is governed by the differential equation

$$\frac{dx}{dt} = r - kx,$$

where r and k are positive constants. The function $x(t)$ describes the concentration of the drug in the bloodstream at any time t. Find the limiting value of $x(t)$ as $t \to \infty$. At what time is the concentration equal to one-half of this limiting value? Assume that $x(0) = 0$.

Solving the differential equation we get

$$\int \frac{dx}{r - kx} = \int dt$$

$$-\frac{1}{k}\ln|r-kx| = t + c$$

$$r - kx = Ce^{-kt}$$

and since $x(0) = 0$ we get $C = r$. Therefore

$$x(t) = \frac{r}{k} - \frac{r}{k}e^{-kt}.$$

and then

$$\lim_{t\to\infty} x(t) = \frac{r}{k}.$$

We now need to find t such that $x(t) = r/(2k)$ and so we need

$$\frac{r}{k} - \frac{r}{k}e^{-kt} = \frac{r}{2k}.$$

Therefore

$$e^{-kt} = \frac{1}{2}$$

and then $-kt = \ln(1/2) = -\ln 2$, which gives

$$t = \frac{\ln 2}{k}.$$