REMARKS ON NONLINEAR UNIFORMLY PARABOLIC EQUATIONS

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Abstract

This paper provides a number of working tools for the discussion of fully nonlinear parabolic equations. These include: a proof that the maximum principle which provides L^∞ estimates of “strong” solutions of extremal equations by L^{n+1} norms of the forcing term over the “contact set” remains valid for viscosity solutions in an L^{n+1} sense, a gradient estimate in L^p for $p < (n+1)(n+2)$ for solutions of extremal equations with forcing terms in L^{n+1} , the use of this estimate in improving the range of p for which the maximum principle first alluded to holds (obtaining some $p < n+1$ - but without the contact set), a proof of the strong solvability of Dirichlet problems for extremal equations with forcing terms in L^p for some $p < n+1$, and the twice parabolic differentiability a.e. of $W^{2,1,p}$ functions for $(n+2)/2 < p$.

0. Introduction. In this work we provide a number of tools for the discussion of nonlinear parabolic equations under appropriate structure conditions. In particular, we fill gaps in the current literature and thereby prove a full generalization of the “maximum principle” (see below) to viscosity solutions of certain extremal fully nonlinear equations with measurable forcing terms. While going about this task, we must resolve certain existence questions, and this is done as well. The results obtained are formulated in terms of standard extremal equations so that they apply to many other equations. In addition, some new proofs of known (to varying degrees) results are given.

Let $0 < \lambda \leq \Lambda$ be constants and define

$$\mathcal{P}^-(X) = -\Lambda \operatorname{trace}(X^+) + \lambda \operatorname{trace}(X^-)$$

for $X \in \mathcal{S}(n)$, the set of real symmetric $n \times n$ matrices. Here $\operatorname{trace}(X^+)$ (respectively, $\operatorname{trace}(X^-)$) is the sum of the positive eigenvalues of X (respectively, $-X$). Let $0 \leq \gamma$ be another constant. We discuss several results concerning solutions of the parabolic inequality

$$(0.1) \quad \mathcal{L}(u) := u_t + \mathcal{P}^-(D^2u) - \gamma|Du| \leq f$$

as well as the equation $\mathcal{L}(u) = f$ in $Q = \Omega \times (0, T]$, where Ω is an open, bounded domain in $\mathbb{R}^n$, $n \geq 2$. Here Du corresponds to the spatial gradient $(u_{x_1}, \dots, u_{x_n})$, D^2u corresponds to the spatial Hessian matrix (u_{x_i, x_j}) and $|p|$ is the Euclidean

length of $p \in \mathbb{R}^n$. The notation “ $\mathcal{P}^-$ ” reflects the fact that this function was introduced by Pucci as well as the property that $\mathcal{P}^-(X)$ is the minimum of $-\text{trace}(AX)$ over $A \in \mathcal{S}(n)$ with $\lambda I \leq A \leq \Lambda I$. The extremal parabolic equations $\mathcal{L}(u) = f$ were first studied by Astesiano [1]; for the modern theory of nonlinear parabolic equations we refer to Krylov [24] and Lieberman [26].

The first version of the maximum principle of concern to us is stated below for easy reference. It follows immediately from Tso’s work [32] which refined a result of Krylov [23] (see also the contemporaneous works of Reye [30] and Nazarov and Ural’tseva [29], and a recent paper of Cabré [3]). In the statement and hereafter, the parabolic boundary of Q is denoted by $\partial_p Q$, the diameter of Ω by $d(\Omega)$, and the diameter of Q by $d(Q)$:

$$(0.2) \quad \partial_p Q = \partial\Omega \times (0, T] \cup \overline{\Omega} \times \{0\}, \quad d(\Omega) = \text{diameter}(\Omega), \quad d(Q) = \text{diameter}(Q).$$

The diameter of Q is measured in the parabolic distance $d((x, t), (y, s)) = (|x - y|^2 + |t - s|)^{\frac{1}{2}}$. The “upper contact set” of $u: Q \rightarrow \mathbb{R}$ is denoted by $,^+(u)$:

$$(0.3) \quad ,^+(u) = \{(x, t) \in Q: \exists p \in \mathbb{R}^n \ u(y, s) \leq u(x, t) + \langle p, y - x \rangle \ \forall y \in \Omega, \ 0 < s \leq t\}.$$

We will say that $u \in W^{2,1,p}(Q)$ (respectively, $u \in W_{\text{loc}}^{2,1,p}(Q)$) if $u_t, Du, D^2u \in L^p(Q)$ (respectively, $L_{\text{loc}}^p(Q)$). In Theorem 0.1 below and elsewhere, given a set A in Euclidean space, $|A|$ will always denote its Lebesgue measure.

Theorem 0.1. *Let $u \in C(\overline{Q})$, $f \in L^{n+1}(Q)$, $u \in W_{\text{loc}}^{2,1,n+1}(Q)$ and (0.1) hold a.e. on Q . Then*

$$(0.4) \quad u \leq \sup_{\partial_p Q} u + d(\Omega)^{\frac{n}{n+1}} C \left(\int_{\Gamma^+(u)} (f^+)^{(n+1)} \right)^{\frac{1}{n+1}},$$

where $C = C(1/\lambda, n, |Q|\gamma^{n+1}/d(\Omega))$ depends only on the indicated quantities and is bounded for bounded arguments.

Hereafter we refer to the inequality (0.4) as “the maximum principle”. Generalizations of Theorem 0.1 involve interpreting (0.1) in a “viscosity” sense which does not require regularity of u . Instead one requires that if φ is a suitable test function satisfying

$$\varphi_t(x, t) + \mathcal{P}^-(D^2\varphi(x, t)) - \gamma|D\varphi(x, t)| - f(x, t) > \delta > 0$$

a.e. in some neighborhood of a point $(\hat{x}, \hat{t})$, then $u - \varphi$ cannot have a maximum at $(\hat{x}, \hat{t})$. Viscosity senses are in contrast with “strong” senses requiring that u belong to some regularity class for which the equation has a pointwise sense and holds a.e. (e.g., the conditions in Theorem 0.1).

More precisely, for each choice of “regularity” class $\mathcal{R} = C(Q)$ or $\mathcal{R} = L^p_{\text{loc}}(Q)$, we say that u is an $\mathcal{R}$ -viscosity solution of (0.1) provided that it satisfies the viscosity definition with all test functions φ satisfying $\varphi_t, D\varphi, D^2\varphi \in \mathcal{R}$. We also say that u is an $\mathcal{R}$ -strong solution of (0.1) provided that $u_t, Du, D^2u \in \mathcal{R}$ and (0.1) holds a.e. Typically $f \in \mathcal{R}$ will hold when these notions are in use, but the notions are sensible even if $f \notin \mathcal{R}$. Finally, we abbreviate the classes $L^p_{\text{loc}}(Q)$, $C(Q)$ to simply L^p , C in this context. Our nomenclature varies a bit from that of Wang [33] (whose results play an essential role in this note) in this regard.

One of our results, Theorem 1.1 below, generalizes Theorem 0.1 by passing from L^{n+1} -strong solutions to L^{n+1} -viscosity solutions. This generalization plays a role in Crandall, Kocan and Świąch [10], wherein full use is made of all that is proved here. An essential step in the proof of Theorem 1.1 is to show that (0.4) holds for C -viscosity solutions of $\mathcal{L}(u) \leq f$ if $f \in C(Q)$, see Lemma 1.2. This result may be found in Wang [33], where the idea of the proof is to show that a certain envelope of u is sufficiently regular and therefore arguments from the proof of Theorem 0.1 can be used. This proof is the parabolic version of arguments of Caffarelli [4] (see Caffarelli and Cabré [5], Lemma 3.3); the main work is to establish the regularity of the envelope by clever use of the (viscosity) differential inequality satisfied by u . Alternatively, as we show, one may approximate arbitrary functions by regular functions via a general process which happens to respect the equation and directly apply Theorem 0.1. We sketch this different and efficient proof of Lemma 1.2 in Section 1. The approximation process involved is due to Lasry and Lions [25]; it is a general tool useful in other circumstances. A key property of the approximation which we will need was established by Jensen [22] without, however, highlighting the connection to the Lasry-Lions procedure. This relation is made clear in [9].

Passing from the case of continuous f to discontinuous f and the L^{n+1} -viscosity setting is a significant step. In Caffarelli and Cabré [5], Remark 3.2, a restricted generalization beyond the continuous case is noted in the elliptic setting. In [33] the corresponding issue arises (for $f \in L^p$, $p > n + 1$), but the author used the existence of strong solutions of $\mathcal{L}(u) = f$ without proof. The difficulties in the elliptic case are treated in [6], Proposition 3.3. It is the parabolic analogue of the arguments of [6] which succeeds in complete generality here.

The second “maximum principle” discussed in this work is the following:

Theorem 0.2. *There exists $\epsilon = \epsilon(n, \lambda, \Lambda, \gamma d(Q)) > 0$ with the following property: for $n + 1 - \epsilon < p$ there is a constant $C = C(\lambda, \Lambda, n, p, \gamma d(Q))$ such that if $f \in L^p(Q)$, and $u \in C(\overline{Q})$ is an L^p -strong solution of*

$$u_t + \mathcal{P}^-(D^2u) - \gamma|Du| \leq f \quad \text{in } Q,$$

then

$$(0.5) \quad u \leq \sup_{\partial_p Q} u + C \left(\int_Q (f^+)^p \right)^{\frac{1}{p}}.$$

This result is in the theme initiated by Fabes and Stroock [16]. A corresponding result for linear operators without first order terms was formulated in terms of the Green's functions by Escoriaza [13], Remark 1; a linear result applies to the nonlinear case via linearization. No proof was given and the author commented that while no proof appeared to be available in print, the result could be obtained by adapting the methods used by Fabes and Stroock in the elliptic case. Moreover, it was noted that this result allowed extension of the regularity theory of [33] to some $p < n + 1$ just as the elliptic theory of [4] was extended in [13] to include some $p < n$.

We will present a complete proof of Theorem 0.2 in Section 4. This proof is different from that to which Escoriaza alluded, and does not rely on linear methods. Instead, we adapt the proof of Fok [18], [17] to the parabolic setting. While Theorem 0.2 is proved in Section 4, it is already used in Sections 1, 2, 3. Moreover, from the results in Section 1 it follows that Theorem 0.2 is still valid if u is not an L^p -strong but just an L^p -viscosity solution of $\mathcal{L}(u) \leq f$, see Remark 1.7.

Hereafter $p_0 = n + 1 - \epsilon(n, \lambda, \Lambda, \gamma d(Q))$ will appear in assumptions in the typical form

$$p_0 < p.$$

This will always mean that (4.1) holds for L^p -strong solutions of the problem under discussion.

In special case when the right hand side of (0.1) depends on the spatial variable only, i.e. $f(x, t) = w(x)$, $w \in L^q(\Omega)$, we can further specialize Theorem 0.2 and replace $\|f^+\|_{L^p(Q)}$ in (4.1) by $\|w^+\|_{L^q(\Omega)}$ for an appropriate $q < n$. This is the content of Theorem 1.6 which shows that this can be achieved for the same range of q and with the same constants as in the maximum principle for elliptic equations, see [16], [13], [6], [18]. Theorem 1.6 is inspired by Escoriaza [14] where an analogous result was obtained for $q = n$ and $\gamma = 0$. The proof in [14] is a refinement of that in [32], while our proof of Theorem 1.6 is completely different.

In various proofs, the existence of certain strong solutions of $\mathcal{L}(u) \geq f$ and $\mathcal{L}(u) = f$ plays a central role. These issues are not completely resolved in the current literature, and we will attend to this matter as well. In particular, see Theorem 2.1, we prove that the Cauchy-Dirichlet problem for $\mathcal{L}(u) = f$ has an L^p -strong solution for $f \in L^p(Q)$ and $p_0 < p$. Our proof of solvability of $\mathcal{L}(u) = f$ applies equally well to more general equations and we believe it to be simpler (given regularity results of Wang [33] assumed herein) and more transparent than

the approaches of Krylov [24], Chapter 3, and of Escauriaza [13] (for the elliptic case).

It is not hard to see that one always has $(n+2)/2 \leq p_0$ since according to [2], Theorem 10.4, $(n+2)/2 < p$ is the range for the imbedding of $W^{2,1,p}(Q)$ into $C(Q)$. We emphasize this fact as it implies that all the $W^{2,1,p}$ functions considered in this paper are continuous. Moreover, it turns out that if $u \in W_{\text{loc}}^{2,1,p}(Q)$ and $(n+2)/2 < p$, then u has an expansion to first order in t and second order in x at almost every point, a fact which we will use in proving the existence of strong solutions. The elliptic version (for $D^2u \in L^p(\Omega)$, $n/2 < p$) is a special case of results of Calderón and Zygmund [7]; a direct proof is given in an appendix to [6]. We did not find a reference for the parabolic version and provide a proof in the appendix, Proposition A.1.

As mentioned before, all the above results play a role in [10]. We cannot summarize the results of [10] here, but some are alluded to at the end of Section 2.

R. DeVore generously showed us how he would prove the elliptic version of the differentiability result in the appendix, and this influenced our presentation. We benefitted from contacts with L. Wang and L. Escauriaza and from comments of the referee and are grateful to them all.

1. The Maximum Principle for L^{n+1} -Viscosity Solutions and Existence for $\mathcal{L}(u) \geq f$. We seek to prove:

Theorem 1.1. *Let $f \in L^{n+1}(Q)$ and $u \in C(\overline{Q})$ be an L^{n+1} -viscosity solution of*

$$(1.1) \quad u_t + \mathcal{P}^-(D^2u) - \gamma|Du| \leq f$$

in Q . Then

$$(1.2) \quad u \leq \sup_{\partial_p Q} u + d(\Omega)^{\frac{n}{n+1}} C \left(\int_{\Gamma^+(u)} (f^+)^{n+1} \right)^{\frac{1}{n+1}},$$

where $C = C(1/\lambda, n, |Q|\gamma^{n+1}/d(\Omega))$ is the same as in Theorem 0.1.

We begin with (a sketch of) the proof in the case of C -viscosity solutions. Recall that this result is contained in Wang [33], but our proof is quite different. The presentation is self-contained in outline, but tracking the details requires considerable familiarity with the “sup-convolution” – see [25] and Section 4 of [9].

Lemma 1.2. *Let $f \in C(Q)$ and $u \in C(\overline{Q})$ be a C -viscosity solution of (1.1) in Q . Then (1.2) holds.*

Proof. “Moving” in from the parabolic boundary of Q a bit, we can assume that u is continuous and satisfies (0.1) in the C -viscosity sense in a parabolic

neighborhood $\tilde{\Omega} \times (-\rho, T]$ of Q – at the end, one passes easily to the limit to remove this assumption. Next, replacing u by the sup-convolution in t ,

$$u^\kappa(x, t) = \sup_{-\rho < s \leq T} \left(u(x, s) - \frac{1}{\kappa} |t - s|^2 \right),$$

where $\kappa > 0$ is sufficiently small, we have

$$u_t^\kappa(x, t) + \mathcal{P}^-(D^2 u^\kappa(x, t)) - \gamma |Du^\kappa(x, t)| \leq f(x, t) + \rho(\kappa)$$

on Q , where $\rho(0+) = 0$. The virtue of this approximation is that u^κ is Lipschitz continuous in t ; if we prove the result in this case we obtain the full result upon letting $\kappa \downarrow 0$. Thus we may assume that u is Lipschitz continuous in t . Denoting now the sup-convolution of $w(x, t)$ in x by w^ϵ ; that is

$$w^\epsilon(x, t) = \sup_{y \in \tilde{\Omega}} \left(w(y, t) - \frac{1}{2\epsilon} |y - x|^2 \right),$$

and the inf-convolution w_δ ; that is

$$w_\delta(x, t) = \inf_{y \in \tilde{\Omega}} \left(w(y, t) + \frac{1}{2\delta} |y - x|^2 \right),$$

we replace u by $v = (u^{\epsilon+\delta})_\delta$. This v satisfies $v_t, Dv, D^2v \in L^\infty(Q)$. Moreover, when δ is sufficiently small (depending on ϵ)

$$v_t(x, t) + \mathcal{P}^-(D^2 v(x, t)) - \gamma |Dv(x, t)| \leq f(x, t) + \rho(\epsilon)$$

holds a.e. for some function ρ such that $\rho(0+) = 0$. To see why this is so one can consult [9], Proposition 4.6. However, since v_t, Dv , and D^2v are bounded, Theorem 0.1 [32] applies directly to v and we conclude in the limit $\epsilon \downarrow 0$ (see [32] or [6] for details related to the passage to the limit). $\square$

Theorem 1.1 will follow from the lemma and the following proposition. Recall that the meaning of “ $p_0 < p$ ” is explained in the introduction. Its precise meaning is provided by Theorem 0.2 ($p_0 = n + 1 - \epsilon(n, \lambda, \Lambda, \gamma d(Q))$).

Proposition 1.3. *Let Ω satisfy a uniform exterior cone condition, let $k > 0$, let $p > p_0(n, \lambda, \Lambda, k)$, $f \in L^p(Q)$, $\psi \in C(\partial_p Q)$, and $\gamma d(Q) \leq k$. Then there exists $u \in C(\overline{Q}) \cap W_{\text{loc}}^{2,1,p}(Q)$ such that $u = \psi$ on $\partial_p Q$ and*

$$(1.3) \quad u_t(x, t) + \mathcal{P}^-(D^2 u(x, t)) - \gamma |Du(x, t)| \geq f(x, t) \quad \text{a.e. in } Q.$$

Moreover, u further satisfies

$$(1.4) \quad |u| \leq \|\psi\|_{L^\infty(\partial_p Q)} + d(Q)^{2-\frac{n+2}{p}} C \|f\|_{L^p(Q)},$$

where $C = C(\lambda, \Lambda, n, p, k)$.

In order to prove Proposition 1.3, we require the following result; it is a consequence of results of Wang [33] and Escauriaza [13] (see also Dong and X.J. Wang [12]), together with simple adaptations of the proof of Lemma 3.1 in [6], to allow $0 < \gamma$. The notation $Q' \subset\subset Q$ means Q' is a relatively open subset of Q whose parabolic closure is contained in Q .

Theorem 1.4. *Let $(n+2)/2 < P$ and assume that there is a constant C_0 such that*

$$(1.5) \quad u \leq \sup_{\partial_p Q} u + C_0 \left(\int_Q ((u_t + \mathcal{P}^-(D^2 u) - \gamma|Du|)^+)^P \right)^{\frac{1}{P}}$$

whenever $u \in C(\overline{Q}) \cap W_{\text{loc}}^{2,1,P}(Q)$. Then for every $p > P$, $Q' \subset\subset Q$ and $u \in W_{\text{loc}}^{2,1,p}(Q)$

$$(1.6) \quad \|u\|_{W^{2,1,p}(Q')} \leq C (\|u\|_{L^\infty(Q)} + \|u_t + \mathcal{P}^-(D^2 u) - \gamma|Du|\|_{L^p(Q)}).$$

Here $C = C(C_0, n, p, P, \gamma, \lambda, \Lambda, d(\Omega), T, \text{distance}(Q', \partial_p Q))$ depends only on the indicated parameters.

Proof of Proposition 1.3. Let $p > p_0$ and $f \in L^p(Q)$. Choose smooth f_j such that

$$(1.7) \quad \int_Q |f - f_j|^p \rightarrow 0$$

and solve

$$(1.8) \quad \begin{cases} u_t^j + \mathcal{P}^-(D^2 u^j) - \gamma|Du^j| = f_j & \text{in } Q \\ u^j = \psi & \text{on } \partial_p Q. \end{cases}$$

The fact that (1.8) can be solved with $u^j \in C(\overline{Q})$, $u_t^j, Du^j, D^2 u^j \in C(Q)$, can be proved by the methods of Krylov [24] (in particular, see Theorem 6.4.3), combined with barrier functions for extremal operators on external cones constructed in Miller [27]. For a different proof, relying on the theory of viscosity solutions (and [24]), we refer the reader to [11].

Let $A \in [\lambda I, \Lambda I]$ mean $A \in \mathcal{S}(n)$ and $\lambda I \leq A \leq \Lambda I$ and $\langle q, p \rangle$ be the Euclidean inner-product of $p, q \in \mathbb{R}^n$. The operator $\mathcal{L}$ is “extremal” in that

$$(1.9) \quad \mathcal{P}^-(X) - \gamma|p| = \inf \{ -\text{trace}(AX) + \langle q, p \rangle : A \in [\lambda I, \Lambda I], q \in \mathbb{R}^n, |q| \leq \gamma \}.$$

Since $(X, p) \mapsto \mathcal{P}^-(X) - \gamma|p|$ is superadditive by (1.9),

$$\begin{aligned} & (u^j - u^k)_t + \mathcal{P}^-(D^2(u^j - u^k)) - \gamma|D(u^j - u^k)| \\ & \leq u_t^j + \mathcal{P}^-(D^2 u^j) - \gamma|Du^j| - (u_t^k + \mathcal{P}^-(D^2 u^k) - \gamma|Du^k|) = f_j - f_k; \end{aligned}$$

therefore from the assumption (4.1) (the meaning of $p > p_0$) we obtain

$$u^j - u^k \leq C \|f_j - f_k\|_{L^p(Q)} \rightarrow 0$$

as $j, k \rightarrow \infty$. Thus $u^j \rightarrow u$ uniformly on Q as $j \rightarrow \infty$ for some $u \in C(\overline{Q})$. Using Theorem 1.4, u_t^j, Du^j, D^2u^j remain bounded in $L^p(Q')$ for $Q' \subset\subset Q$ and thus $u \in W_{\text{loc}}^{2,1,p}(Q)$.

In view of (1.9) we have

$$u_t^j - \text{trace}(AD^2u^j) + \langle q, Du^j \rangle \geq f_j$$

a.e. for $A \in [\lambda I, \Lambda I]$, $|q| \leq \gamma$. We may pass to the weak limit in $L_{\text{loc}}^p(Q)$ to conclude that

$$u_t - \text{trace}(AD^2u) + \langle q, Du \rangle \geq f$$

a.e. Then “infing” over A, q yields $\mathcal{L}(u) \geq f$ a.e. Finally, (1.4) holds for each u^j with f replaced by f_j , and hence in the limit. $\square$

We are now able to prove Theorem 1.1.

Proof of Theorem 1.1. Since Ω can be approximated from the inside by arbitrarily regular subdomains, without loss of generality we may assume that Ω satisfies the assumption of Proposition 1.3. Let $f \in L^{n+1}(Q)$ and $u \in C(\overline{Q})$ be an L^{n+1} -viscosity solution of $\mathcal{L}(u) \leq f$ in Q . Let $g \in L^{n+1}(Q)$ and $v \in C(\overline{Q}) \cap W_{\text{loc}}^{2,1,n+1}(Q)$ be the strong solution of

$$v_t + \mathcal{P}^-(D^2v) - \gamma|Dv| \geq g \quad \text{in } Q, \quad v = 0 \quad \text{on } \partial_p Q,$$

whose existence is asserted in Proposition 1.3. Formally,

$$\mathcal{L}(u - v) \leq \mathcal{L}(u) - \mathcal{L}(v) \leq f - g$$

by the superadditivity of $\mathcal{L}$; since $v \in W_{\text{loc}}^{2,1,n+1}(Q)$ a check of the definition shows that in fact

$$(1.10) \quad \mathcal{L}(u - v) \leq f - g$$

in the L^{n+1} -viscosity sense. Choose g so that $f - g \in C(Q)$. Then $u - v$ is evidently also a C -viscosity solution of (1.10) and by Lemma 1.2 we obtain

$$u - v \leq \sup_{\partial_p Q} (u - v) + d(\Omega)^{\frac{n}{n+1}} C \left(\int_{\Gamma^+(u-v)} ((f - g)^+)^{(n+1)} \right)^{\frac{1}{n+1}},$$

where C is from (0.4). It remains to let $g \rightarrow 0$ in $L^{n+1}(Q)$ so that $v \rightarrow 0$ in $L^\infty(Q)$ because $|v|$ is estimated by the L^{n+1} norm of g . $\square$

Next we will use Proposition 1.3 to obtain an extension of Theorem 0.2. First recall the elliptic maximum principle for strong solutions, see [16], [13], [6], [18].

Theorem 1.5. *There exists $\delta = \delta(n, \lambda, \Lambda, \gamma d(\Omega)) > 0$ with the following property: for $q > n - \delta$ there is a constant $B = B(\lambda, \Lambda, n, q, \gamma d(\Omega))$ such that if $w \in L^q(\Omega)$, and $v \in C(\overline{\Omega}) \cap W_{\text{loc}}^{2,q}(\Omega)$ satisfies*

$$\mathcal{P}^-(D^2v(x)) - \gamma|Dv(x)| \leq w(x) \quad \text{for a.e. } x \in \Omega,$$

then

$$(1.11) \quad v \leq \sup_{\partial\Omega} v + B \left(\int_{\Omega} (w^+)^q \right)^{\frac{1}{q}}.$$

We will denote $q_0 = n - \delta(n, \lambda, \Lambda, \gamma d(\Omega))$. As explained in the introduction, the following result is inspired by and improves upon [14].

Theorem 1.6. *Suppose that $u \in C(\overline{Q})$ is an L^p -strong solution of*

$$u_t(x, t) + \mathcal{P}^-(D^2u(x, t)) - \gamma|Du(x, t)| \leq f(x, t) + w(x) \quad \text{for } (x, t) \in Q,$$

where $f \in L^p(Q)$, $p > p_0$, and $w \in L^q(\Omega)$, $q > q_0$. Then

$$(1.12) \quad u \leq \sup_{\partial_p Q} u + C \left(\int_Q (f^+)^p \right)^{\frac{1}{p}} + B \left(\int_{\Omega} (w^+)^q \right)^{\frac{1}{q}},$$

with constants C and B as in Theorems 0.2 and 1.5, respectively.

Proof. Since instead of Q we can consider its slightly smaller subdomain, without loss of generality we may assume that u is a strong solution of $\mathcal{L}(u) \leq f + w$ in a neighborhood of $\overline{Q}$, and that $\partial\Omega$ is as regular as we please. Let u_n be standard mollifications of u . It follows that u_n satisfy

$$\mathcal{L}(u_n) \leq f_n + w \quad \text{a.e. in } Q,$$

where $f_n \rightarrow f^+$ in $L^p(Q)$. We can assume that $f_n, w \geq 0$. Moreover, since $u_n \in C^\infty(\overline{Q})$, $\mathcal{L}(u_n)$ is bounded and therefore there is $w_n \in L^\infty(\Omega)$ such that $0 \leq w_n \leq w$ and

$$\mathcal{L}(u_n) \leq f_n + w_n \quad \text{a.e. in } Q.$$

From [6], Corollary 3.10, there is $v_n \in C(\overline{\Omega}) \cap W_{\text{loc}}^{2,p}(\Omega)$ such that

$$\mathcal{P}^-(D^2v_n) - \gamma|Dv_n| = w_n \quad \text{a.e. in } \Omega$$

and $v_n = 0$ on $\partial\Omega$. From Theorem 1.5 we have

$$(1.13) \quad 0 \leq v_n \leq B\|w_n\|_{L^q(\Omega)} \leq B\|w\|_{L^q(\Omega)}.$$

Then $u_n - v_n \in W_{\text{loc}}^{2,1,p}(Q)$ is an L^p -strong solution of $\mathcal{L}(u_n - v_n) \leq f_n$ and therefore by (4.1)

$$u_n - v_n \leq \sup_{\partial_p Q} (u_n - v_n) + C\|f_n\|_{L^p(Q)},$$

and then from (1.13)

$$u_n \leq \sup_{\partial_p Q} u_n + C\|f_n\|_{L^p(Q)} + B\|w\|_{L^q(\Omega)}.$$

Letting $n \rightarrow \infty$ concludes the proof. $\square$

Remark 1.7. Using Proposition 1.3 it is straightforward to extend Theorem 0.2 from L^p -strong to L^p -viscosity solutions, $p > p_0$. The proof consists in solving strongly the extremal inequality

$$\mathcal{L}(v) \geq f \quad \text{in } Q, \quad v = 0 \quad \text{on } \partial_p Q;$$

then Lemma 1.2 applied to $u - v$ together with estimate (1.4) give the result.

Similarly, the estimate (1.12) of Theorem 1.6 holds for L^p -viscosity solutions of

$$\mathcal{L}(u) \leq f + g$$

if $f, g \in L^p(Q)$, $p > p_0$, and w defined as

$$w(x) = \text{ess sup}_{0 \leq t \leq T} g(x, t)$$

satisfies $w^+ \in L^q(\Omega)$, $q > q_0$.

2. Existence of Strong Solutions for $\mathcal{L}(u) = f$. As a consequence of Proposition 1.3 one obtains existence of solutions of the Cauchy-Dirichlet problem for the equation $\mathcal{L}(u) = f$. We point out that if $n+1 \leq p$ this result follows directly by passing to weak limit in the proof of Proposition 1.3 via results of Chapter 3 of Krylov [24] (which in turn employs ideas of Evans [15] used for related purposes). We present another approach which is more general. It applies, for example, to establish pointwise properties of viscosity solutions and is more relevant to the general theory developed in [10].

Theorem 2.1. *Let Ω be as in Proposition 1.3, let $p > p_0$, $f \in L^p(Q)$, and $\psi \in C(\partial_p Q)$. Then there exists $u \in C(\overline{Q}) \cap W_{\text{loc}}^{2,1,p}(Q)$ such that $u = \psi$ on $\partial_p Q$ and*

$$(2.1) \quad u_t(x, t) + \mathcal{P}^-(D^2 u(x, t)) - \gamma|Du(x, t)| = f(x, t) \quad \text{a.e. in } Q.$$

Moreover

$$\|u\|_{W^{2,1,p}(Q')} \leq C \left(\|u\|_{L^\infty(Q)} + \|f\|_{L^p(Q)} \right),$$

where $C = C(n, p, \gamma, \lambda, \Lambda, d(\Omega), T, \text{distance}(Q', \partial_p Q))$.

We already know how to solve $\mathcal{L}(u) \geq f$, see Proposition 1.3. We will show that u constructed in the course of its proof in fact solves $\mathcal{L}(u) = f$.

For $r > 0$ and $(x, t) \in \mathbb{R}^{n+1}$ we will write $Q_r(x, t)$ for the parabolic cylinder $B_r(x) \times (t - r^2, t]$, where $B_r(x)$ denotes the open Euclidean ball in $\mathbb{R}^n$ centered at x of radius r .

Remark 2.2. Observe that if $f \in L^p_{\text{loc}}(\mathbb{R}^{n+1})$, $p \in [1, \infty)$, then for a.e. $(\hat{x}, \hat{t})$

$$(2.2) \quad \lim_{\substack{(z,s) \rightarrow (\hat{x}, \hat{t}) \\ r \downarrow 0}} \frac{1}{|Q_r(z,s)|} \int_{Q_r(z,s)} |f - f(\hat{x}, \hat{t})|^p dx dt = 0;$$

in particular,

$$(2.3) \quad \lim_{r \downarrow 0} \frac{1}{|Q_r(\hat{x}, \hat{t})|} \int_{Q_r(\hat{x}, \hat{t})} |f - f(\hat{x}, \hat{t})|^p dx dt = 0.$$

This follows in the standard way from a version of Vitali's covering theorem with Q_r 's replacing the Euclidean balls, see e.g. Remark I.3.1 in de Guzmán [21]. The covering theorem needed has essentially the same proof as is given for Euclidean balls.

Proof of Theorem 2.1. We will show that the $u \in C(\overline{Q}) \cap W^{2,1,p}_{\text{loc}}(Q)$ constructed in the course of the proof of Proposition 1.3 does the job. We have $u_t, Du, D^2u \in L^p_{\text{loc}}(Q)$ by construction. It follows from the appendix that almost every $(\hat{x}, \hat{t}) \in Q$ is a point of differentiability – twice in x , once in t – of u , and a Lebesgue point of f in the sense of (2.3) as well. More precisely, the differentiability proved in Proposition A.1 at almost every $(\hat{x}, \hat{t})$ is that if

$$\varphi(x, t) = u(\hat{x}, \hat{t}) + u_t(\hat{x}, \hat{t})(t - \hat{t}) + \langle Du(\hat{x}, \hat{t}), x - \hat{x} \rangle + \frac{1}{2} \langle D^2u(\hat{x}, \hat{t})(x - \hat{x}), x - \hat{x} \rangle$$

then

$$(2.4) \quad u(x, t) - \varphi(x, t) = o(|x - \hat{x}|^2 + |t - \hat{t}|) \quad \text{as } (x, t) \rightarrow (\hat{x}, \hat{t}).$$

We will show that at these points $(\hat{x}, \hat{t})$, points at which (2.4) and (2.3) hold, one has

$$(2.5) \quad u_t(\hat{x}, \hat{t}) + \mathcal{P}^-(D^2u(\hat{x}, \hat{t})) - \gamma |Du(\hat{x}, \hat{t})| \leq f(\hat{x}, \hat{t}).$$

Together with the inequality $\mathcal{L}(u) \geq f$ already proven, this will prove the theorem.

If (2.5) fails then

$$(2.6) \quad \varphi_t(\hat{x}, \hat{t}) + \mathcal{P}^-(D^2\varphi(\hat{x}, \hat{t})) - \gamma|D\varphi(\hat{x}, \hat{t})| > f(\hat{x}, \hat{t}).$$

By (2.4), if we replace $\varphi(x, t)$ by $\varphi(x, t) + 3\eta(\hat{t} - t + |x - \hat{x}|^2)$ with sufficiently small $\eta > 0$, by (2.6) we can achieve that if r is sufficiently small then for all $(x, t) \in Q_r = Q_r(\hat{x}, \hat{t})$

$$(2.7) \quad \varphi_t(x, t) + \mathcal{P}^-(D^2\varphi(x, t)) - \gamma|D\varphi(x, t)| \geq f(\hat{x}, \hat{t}),$$

while

$$(2.8) \quad u(x, t) - \varphi(x, t) \leq -2\eta(\hat{t} - t + |x - \hat{x}|^2),$$

with equality at $(\hat{x}, \hat{t})$.

Let u^j and f_j be as in the proof of Proposition 1.3; that is, u^j solve (1.8) strongly and (1.7) holds, while $u^j \rightarrow u$ uniformly. For $r > 0$ we use Proposition 1.3 to find $\xi = \xi_r \in C(\overline{Q_r}) \cap W_{\text{loc}}^{2,1,n+1}(Q_r)$ satisfying

$$(2.9) \quad \xi_t + \mathcal{P}^-(D^2\xi) - \gamma|D\xi| \geq (f - f(\hat{x}, \hat{t}))^+ \quad \text{a.e. in } Q_r$$

and $\xi \equiv 0$ on $\partial_p Q_r$. By construction, see (1.4), ξ^+ is estimated by

$$Cr^{2-\frac{n+2}{p}} \left(\int_{Q_r} |f - f(\hat{x}, \hat{t})|^p \right)^{\frac{1}{p}} = Cr^{2-\frac{n+2}{p}} |Q_r|^{\frac{1}{p}} o(1) = o(r^2)$$

as $r \downarrow 0$, where the first equality above followed from the Lebesgue point assumption. Thus taking r sufficiently small we can assume that (2.7) and (2.8) hold and ξ satisfies

$$(2.10) \quad \xi \leq \eta r^2 \quad \text{in } Q_r.$$

Now consider $v^j := u^j - \varphi - \xi$. Note that by (2.10)

$$v^j(\hat{x}, \hat{t}) = u^j(\hat{x}, \hat{t}) - u(\hat{x}, \hat{t}) - \xi(\hat{x}, \hat{t}) \geq -\|u - u^j\|_{L^\infty(Q_r)} - \eta r^2,$$

while on $\partial_p Q_r$ by (2.8)

$$v^j(x, t) \leq \|u - u^j\|_{L^\infty(Q_r)} - 2\eta r^2,$$

and therefore

$$(2.11) \quad v^j(\hat{x}, \hat{t}) - \sup_{\partial_p Q_r} v^j \geq -2\|u - u^j\|_{L^\infty(Q_r)} + \eta r^2.$$

Since by (2.7) and (2.9) v_j solves

$$v_t^j + \mathcal{P}^-(D^2 v^j) - \gamma |D v^j| \leq f_j - f(\hat{x}, \hat{t}) - (f - f(\hat{x}, \hat{t}))^+ \leq f_j - f$$

strongly in $Q_r(\hat{x}, \hat{t})$, (1.7) and the generalized maximum principle (4.1) yield

$$\lim_{j \rightarrow \infty} \left(\sup_{Q_r} v^j - \sup_{\partial_p Q_r} v^j \right) = 0,$$

which given $\|u - u^j\|_{L^\infty(Q_r)} \rightarrow 0$, contradicts (2.11). The proof is complete. The dependence of constant C follows from (1.6) and Theorem 0.2. $\square$

The proof just given is in fact general. Using Proposition 1.3 as above, one can show that if solutions u^j of equations $u_t^j + G_j(x, t, Du^j, D^2 u^j) \leq 0$ with certain structure converge uniformly to a limit u and $G_j \rightarrow G$ suitably, then for almost every point $(\hat{x}, \hat{t})$ at which u satisfies

$$u(x, t) \leq u(\hat{x}, \hat{t}) + a(t - \hat{t}) + \langle p, x - \hat{x} \rangle + \frac{1}{2} \langle X(x - \hat{x}), x - \hat{x} \rangle + o(|x - \hat{x}|^2 + |t - \hat{t}|)$$

for some $a \in \mathbb{R}$, $p \in \mathbb{R}^n$, $X \in \mathcal{S}(n)$, one has $a + G(\hat{x}, \hat{t}, p, X) \leq 0$. However, as there are many other related issues best presented as a whole, we defer further details to [10], where a general theory is developed. We will foreshadow a bit. If $F: \mathbb{R}^n \times \mathcal{S}(n) \rightarrow \mathbb{R}$ satisfies

$$\mathcal{P}^-(X - Y) - \gamma |p - q| \leq F(p, X) - F(q, Y) \leq \mathcal{P}^+(X - Y) + \gamma |p - q|,$$

where $\mathcal{P}^+(X) = -\mathcal{P}^-(-X)$, then the Cauchy-Dirichlet problem for

$$u_t + F(Du, D^2 u) = f$$

is uniquely solvable with equation understood in the viscosity sense (an example due to Nadirashvili [28] shows that classical solutions do not exist in general), the solutions possess pointwise derivatives a.e. which satisfy the equation a.e., and a pointwise characterization of the solutions can be given. All this and much more is found in [10] (in which the current results are used vigorously).

3. Aside on Good Solutions. A main point of the next section is that an estimate on Du for some solutions of $u_t + \mathcal{P}^-(D^2 u) - \gamma |Du| = f$, $f \in L^{n+1}$, implies that (1.5) holds for P in a range $(n+1) - \epsilon < P$. Roughly speaking, it would be convenient to have Theorem 2.1 available with $p = n+1$ for this proof, but it is not available in this line of argument since we seek to prove (1.4). We do have, however, a convenient substitute.

As noted already, if u, v are L^{n+1} -strong solutions of

$$(3.1) \quad u_t + \mathcal{P}^-(D^2 u) - \gamma |Du| \leq f, \quad v_t + \mathcal{P}^-(D^2 v) - \gamma |Dv| \geq g,$$

then, by superadditivity, $u - v$ is an L^{n+1} -strong solution of

$$(3.2) \quad (u - v)_t + \mathcal{P}^-(D^2(u - v)) - \gamma|D(u - v)| \leq f - g.$$

It follows that if $u^j \in C(\overline{Q})$ are L^{n+1} -strong solutions of

$$u_t^j + \mathcal{P}^-(D^2 u^j) - \gamma|D u^j| = f_j$$

then

$$|u^j - u^k| \leq \sup_{\partial_p Q} |u^j - u^k| + C \|f_j - f_k\|_{L^{n+1}(Q)}.$$

Hence if $f_j \rightarrow f$ in $L^{n+1}(Q)$ and $u^j \rightarrow \psi$ uniformly on $\partial_p Q$, then $u^j \rightarrow u$ uniformly on $\overline{Q}$ for some u , obviously unique. We call u the L^{n+1} -good solution of $\mathcal{L}(u) = f$ in Q , $u = \psi$ on $\partial_p Q$.

Remark 3.1. If Ω satisfies a uniform exterior cone condition, $f \in L^{n+1}(Q)$ and $\psi \in C(\partial_p Q)$, then $u_t + \mathcal{P}^-(D^2 u) - \gamma|Du| = f$ in Q , $u = \psi$ on $\partial_p Q$ has a unique L^{n+1} -good solution. The proof is obvious (u is the limit of the u^j of (1.8)) and does not require Theorem 1.4.

Similarly, $u \in C(Q)$ is an L^{n+1} -good solution of

$$u_t + \mathcal{P}^-(D^2 u) - \gamma|Du| \leq f$$

in Q if there are $f_j \in L^{n+1}(Q)$, $u^j \in C(Q)$ such that u^j is an L^{n+1} -strong solution of $\mathcal{L}(u^j) \leq f_j$ and $f_j \rightarrow f$ in $L^{n+1}(Q)$ and $u^j \rightarrow u$ in $L^\infty(Q)$. Good solutions of $\mathcal{L}(u) \geq f$ are defined similarly. It is evident that if (3.1) holds in the good sense, then so does (3.2), while (0.4) is valid for good solutions of (0.1).

4. A Proof of the Generalized Maximum Principle. In this section we prove the generalized maximum principle used in Sections 2 and 3. It is important to note that we will use Theorem 0.1 and the discussion of Section 3; we will *not* use any other results from preceding developments. We are out to prove:

Theorem 0.2. *There exists $\epsilon = \epsilon(n, \lambda, \Lambda, \gamma d(Q)) > 0$ with the following property: if $f \in L^P(Q)$ and $u \in C(\overline{Q})$ is an L^P -strong solution of*

$$u_t + \mathcal{P}^-(D^2 u) - \gamma|Du| \leq f \quad \text{in } Q,$$

where $n + 1 - \epsilon < P$, then

$$(4.1) \quad u \leq \sup_{\partial_p Q} u + C \left(\int_Q (f^+)^P \right)^{\frac{1}{P}}.$$

Here $C = C(\lambda, \Lambda, n, P, \gamma d(Q))$.

Note that we are now using “ P ” above to save “ p ” for use in the course of argument. As before, $Q_r(x, t)$ will denote the parabolic cylinder $B_r(x) \times (t-r^2, t]$, where $B_r(x)$ is the open Euclidean ball in $\mathbb{R}^n$ centered at x of radius r . In addition, Q_r denotes $Q_r(0, 0)$.

The following regularity result contains information used in the proof. The full result is of independent interest. The proof uses arguments similar to those employed to obtain the general gradient regularity result in the elliptic case in Świąch [31] and its parabolic counterpart in [10]. At the core of these proofs are techniques used by Caffarelli [4] to prove interior $C^{1,\alpha}$ estimates for elliptic equations and later generalized to the parabolic setting by Wang [33]. Our proof appears to be longer than typical of the area, but this is simply because we made it easy to follow.

Lemma 4.1. *Let $f \in L^{n+1}(Q_{2r})$ and $u \in C(\overline{Q_{2r}})$ be an L^{n+1} -good solution of*

$$(4.2) \quad u_t + \mathcal{P}^-(D^2u) = f \quad \text{in } Q_{2r}, \quad u = 0 \quad \text{on } \partial_p Q_{2r}.$$

Then for $1 \leq p < (n+2)(n+1)$, $Du \in L^p(Q_{3r/2})$ and

$$(4.3) \quad \|Du\|_{L^p(Q_{3r/2})} \leq C_1 r^{\frac{(n+1)(n+2)-p}{(n+1)p}} \|f\|_{L^{n+1}(Q_{2r})},$$

where $C_1 = C_1(p, n, \lambda, \Lambda)$.

Proof of Lemma 4.1. The estimate (4.3) for $r = 1$ implies the general case via scaling. Moreover, approximating f , it is enough to prove the lemma for smooth f and a classical solution u . It is also enough to prove it for $Q_{3r/2}$ replaced by Q_r of the left of (4.3).

While it is well known that the equation $h_t + \mathcal{P}^-(D^2h) = 0$ has classical solutions with interior first and second derivative estimates, we merely require an interior $C^{1+\alpha}$ estimate (see [34], Theorem 4.8, where a suitable estimate is stated for a general equation). The precise result we use is that for some $0 < \bar{\alpha} = \bar{\alpha}(n, \lambda, \Lambda)$ and $K = K(n, \lambda, \Lambda)$, any solution h of $h_t + \mathcal{P}^-(D^2h) = 0$ in Q_{2r} satisfies: for every $(y, s) \in Q_r$ and $(x, t) \in Q_{2r}$

$$(4.4) \quad r|Dh(y, s)| + r^{(1+\bar{\alpha})} \frac{|h(x, t) - h(y, s) - \langle Dh(y, s), x - y \rangle|}{(|y - x| + |s - t|^{1/2})^{1+\bar{\alpha}}} \leq \frac{K}{2^{1+\bar{\alpha}}} \|h\|_{L^\infty(Q_{2r})}.$$

The right hand side has been rigged for future simplicity.

Let (4.2) hold with $r = 1$. We will first prove that for a.e. $(y, s) \in Q_1$, $Du(y, s)$ (taken in the classical sense) exists and an analogue of (4.4) holds at (y, s) . The condition on (y, s) concerns the behavior of f near (y, s) – we assume that for some $0 < \alpha < \bar{\alpha}$

$$(4.5) \quad N = N(y, s) = \sup_{r \leq 1} \left\{ r^{1-\alpha} \left(\frac{1}{r^{n+2}} \int_{Q_r(y, s)} |f(z, \tau)|^{n+1} dz d\tau \right)^{\frac{1}{n+1}} \right\} < \infty.$$

Without loss of generality, assume that $(y, s) = (0, 0)$.

We study the approximation of u by affine functions $l(x) = a + \langle b, x \rangle$ of x near $(0, 0)$ on a sequence Q_{μ^k} of cylinders (where $0 < \mu \leq 1/2$ will be chosen later). It will turn out that the function

$$(4.6) \quad l_k(x) = h_k(0, 0) + \langle Dh_k(0, 0), x \rangle,$$

where h_k is the solution of

$$(4.7) \quad \begin{cases} (h_k)_t + \mathcal{P}^-(D^2 h_k) = 0 & \text{in } Q_{\mu^{k-1}} \\ h_k = u & \text{on } \partial_p Q_{\mu^{k-1}} \end{cases}$$

is about as good as any other l on Q_{μ^k} . The following analysis establishes this fact and more.

We need one bit more of notation. If $h(x, t)$ is differentiable in x at $(x, t) = (0, 0)$, we set

$$(4.8) \quad L(h)(x) = h(0, 0) + \langle Dh(0, 0), x \rangle;$$

note that $L(l) = l$ if l is affine in x , while (4.6) amounts to $l_k = L(h_k)$. Let l be affine and h_k be given by (4.7). We have

$$(4.9) \quad \begin{aligned} \|u - L(h_k)\|_{L^\infty(Q_{\mu^k})} &\leq \|u - h_k\|_{L^\infty(Q_{\mu^k})} + \|(h_k - l) - L(h_k - l)\|_{L^\infty(Q_{\mu^k})} \\ &\leq \|u - h_k\|_{L^\infty(Q_{\mu^{k-1}})} + \|(h_k - l) - L(h_k - l)\|_{L^\infty(Q_{\mu^k})}. \end{aligned}$$

To deal with the first term on the right, we notice that since $w = u - h_k$ and $w = h_k - u$ both satisfy $w_t + \mathcal{P}^-(D^2 w) \leq |f|$ and vanish on $\partial_p Q_{\mu^{k-1}}$, the maximum principle yields

$$(4.10) \quad \|u - h_k\|_{L^\infty(Q_{\mu^{k-1}})} \leq C\mu^{\frac{(k-1)n}{n+1}} \left(\int_{Q_{\mu^{k-1}}} |f(z, \tau)|^{n+1} dz d\tau \right)^{\frac{1}{n+1}}.$$

To deal with the second term, we notice that $h = h_k - l$ satisfies $h_t + \mathcal{P}^-(D^2 h) = 0$ and $h = u - l$ on $\partial_p Q_{\mu^{k-1}}$, so that

$$|h| \leq \|u - l\|_{L^\infty(Q_{\mu^{k-1}})}.$$

Thus by (4.4) with $(y, s) = (0, 0)$, $r = \mu^{k-1}$ and the assumption $\mu \leq 1/2$, for $(x, t) \in Q_{\mu^k}$

$$\begin{aligned} \mu^{(k-1)(1+\bar{\alpha})} \frac{|h(x, t) - h(0, 0) - \langle Dh(0, 0), x \rangle|}{(|x| + |t|^{1/2})^{1+\bar{\alpha}}} &\leq \frac{K}{2^{1+\bar{\alpha}}} \|h\|_{L^\infty(Q_{\mu^{k-1}})} \\ &= \frac{K}{2^{1+\bar{\alpha}}} \|u - l\|_{L^\infty(Q_{\mu^{k-1}})}. \end{aligned}$$

This implies

$$(4.11) \quad \|(h_k - l) - L(h_k - l)\|_{L^\infty(Q_{\mu^k})} \leq K\mu^{(1+\bar{\alpha})}\|u - l\|_{L^\infty(Q_{\mu^{k-1}})}.$$

Using (4.11) and (4.10) on the right of (4.9) results in

$$\begin{aligned} \|u - L(h_k)\|_{L^\infty(Q_{\mu^k})} &\leq C\mu^{\frac{(k-1)n}{n+1}} \left(\int_{Q_{\mu^{k-1}}} |f(z, \tau)|^{n+1} dz d\tau \right)^{\frac{1}{n+1}} \\ &\quad + K\mu^{1+\bar{\alpha}}\|u - l\|_{L^\infty(Q_{\mu^{k-1}})}. \end{aligned}$$

This estimate is correct for any choice for l whatsoever; in particular for $l = L(h_{k-1})$ if $k > 1$ and $l = 0$ if $k = 1$ – we make these choices, recall $l_k(x) = L(h_k)(x)$ for $k \geq 1$ and put $l_0 = 0$. Dividing both sides of the above inequality by $\mu^{k(1+\alpha)}$ and putting $A_k = \|u - l_k\|_{L^\infty(Q_{\mu^k})} / \mu^{k(1+\alpha)}$, we find

$$(4.12) \quad A_k \leq B_{k-1} + K\mu^{(\bar{\alpha}-\alpha)}A_{k-1} = B_{k-1} + \theta A_{k-1} \leq B + \theta A_{k-1},$$

where $\theta = K\mu^{(\bar{\alpha}-\alpha)}$,

$$B_{k-1} = C\mu^{-(1+\alpha)}\mu^{(k-1)(1-\alpha)} \left(\frac{1}{\mu^{(k-1)(n+2)}} \int_{Q_{\mu^{k-1}}} |f(z, \tau)|^{n+1} dz d\tau \right)^{\frac{1}{n+1}}$$

and B is chosen to bound the B_k :

$$B = C\mu^{-(1+\alpha)} \sup_{r \leq 1} \left\{ r^{1-\alpha} \left(\frac{1}{r^{n+2}} \int_{Q_r(y, s)} |f(z, \tau)|^{n+1} dz d\tau \right)^{\frac{1}{n+1}} \right\}.$$

Solving the recursion (4.12), we find

$$A_k \leq B \sum_{j=0}^{k-1} \theta^j + \theta^k A_0 = B \sum_{j=0}^{k-1} \theta^j + \theta^k \|u\|_{L^\infty(Q_1)}.$$

To conclude, choose μ so that $\theta = K\mu^{(\bar{\alpha}-\alpha)} < 1$, and find there is a constant C_1 such that

$$(4.13) \quad \|u - l_k\|_{L^\infty(Q_{\mu^k})} = \mu^{k(1+\alpha)} A_k \leq C_1 \mu^{k(1+\alpha)} M,$$

where

$$M = M(y, s) = \|u\|_{L^\infty(Q_2)} + N$$

and N is given by (4.5).

From (4.15) it follows that

$$\|l_k - l_{k-1}\|_{L^\infty(Q_{\mu^k})} \leq 2C_1 M \mu^{(k-1)(1+\alpha)}$$

and from this, writing $l_k(x) = a_k + \langle b_k, x \rangle$,

$$(4.14) \quad |a_k - a_{k-1}| \leq 2C_1 M \mu^{(k-1)(1+\alpha)}, \quad \mu^k |b_k - b_{k-1}| \leq 4C_1 M \mu^{(k-1)(1+\alpha)}.$$

It follows that

$$a_\infty = \lim_{k \rightarrow \infty} a_k, \quad b_\infty = \lim_{k \rightarrow \infty} b_k$$

exist and

$$|a_k - a_\infty| \leq 2C_1 M \sum_{j=k}^{\infty} \mu^{j(1+\alpha)} \leq MR \mu^{k(1+\alpha)};$$

similarly,

$$\mu^k |b_k - b_\infty| \leq MR \mu^{k(1+\alpha)},$$

where R is determined via the geometric series. Thus

$$\|l_k - l_\infty\|_{L^\infty(Q_{\mu^k})} \leq 2RM \mu^{k(1+\alpha)}.$$

Combining this last estimate with (4.15), with a different constant R , we have, for $k = 0, 1, 2, \dots$,

$$(4.15) \quad \|u - l_\infty\|_{L^\infty(Q_{\mu^k})} \leq RM \mu^{k(1+\alpha)}.$$

Thus $Du(0, 0) = Dl_\infty = b_\infty$ exists. In view of (4.14) and $b_0 = 0$, we also have $|b_\infty| \leq MR$ (for some controlled R). The above proof gives us a $C^{1+\alpha}$ -pointwise estimate at (y, s) , however we will only use the estimate on the size of $Du(y, s)$. We have obtained

$$(4.16) \quad |Du(y, s)| \leq R \left(\|u\|_{L^\infty(Q_2)} + \sup_{r \leq 1} \left\{ r^{(1-\alpha)} \left(\frac{1}{r^{n+2}} \int_{Q_r(y, s)} |f(z, \tau)|^{n+1} dz d\tau \right)^{\frac{1}{n+1}} \right\} \right).$$

We now integrate (4.16) to obtain, for $p \geq 1$,

$$(4.17) \quad \int_{Q_1} |Du(y, s)|^p dy ds \leq (2R)^p \left(|Q_1| \|u\|_{L^\infty(Q_2)}^p + \int_{Q_1} |N(y, s)|^p dy ds \right).$$

Next, we are going to estimate $\|N\|_{L^p(Q_1)}$. The main point of the estimate we summarize with a lemma concerning the mapping

$$\mathcal{M}_\kappa(g)(y, s) = \sup_{r \leq 1} r^\kappa \left(\frac{1}{r^{n+2}} \int_{Q_r(y, s)} |g(x, t)| dx dt \right),$$

which takes $g \in L^1(Q_2)$ into a function on Q_1 . The range of κ will be $0 \leq \kappa \leq n+2$. Note that $\mathcal{M}_0$ is basically a maximal function – it follows that $\mathcal{M}_0$ is of weak type 1-1. On the other hand $\mathcal{M}_{(n+2)}(g)$ is estimated in $L^\infty(Q_1)$ by $\|g\|_{L^1(Q_2)}$. Interpolation between these extremes will prove:

Lemma 4.2. *If $1 \leq q < (n+2)/(n+2-\kappa)$, then there is a constant $C = C(n, \kappa, q)$ such that*

$$\|\mathcal{M}_\kappa(g)\|_{L^q(Q_1)} \leq C\|g\|_{L^1(Q_2)}.$$

We use the lemma and then prove it following the theorem. With $\kappa = (1-\alpha)(n+1)$ and $g = |f|^{n+1}$ we have

$$N(y, s)^{(n+1)} = \mathcal{M}_{(1-\alpha)(n+1)}(f^{n+1})(y, s).$$

Thus by Lemma 4.2, if $p = q(n+1)$

$$(4.18) \quad \left(\int_{Q_1} N(y, s)^p dy ds \right)^{\frac{1}{p}} = \left(\int_{Q_1} N(y, s)^{q(n+1)} dy ds \right)^{\frac{1}{q(n+1)}} \leq C^{1/(n+1)} \|f\|_{L^{n+1}(Q_2)} \Bigg|$$

for $q < (n+2)/((n+2) - (1-\alpha)(n+1))$ - or $p < (n+1)(n+2)/((n+2) - (1-\alpha)(n+1))$. We are still free to choose α as small as desired, and therefore obtain the range $1 \leq p < (n+1)(n+2)$. In (4.18) the constant C is determined by p and α – which itself depends on p . Combined, (4.17) and (4.18) give the desired estimate (4.3). $\square$

Proof of Lemma 4.2. Put

$$(4.19) \quad \eta = \frac{n+2-\kappa}{n+2} \quad \text{so that} \quad \kappa - (1-\eta)(n+2) = 0 \quad \text{and} \quad \eta q < 1.$$

We have

$$\begin{aligned} & r^\kappa \left(\frac{1}{r^{n+2}} \int_{Q_r(y, s)} |g(x, t)| dx dt \right) \\ &= r^\kappa \left(\frac{1}{r^{n+2}} \int_{Q_r(y, s)} |g(x, t)| dx dt \right)^\eta \left(\frac{1}{r^{n+2}} \int_{Q_r(y, s)} |g(x, t)| dx dt \right)^{(1-\eta)} \\ &\leq r^{\kappa - (1-\eta)(n+2)} \mathcal{M}_0(g)(y, s)^\eta \left(\int_{Q_r(y, s)} |g(x, t)| dx dt \right)^{(1-\eta)} \\ &\leq \mathcal{M}_0(g)(y, s)^\eta \|g\|_{L^1(Q_2)}^{(1-\eta)}. \end{aligned}$$

We conclude that

$$(4.20) \quad \|\mathcal{M}_\kappa(g)\|_{L^q(Q_1)} \leq \left(\int_{Q_1} \mathcal{M}_0(g)^{\eta q} \right)^{\frac{1}{q}} \|g\|_{L^1(Q_2)}^{(1-\eta)}.$$

Since $\mathcal{M}_0$ is of weak type 1-1, for $0 < \beta < 1$ there is a constant C_β such that

$$\int_{Q_1} \mathcal{M}_0(g)^\beta \leq C_\beta \|g\|_{L^1(Q_2)}^\beta.$$

Since $\eta q < 1$ we may use this in (4.20) to find $\|\mathcal{M}_\kappa(g)\|_{L^q(Q_1)} \leq (C_{\eta q})^{\frac{1}{q}} \|g\|_{L^1(Q_2)}$. $\square$

Remark 4.3. We outline an alternative strategy of proving (4.3). It was observed by Fok [17], Theorem A.1.4, that by modifying the proof of Caffarelli [5] it is possible to show that solutions to elliptic equations $\mathcal{P}^-(D^2u) = f$, where $f \in C(\Omega) \cap L^n(\Omega)$, satisfy interior $W^{2,p}$ estimates for all $p < n$. Using similar ideas in conjunction with [33] it can be shown that if $f \in C(Q_2) \cap L^{n+1}(Q_2)$ then solutions of the parabolic equation $u_t + \mathcal{P}^-(D^2u) = f$ in Q_2 satisfy for every $1 \leq p < n+1$ the estimate

$$\|u\|_{W^{2,1,p}(Q_1)} \leq C \left(\|u\|_{L^\infty(\partial_p Q_2)} + \|f\|_{L^{n+1}(Q_2)} \right),$$

where $C = C(p, n, \lambda, \Lambda)$. Invoking the imbedding theorem for anisotropic Sobolev spaces [2], Theorem 10.2, we conclude that $Du \in L^q(Q_1)$ for all $q \in [1, (n+1)(n+2))$, which yields (4.3).

We turn to the proof of the generalized maximum principle.

Proof of Theorem 0.2. We begin with simple reductions. Firstly, approximating u by smooth functions it is enough to show (4.1) for $f \in L^{n+1}(Q)$. Secondly, replacing u by $u - \sup_{\partial_p Q} u$, it suffices to prove the result when $u \leq 0$ on $\partial_p Q$. Via translation we can assume that Q (originally $\Omega \times (0, T]$) satisfies

$$Q \subset Q_R$$

for some R . Let then $f \in L^{n+1}(Q)$ and define g on Q_{2R} by

$$g = f^+ \quad \text{on } Q, \quad g = 0 \quad \text{on } Q_{2R} \setminus Q.$$

The problem

$$v_t + \mathcal{P}^-(D^2v) - \gamma|Dv| = g \quad \text{on } Q_{2R}, \quad v = 0 \quad \text{on } Q_{2R}$$

has an L^{n+1} -good solution v on Q_{2R} and $v \geq 0$ on Q_{2R} . It follows that $w = u - v$ is an L^{n+1} -good solution of

$$w_t + \mathcal{P}^-(D^2w) - \gamma|Dw| \leq 0 \quad \text{in } Q, \quad w \leq 0 \quad \text{on } \partial_p Q.$$

Thus $u \leq v$ in Q .

In view of the above, it suffices to prove the following: If $g \in L^{n+1}(Q_{2R})$ and v is the L^{n+1} -good solution of

$$(4.21) \quad v_t + P^-(D^2v) - \gamma|Dv| = g \quad \text{on } Q_{2R}, \quad v = 0 \quad \text{on } \partial_p Q_{2R},$$

then for $(x, t) \in Q_{2R}$

$$(4.22) \quad v(x, t) \leq C\|g^+\|_{L^P(Q_R)} \quad \text{provided } g = 0 \quad \text{on } Q_{2R} \setminus Q_R,$$

where C, P are as in Theorem 0.2.

Let $\mathcal{S}$ be the solution operator for (4.21); that is $v = \mathcal{S}(g)$ is the L^{n+1} -good solution. The (formal) inequality $\mathcal{L}(w) + \mathcal{L}(v) \leq \mathcal{L}(v+w)$, implies the (rigorous) relation $\mathcal{S}(g+h) \leq \mathcal{S}(g) + \mathcal{S}(h)$ for $g, h \in L^{n+1}(Q_{2R})$; thus the evaluation map

$$E_{(x,t)}(g) = \mathcal{S}(g)(x, t)$$

from $L^{n+1}(Q_{2R})$ into $\mathbb{R}$ is subadditive for each $(x, t) \in Q_{2R}$. Since also $\mathcal{S}$ is positive homogeneous of degree 1, both it and $E_{(x,t)}$ are sublinear. In addition, by the maximum principle,

$$(4.23) \quad |E_{(x,t)}(g)| \leq C_0\|g\|_{L^{n+1}(Q_{2R})}.$$

By the Hahn-Banach Theorem if

$$\mathcal{KF}(x, t) = \left\{ k \in L^{\frac{n+1}{n}}(Q_{2R}): \int_{Q_{2R}} k(y, s)g(y, s) dyds \leq E_{(x,t)}(g) \text{ for } g \in L^{n+1}(Q_{2R}) \right\},$$

then

$$(4.24) \quad E_{(x,t)}(g) = \max_{k \in \mathcal{KF}(x,t)} \int_{Q_{2R}} k(y, s)g(y, s) dyds.$$

We call “ $\mathcal{KF}(x, t)$ ” the “kernel functions” at $(x, t) \in Q_{2R}$. As kernel functions are nonnegative (see below), the estimate (4.22) follows from (4.24) if we can prove

$$(4.25) \quad \int_{Q_R} |k(y, s)|^{P/(P-1)} dyds \leq C$$

for $k \in \mathcal{KF}(x, t)$.

In turn, (4.25) will follow upon demonstrating a reverse Hölder inequality satisfied by the kernel functions. We begin with some simple observations and then turn to two straightforward estimates which imply (4.25).

The relations (4.23), $E_{(x,t)}(g) \leq 0$ if $g \leq 0$ and $E_{(x,t)}(g) = 0$ if $g(y, s) = 0$ for $s \leq t$, are equivalently stated in terms of the kernel functions as follows: for $k \in \mathcal{KF}(x, t)$

$$(4.26) \quad \begin{cases} \text{(i)} \quad \|k\|_{L^{((n+1)/n)}(Q_{2R})} \leq C_0, \\ \text{(ii)} \quad 0 \leq k \quad \text{a.e.}, \\ \text{(iii)} \quad k(y, s) = 0 \quad \text{for a.e. } (y, s) \in Q_{2R} \times (t, 0]. \end{cases}$$

From (4.24) we deduce

$$(4.27) \quad \int_{Q_{2R}} k(y, s) (w_t(y, s) + \mathcal{P}^-(D^2 w(y, s)) - \gamma |Dw(y, s)|) dy ds \leq w(x, t)$$

for $k \in \mathcal{KF}(x, t)$ whenever w is “smooth” in Q_{2R} and vanishes on $\partial_p Q_{2R}$ – here “smooth” includes $w \in C(\overline{Q_{2R}})$, $w_t, D^2 w$ in $L^{n+1}(Q_{2R})$.

We begin with the first main estimate – (4.30) below – in a standard way. Below $(x, t) \in Q_{2R}$ is fixed and $k \in \mathcal{KF}(x, t)$.

Fix $(z, \tau) \in Q_{2R}$, $r > 0$, and assume $Q_{2r}(z, \tau) \subset Q_{2R}$. We want to estimate $\int_{Q_{2r}(z, \tau)} k$ when $(x, t) \in Q_{3r/2}(z, \tau)$. Put

$$w(y, s) = \begin{cases} -(s - \tau + 4r^2)(|z - y|^2 - 4r^2)^2 & \text{in } \tau - 4r^2 \leq s \leq \tau, \quad |z - y|^2 \leq 4r^2 \\ 0 & \text{otherwise.} \end{cases}$$

Note that on the support of $k(y, s)w(y, s)$ we have $\tau - 4r^2 \leq s \leq \tau$, so $|s - \tau + 4r^2| \leq 4r^2$. Hence, by simple calculations, we have

$$(4.28) \quad |w_t(y, s) + \mathcal{P}^-(D^2 w(y, s)) - \gamma |Dw(y, s)|| \leq C_2 r^4.$$

In addition, for $(x, t) \in Q_{3r/2}(z, \tau)$, $w(x, t) \leq -r^6$. From this and (4.27)

$$(4.29) \quad \begin{aligned} r^6 \leq -w(x, t) &\leq - \int_{Q_{2R}} k(y, s) (w_t(y, s) + \mathcal{P}^-(D^2 w(y, s)) - \gamma |Dw(y, s)|) dy ds \\ &\leq C_2 r^4 \int_{Q_{2r}(z, \tau)} k(y, s) dy ds. \end{aligned}$$

We conclude: For $k \in \mathcal{KF}(x, t)$, $(x, t) \in Q_{3r/2}(z, \tau)$, $Q_{2r}(z, \tau) \subset Q_{2R}$,

$$(4.30) \quad r^2 \leq C_2 \int_{Q_{2r}(z, \tau)} k(y, s) ds dy.$$

We proceed to the second estimate, (4.32) below. Hereafter, we do not display the “variables of integration”. Assuming $Q_{2r}(z, \tau) \subset Q_{2R}$, choose

$$(4.31) \quad f \in C_0^\infty(Q_r(z, \tau)) \quad \text{with} \quad \|f\|_{L^{n+1}(Q_r(z, \tau))} = 1$$

and solve

$$w_t + \mathcal{P}^-(D^2 w) = f \quad \text{in } Q_{2r}(z, \tau), \quad w|_{\partial_p Q_{2r}(z, \tau)} = 0.$$

This problem has a classical solution as all the ingredients are smooth. Then, if $\varphi \equiv 1$ on $Q_r(z, \tau)$ is smooth, nonnegative and supported in $Q_{3r/2}(z, \tau)$,

$$\begin{aligned}
(4.32) \quad \int_{Q_r(z, \tau)} k f &= \int_{Q_r(z, \tau)} k (w_t + \mathcal{P}^-(D^2 w)) = \int_{Q_{2r}(z, \tau)} k \varphi (w_t + \mathcal{P}^-(D^2 w)) \\
&= \int_{Q_{2r}(z, \tau)} k ((\varphi w)_t + \mathcal{P}^-(D^2(\varphi w) - w D^2 \varphi - 2D\varphi \otimes Dw) - \varphi_t w) \\
&\leq \int_{Q_{2r}(z, \tau)} k ((\varphi w)_t + \mathcal{P}^-(D^2(\varphi w)) - \gamma |D(\varphi w)|) \\
&\quad + C_3 \int_{Q_{2r}(z, \tau)} k ((|\varphi_t| + |D\varphi| + |D^2\varphi|)|w| + (|D\varphi| + \varphi)|Dw|) \\
&\leq C_3 \int_{Q_{2r}(z, \tau)} k ((|\varphi_t| + |D\varphi| + |D^2\varphi|)|w| + (|D\varphi| + \varphi)|Dw|) \\
&\quad + \varphi(x, t)w(x, t).
\end{aligned}$$

For this we used that $\mathcal{P}^-(X)$ is Lipschitz continuous in X denoting $|X|$ to be any norm on $\mathcal{S}(n)$ – the choice of which will affect C_3 . In addition we note that, by the maximum principle estimate,

$$(4.33) \quad |w| \leq C_4 r^{\frac{n}{n+1}}.$$

We proceed, term by term. First, letting $0 \leq \varphi \leq 1$ and using (4.33)

$$|\varphi(x, t)w(x, t)| \leq \begin{cases} 0 & \text{if } (x, t) \notin Q_{3r/2} \\ C_4 r^{\frac{n}{n+1}} & \text{if } (x, t) \in Q_{3r/2}. \end{cases}$$

In both situations, by (4.30),

$$(4.34) \quad |\varphi(x, t)w(x, t)| \leq C_4 C_2 r^{-\frac{n+2}{n+1}} \int_{Q_{2r}(z, \tau)} k.$$

Regarding φ , we choose it so that

$$(4.35) \quad r^2(|\varphi_t| + |D^2\varphi|) + (r^2 + r)|D\varphi| + (1 + r)\varphi \leq C_5$$

for some C_5 . Noting again (4.33), we conclude that

$$(4.36) \quad \int_{Q_{2r}(z, \tau)} k (|\varphi_t| + |D\varphi| + |D^2\varphi|) |w| \leq C_4 C_5 r^{-\frac{n+2}{n+1}} \int_{Q_{2r}(z, \tau)} k.$$

The remaining terms are treated using (4.3) and (4.35) thus: for $1 \leq p < (n+1)(n+2)$

$$(4.37) \quad \begin{aligned} \int_{Q_{2r}} k(|D\varphi| + \varphi)|Dw| &\leq C_5 \frac{1}{r} \left(\int_{Q_{2r}} k^{\frac{p}{p-1}} \right)^{\frac{p-1}{p}} \left(\int_{Q_{3r/2}} |Dw|^p \right)^{1/p} \\ &\leq C_5 C_1 r^{\frac{(n+2)(n+1-p)}{p(n+1)}} \left(\int_{Q_{2r}} k^{\frac{p}{p-1}} \right)^{\frac{p-1}{p}}. \end{aligned}$$

All told, taking the supremum on the left over f 's satisfying (4.31) in (4.32), using (4.34), (4.36) and (4.37), we arrive at

$$\begin{aligned} \left(\int_{Q_r(z, \tau)} k^{\frac{n+1}{n}} \right)^{\frac{n}{n+1}} &\leq C_4(C_2 + C_3 C_5) r^{-\frac{n+2}{n+1}} \int_{Q_{2r}(z, \tau)} k \, ds dy \\ &\quad + C_3 C_5 C_1 r^{\frac{(n+2)(n+1-p)}{p(n+1)}} \left(\int_{Q_{2r}(z, \tau)} k^{\frac{p}{p-1}} \right)^{\frac{p-1}{p}} \\ &\leq \left(C_4(C_2 + C_3 C_5) |Q_{2R}|^{1/p} + C_3 C_5 C_1 \right) r^{\frac{(n+2)(n+1-p)}{p(n+1)}} \left(\int_{Q_{2r}(z, \tau)} k^{\frac{p}{p-1}} \right)^{\frac{p-1}{p}}. \end{aligned}$$

This result may be restated as: for any $Q_{2r}(z, \tau) \subset Q_{2R}$, $W = k^{\frac{p}{p-1}}$ satisfies

$$\left(\frac{1}{r^{n+2}} \int_{Q_r(z, \tau)} W^q \right)^{1/q} \leq b \left(\frac{1}{r^{n+2}} \int_{Q_{2r}(z, \tau)} W \right),$$

where

$$q = \frac{(n+1)(p-1)}{np}, \quad b = \left(C_4(C_2 + C_3 C_5) |Q_{2R}|^{1/p} + C_3 C_5 C_1 \right)^{p/(p-1)}$$

and that is “reverse Hölder” if $q > 1$ (or $p > n+1$). The parabolic version of (for example) [19] Theorem 6.3 provides a bound of the form

$$(4.38) \quad \left(\frac{1}{|Q_R|} \int_{Q_R} W^\alpha \right)^{1/\alpha} \leq C_6 \left(\frac{1}{|Q_{2R}|} \int_{Q_{2R}} W^q \right)^{1/q}$$

for

$$\frac{(n+1)(p-1)}{np} \leq \alpha < \frac{(n+1)(p-1)}{np} + \kappa,$$

where κ is of order $1/b^q$ and C_6 depends on b, q, n and α . Restating this in terms of k amounts to

$$\left(\int_{Q_R} k^\theta \right)^{1/\theta} \leq C_7 \left(\int_{Q_{2R}} k^{\frac{n+1}{n}} \right)^{\frac{n}{n+1}},$$

where

$$\frac{n+1}{n} \leq \theta < \frac{n+1}{n} + \frac{p}{p-1} \kappa.$$

The new C_7 depends on b, q, n, R and θ . The parameter κ itself depends on the choice of p , via the constant C_1 of (4.3). Finally, expressing these relations in terms of $P = \theta/(\theta - 1)$ to correspond to (4.25), we find the range

$$n+1 - \frac{n^2 p \kappa}{p-1 + np \kappa} < P \leq n+1.$$

For any choice of $p > n+1$ we may use $\epsilon = n^2 p \kappa / (p-1 + np \kappa)$ and the proof is complete. The dependence of ϵ on $\gamma d(Q)$ follows from scaling. If u solves (0.1) in Q_r then

$$v(x, t) = u(rx, r^2 t)$$

solves

$$v_t + \mathcal{P}^-(D^2 v) - r\gamma |Dv| \leq r^2 f(rx, r^2 t)$$

in Q_1 . □

A. Appendix: Pointwise Differentiability of $W^{2,1,p}$ Functions. While we do not know that the proposition below has been proved elsewhere, it may well have been. In any case, the presentation below is clean and self-contained. The formulas used in the proof of the following lemma are related to some given in great generality in the book of Besov et al. [2]. Given the lemma, the proposition is finally proved by arguments we learned from R. DeVore.

Proposition A.1. *Let $(n+2)/2 < p$ and $v \in W_{\text{loc}}^{2,1,p}(Q)$. Let $(\hat{x}, \hat{t})$ be a Lebesgue point (in the sense of (2.2)) of v , Dv (in L^1) and $D^2 v$, v_t (in L^p). Then as $(x, t) \rightarrow (\hat{x}, \hat{t})$ we have*

$$(A.1) \quad \left| v(x, t) - \left(v(\hat{x}, \hat{t}) + v_t(\hat{x}, \hat{t})(t - \hat{t}) + \langle Dv(\hat{x}, \hat{t}), x - \hat{x} \rangle + \frac{1}{2} \langle D^2 v(\hat{x}, \hat{t})(x - \hat{x}), x - \hat{x} \rangle \right) \right| \leq o(|x - \hat{x}|^2 + |t - \hat{t}|).$$

We will write K for any translate of the “unit cube” in $\mathbb{R}^{n+1}$; that is any translate of

$$K = \{(x, t): 0 \leq x_i \leq 1, i = 1, \dots, n, 0 \leq t \leq 1\}.$$

Lemma A.2. *For $u \in W^{2,1,p}(K)$, $(n+2)/2 < p$, denote*

$$L_K(u)(x) = \int_K (u(y, s) - \langle y, Du(y, s) \rangle) dy ds + \int_K \langle x, Du(y, s) \rangle dy ds.$$

If $(n+2)/2 < p$ then there is a constant $C = C(p, n)$ such that

$$|u(x, t) - L_K(u)(x)| \leq C (\|D^2 u\|_{L^p(K)} + \|u_t\|_{L^p(K)}) \quad \text{for } (x, t) \in K.$$

Proof. First, write

$$L_K(u)(x) - u(x, t) = \int_K (u(y, s) + \langle Du(y, s), x - y \rangle - u(x, t)) \, dy \, ds$$

and

(A.2)

$$\begin{aligned} & u(y, s) + \langle Du(y, s), x - y \rangle - u(x, t) \\ &= \int_0^1 \frac{d}{d\tau} \left(\tau \langle Du(x + \tau(y - x), t + \tau^2(s - t)), x - y \rangle \right. \\ & \quad \left. + u(x + \tau(y - x), t + \tau^2(s - t)) \right) d\tau \\ &= \int_0^1 \tau \langle D^2 u(x + \tau(y - x), t + \tau^2(s - t))(y - x), x - y \rangle d\tau \\ & \quad + \int_0^1 \tau \langle Du_t(x + \tau(y - x), t + \tau^2(s - t)), x - y \rangle 2\tau(s - t) d\tau \\ & \quad + \int_0^1 u_t(x + \tau(y - x), t + \tau^2(s - t)) 2\tau(s - t) d\tau. \end{aligned}$$

Next integrate each side over K (with respect to (y, s)) and change integration variables in the resulting terms on the right according to

$$z = x + \tau(y - x), \quad h = t + \tau^2(s - t), \quad \tau = \tau;$$

the inverse transformation is

$$y = \frac{z - (1 - \tau)x}{\tau}, \quad s = \frac{h - (1 - \tau^2)t}{\tau^2}, \quad \tau = \tau,$$

while the Jacobian determinant is

$$J = \frac{1}{\tau^n \tau^2} = \frac{1}{\tau^{n+2}}.$$

The range of integration is

$$(1 - \tau)x_i \leq z_i \leq (1 - \tau)x_i + \tau, \quad (1 - \tau^2)t \leq h \leq (1 - \tau^2)t + \tau^2, \quad 0 \leq \tau \leq 1.$$

We have to examine the three resulting terms arising from this procedure on the right of (A.2). It will suffice to let K be the unit cube and consider the

case $x = 0$, $t = 0$, which we do for simplicity in writing. After the integration and the change of variables, the first term to estimate is

$$\begin{aligned} & \int_K \int_0^1 \tau \langle D^2 u(\tau y, \tau^2 s) y, y \rangle d\tau dy ds \\ &= \int_0^1 \int_0^{\tau^2} \int_0^\tau \cdots \int_0^\tau \frac{1}{\tau^{n+2}} \frac{1}{\tau} \langle D^2 u(z, h) z, z \rangle dz_1 \cdots dz_n dh d\tau. \end{aligned}$$

Now with $q = p/(p-1)$ we have

$$\begin{aligned} & \left| \int_0^{\tau^2} \int_0^\tau \cdots \int_0^\tau \langle D^2 u(z, h) z, z \rangle dz_1 \cdots dz_n dh \right| \\ & \leq \left| \int_0^{\tau^2} \int_0^\tau \cdots \int_0^\tau |z|^{2q} dz_1 \cdots dz_n dh \right|^{1/q} \|D^2 u\|_{L^p(K)} \\ & \leq C \tau^{((2q+n+2)/q)} \|D^2 u\|_{L^p(K)}. \end{aligned}$$

Since

$$p > \frac{n+2}{2} \iff q < \frac{n+2}{n} \iff \frac{2q+n+2}{q} > 2+n$$

we have that

$$\left| \int_0^1 \int_0^{\tau^2} \int_0^\tau \cdots \int_0^\tau \frac{1}{\tau^{n+2}} \frac{1}{\tau} \langle D^2 u(z, h) z, z \rangle dz_1 \cdots dz_n dt d\tau \right| \leq C \|D^2 u\|_{L^p(K)}.$$

Similarly,

$$\left| \int_0^1 \int_0^{\tau^2} \int_0^\tau \cdots \int_0^\tau \frac{1}{\tau^{n+2}} \frac{1}{\tau} u_t(z, h) h dz_1 \cdots dz_n dh d\tau \right| \leq C \|u_t\|_{L^p(K)},$$

corresponding to the third term of (A.2). The second term is treated differently. It consists of a sum of terms of which

$$\int_0^1 \int_0^{\tau^2} \int_0^\tau \cdots \int_0^\tau \frac{1}{\tau^{n+2}} \frac{1}{\tau} z_1 \frac{\partial}{\partial z_1} u_t(z_1, z_2, \dots, z_n, h) h dz_1 \cdots dz_n dh d\tau$$

is typical. Integration by parts in z_1 yields one term as above and the boundary contribution

$$\int_0^1 \int_0^{\tau^2} \int_0^\tau \cdots \int_0^\tau \frac{1}{\tau^{n+2}} u_t(\tau, z_2, \dots, z_n, h) h dz_2 \cdots dz_n dh d\tau.$$

Thus we are done if

$$\int_0^1 \int_0^{\tau^2} \int_0^{\tau} \cdots \int_0^{\tau} \frac{1}{\tau^{q(n+2)}} h^q dz_2 \cdots dz_n dh d\tau < \infty$$

and this amounts again to $q < (n+2)/n$. $\square$

As a corollary, scaling parabolically, we have that if $K_r = [0, r]^n \times [0, r^2]$ then

$$(A.3) \quad |u(x, t) - L_{K_r}(u)(x)| \leq Cr^2 \left(\frac{1}{r^{n+2}} \int_{K_r} (|D^2 u|^p + |u_t|^p) \right)^{1/p},$$

where

$$L_{K_r}(u)(x) = \frac{1}{r^{n+2}} \int_{K_r} (u(y, s) - \langle Du(y, s), y \rangle) dy ds + \frac{1}{r^{n+2}} \int_{K_r} \langle Du(y, s), x \rangle dy ds.$$

Proof of Proposition A.1. Let

$$\begin{aligned} u(x, t) &= v(x, t) - \left(v(\hat{x}, \hat{t}) + \langle Dv(\hat{x}, \hat{t}), x - \hat{x} \rangle \right. \\ &\quad \left. + v_t(\hat{x}, \hat{t})(t - \hat{t}) + \frac{1}{2} \langle D^2 v(\hat{x}, \hat{t})(x - \hat{x}), x - \hat{x} \rangle \right) \end{aligned}$$

and apply (A.3) to u with the (parabolic) cube centered at $(\hat{x}, \hat{t})$, as all cubes below will be, to obtain:

$$(A.4) \quad \|u - L_{K_r}(u)\|_{L_\infty(K_r)} \leq \left| Cr^2 \left(\frac{1}{r^{n+2}} \int_{K_r} |D^2 v(y, s) - D^2 v(\hat{x}, \hat{t})|^p + |v_t(y, s) - v_t(\hat{x}, \hat{t})|^p dy ds \right)^{1/p} = r^2 \epsilon(r) \right|$$

Here $\epsilon(r)$ tends to zero as $r \downarrow 0$. This is the result modulo the unwanted linear term $L_{K_r}(u)$.

We next remove the unwanted linear term. Let $\eta(r) = \sup_{0 \leq s \leq r} \epsilon(s)$. Then, $\eta(r) \rightarrow 0$ as $r \downarrow 0$. Let K_k be $K_{2^{-k}r}$ and denote $R_k := L_{K_k}(u) - L_{K_{k-1}}(u)$. From (A.4)

$$\|R_k\|_{L_\infty(K_k)} \leq C 2^{-2k} r^2 \eta(r).$$

From the definition of $L_{K_k}(u)$, it tends to 0 in $L^\infty(K_r)$. Also, owing to R_k being linear in x ,

$$\|R_k\|_{L_\infty(K_r)} \leq C 2^k \|R_k\|_{L_\infty(K_k)} \leq C 2^{-k} r^2 \eta(r).$$

Hence,

$$\|L_{K_r}(u)\|_{L_\infty(K_r)} \leq \sum_{k=1}^{\infty} 2^k \|R_k\|_{L_\infty(K_k)} \leq C r^2 \eta(r)$$

which shows that (A.4) still holds if $L_{K_r}(u)$ is removed. This finishes it. $\square$

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