L^p -THEORY FOR FULLY NONLINEAR UNIFORMLY PARABOLIC EQUATIONS

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Introduction

The theory of fully nonlinear uniformly parabolic equations in spaces with Hölder continuous derivatives is very well developed and can be found in the books of Krylov [21], Dong [11] and Lieberman [23]. Less is known when the equations themselves involve discontinuous functions and solutions are investigated from the point of view of Sobolev spaces. Basic results such as maximum principles, interior Hölder continuity of solutions, convergence results, and some derivative estimates can be found for instance in Krylov [21], [19]. However a lack of adequate regularity results inhibited further development. This obstacle was overcome in Wang [30], [31] wherein $W^{2,1,p}$ and $C^{1+\alpha}$ estimates are obtained for solutions of parabolic analogues of the elliptic equations treated in the fundamental work of Caffarelli [4]. Nonetheless the theory is not yet complete. In the current paper we fill some remaining gaps and present a comprehensive treatment of the L^p -theory for fully nonlinear parabolic equations that will complete a line of development begun in [9]. The end result is a complete infrastructure for the discussion of solutions for a class of fully nonlinear uniformly parabolic equations which depend only measurably on “space time”. The general theory is used to obtain new results on the existence and uniqueness of both generalized (L^p -viscosity) solutions and strong (or $W_{\text{loc}}^{2,1,p}$ pointwise a.e.) solutions of these problems. The framework

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presented here parallels the corresponding elliptic version contained in the three works [6], [8] and [28].

Under structure conditions explained in Section 1, the equation discussed is

$$(PE) \quad u_t + F(x, t, u, Du, D^2u) = f(x, t),$$

where $F: Q \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}(n) \rightarrow \mathbb{R}$, $Q = \Omega \times (0, T]$, $\Omega \subset \mathbb{R}^n$, $\mathcal{S}(n)$ is the space of symmetric real $n \times n$ matrices, and $Du \approx (u_{x_1}, \dots, u_{x_n})$, $D^2u \approx (u_{x_i x_j})$ correspond to the spatial gradient and Hessian matrix of u . In general, $F(x, t, r, p, X)$ needs only be measurable in x, t , but for almost all (x, t) it will be uniformly continuous in r, p, X , uniformly in (x, t) . The “forcing” term f will belong to some $L^p(Q)$. Uniform parabolicity amounts to requiring that the map $X \rightarrow F(x, t, r, p, X)$ be uniformly strictly decreasing at a linear rate in the natural ordering on $\mathcal{S}(n)$ as well as uniformly Lipschitz continuous. This is made quantitative in Section 1.

Three notions of solution considered in this work are: L^p -viscosity solutions, the definition of which requires no regularity of u except continuity (see Section 1), L^p -strong solutions (functions $u \in W_{\text{loc}}^{2,1,p}(Q)$ for which (PE) holds a.e.), and pointwise a.e. solutions (functions u which satisfy (PE) at almost every point with Du , D^2u understood pointwise and *not* in the sense of distributions – u need not lie in any space $W_{\text{loc}}^{2,1,p}(Q)$). Relations among these notions are established. For p in a range determined by the equation one has: if $u \in W_{\text{loc}}^{2,1,p}(Q)$ is an L^p -viscosity solution then it is an L^p -strong solution; if u is an L^p -strong solution then it is an L^p -viscosity solution; if u is an L^p -viscosity solution then it is parabolically twice pointwise differentiable a.e. and is a pointwise a.e. solution; if u is an L^p -viscosity solution of (PE) and G also satisfies the structure conditions, then u is an L^p -viscosity solution of $u_t + G(x, t, u, Du, D^2u) = g(x, t)$, where $g \in L^p$, if and only if it is a pointwise a.e. solution.

The last assertion above is striking. It allows manipulation of equations solved in the L^p -viscosity sense as if they are satisfied pointwise. As the reader can imagine, this is very convenient when discussing regularity. It is a feature of this work that the restrictions on p needed for the assertions above to hold are always of the form $p_0 < p$, where $p_0 < n + 1$ is determined by the equation (see Section 2).

Applications of the theory include existence and uniqueness assertions. To give concrete examples, let us consider (PE) when F has the “sup-inf”

structure associated with stochastic differential games. That is, we put

$$G(x, t, r, p, X) = \sup_{\mu} \inf_{\nu} \left(-\text{trace}(A^{\mu, \nu}(x, t)X) + \langle B^{\mu, \nu}(x, t), p \rangle + c^{\mu, \nu}(x, t)r - f^{\mu, \nu}(x, t) \right),$$

and

$$(0.1) \quad \begin{aligned} F(x, t, r, p, X) &= G(x, t, r, p, X) - G(x, t, 0, 0, 0), \\ f(x, t) &= -G(x, t, 0, 0, 0); \end{aligned}$$

here μ, ν range over countable index sets, $\langle \cdot, \cdot \rangle$ is the inner-product of $\mathbb{R}^n$, while the matrices $A^{\mu, \nu} = (a_{i,j}^{\mu, \nu}) \in \mathcal{S}(n)$, the vectors $B^{\mu, \nu} = (b_1^{\mu, \nu}, \dots, b_n^{\mu, \nu}) \in \mathbb{R}^n$, and the scalars $c^{\mu, \nu}, f^{\mu, \nu}$ are all measurable in (x, t) and further satisfy

$$0 < \lambda I \leq A^{\mu, \nu}(x, t) \leq \Lambda I, \quad |B^{\mu, \nu}(x, t)| \leq \gamma, \quad 0 \leq c^{\mu, \nu}(x, t) \leq \gamma, \quad f(x, t) \in L^p(Q)$$

for some positive constants $0 < \lambda \leq \Lambda$ and γ , uniformly in μ, ν , and $n + 1 \leq p$. The passage from G to F in (0.1) is to achieve the normalization $F(x, t, 0, 0, 0) = 0$ which will be used throughout this work. We then have that for each continuous function ψ on the parabolic boundary $\partial_p Q$ of Q , there is a $u \in C(\overline{Q})$ satisfying $u = \psi$ on $\partial_p Q$ which is an L^p -viscosity solution of (PE) in Q . Moreover, if the $A^{\mu, \nu} = A^\mu$ are independent of ν (so that $F(x, t, r, p, X)$ is convex in X) and are continuous, uniformly in μ while $\sup_{\mu, \nu} |f^{\mu, \nu}| \in L^p(Q)$, then the solution u is unique and is an L^p -strong solution of (PE) in Q . Of course, these statements admit technical variations.

In order to prove existence one approximates and in order to then take limits one requires general results to the effect that this can be done. It is shown in Section 6 that the limit of a uniformly convergent sequence of L^p -viscosity solutions of a sequence of equations of the form (PE) solves the limit equation in considerable generality. To obtain the uniform convergence, estimates of moduli of continuity are required. In Section 5 Krylov-Safonov style Hölder continuity in the interior is proved generalizing results of Wang; coupling this with barriers at the boundary one obtains the required compactness of approximations (see [10], Section 4). To obtain strong solutions, gradient estimates on solutions of (PE) are required. In Section 7 we provide the parabolic generalization of the estimates of Świąch [28] in the elliptic case. These estimates are one of the primary contributions of this work and present a rather complete picture. If $p > n + 2$ and F is sufficiently regular in x, t , results of Wang [31] show that solutions of (PE) are in $C_{\text{loc}}^{1+\alpha}$ for some $\alpha > 0$. This is no longer

true if $p \leq n + 2$. However we prove that if $p_0 < p \leq n + 2$ then Du belongs to L^q_{loc} for $q < p(n + 2)/(n + 2 - p)$.

While we attempted to write this paper so that a reader might scan the results it contains without reference to other works, it otherwise depends strongly on [9] for some perspective on the literature and for preliminary results. Results from [9] are recounted in Section 2 after a discussion of preliminaries in Section 1. In addition, the issue of existence of L^p -viscosity solutions has been treated separately in [10] making use of results of Sections 5 and 6 as mentioned above.

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Section 1. Notation, Definitions and Basic Assumptions

To begin, Ω will be a bounded open domain in $\mathbb{R}^n$, $T > 0$ and Q will be the associated parabolic cylinder:

$$Q = \Omega \times (0, T].$$

The reader may want to scan the topics below and return to the details as needed.

Structure Conditions on F

We first discuss the structure conditions to be placed on $F: Q \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}(n) \rightarrow \mathbb{R}$. These are parameterized by constants $0 < \lambda \leq \Lambda$, $0 \leq \gamma$ and a “modulus” ω (continuous subadditive function $\omega: [0, \infty) \rightarrow [0, \infty)$ such that $\omega(0) = 0$) and involve the “Pucci extremals” $\mathcal{P}^+, \mathcal{P}^-$ associated with λ, Λ . These are defined on $\mathcal{S}(n)$ according to

$$(1.1) \quad \begin{cases} \mathcal{P}^-(X) = -\Lambda \operatorname{trace}(X^+) + \lambda \operatorname{trace}(X^-), \\ \mathcal{P}^+(X) = -\lambda \operatorname{trace}(X^+) + \Lambda \operatorname{trace}(X^-) \end{cases}$$

for $X \in \mathcal{S}(n)$, the set of real symmetric $n \times n$ matrices. Here $\operatorname{trace}(X^+)$ (respectively, $\operatorname{trace}(X^-)$) is the sum of the positive eigenvalues of X (respectively, $-X$).

The basic structure condition on $F(x, t, r, p, X)$ with these parameters is that for almost all $(x, t) \in Q$

$$(SC) \quad \begin{cases} F \text{ is nondecreasing in } r, F(x, t, 0, 0, 0) = 0, \text{ and} \\ \mathcal{P}^-(X - Y) - \gamma|p - q| - \omega(|s - r|) \\ \leq F(x, t, r, p, X) - F(x, t, s, q, Y) \\ \leq \mathcal{P}^+(X - Y) + \gamma|p - q| + \omega(|r - s|) \end{cases}$$

for $X, Y \in \mathcal{S}(n)$, $p, q \in \mathbb{R}^n$ and $r, s \in \mathbb{R}$.

Let us illuminate the content of (SC). First, the F of (0.1) satisfies (SC) with the same λ, Λ, γ as occurs in the conditions on its ingredients and $\omega(r) = \gamma r$, as one can confirm along the way. To continue, we use the norm

$$(1.2) \quad \|X\| = \operatorname{trace}(X^+) + \operatorname{trace}(X^-)$$

on $\mathcal{S}(n)$ - that is, $\|X\|$ is the sum of the absolute values of the eigenvalues of X . (SC) is the conjunction of the following requirements: $F(x, t, 0, 0, 0) = 0$ (which is a normalization), $F(x, t, r, p, X)$ is Lipschitz continuous in $p \in \mathbb{R}^n$ with constant γ uniformly in the other variables; $F(x, t, r, p, X)$ is nondecreasing in r and has ω as a modulus of continuity uniformly in the other variables; $F(x, t, r, p, X + P) \leq F(x, t, r, p, X) - \lambda\|P\|$ if P is a positive matrix; $F(x, t, r, p, X)$ is Lipschitz continuous in $X \in \mathcal{S}(n)$ with constant Λ uniformly in the other variables. Checking this assertion will render (SC) transparent to the reader. No restriction is placed on the dependence of F in (x, t) , all that is assumed in this regard is measurability and that (SC) holds *for almost*

all $(x, t) \in Q$. The reader should check that $\mathcal{P}^+(X) + \gamma|p|$ is the largest and $\mathcal{P}^-(X) - \gamma|p|$ is the smallest choice for F satisfying (SC) with $\omega = 0$; similarly, it is instructive to check that F in (0.1) satisfies (SC).

Point Sets, Parabolic Distance

Concerning the bounded open domain Ω in $\mathbb{R}^n$, the results of this note will either demand no further condition on Ω (typical of results of “maximum principle” type) or that Ω satisfy a uniform exterior cone condition (typical of results involving existence assertions or arguments) which guarantees the existence of barriers. In this regard, see [10]. The boundary of Ω is $\partial\Omega$ and its closure is $\bar{\Omega}$. The open ball in $\mathbb{R}^n$ of radius r and center 0 is denoted by B_r .

The parabolic boundary of $Q = \Omega \times (0, T]$ is denoted by $\partial_p Q$:

$$\partial_p Q = (\partial\Omega \times (0, T]) \cup (\bar{\Omega} \times \{0\}).$$

We will employ parabolic neighborhoods of points $(x, t) \in Q$. Set

$$Q_r = B_r \times (-r^2, 0], \quad Q_r(x, t) = Q_r + (x, t).$$

It is clear what set $\partial_p Q_r(x, t)$ denotes.

The parabolic distance between (x, t) and (y, s) in $\mathbb{R}^n \times \mathbb{R}$ is

$$d((x, t), (y, s)) = (|x - y|^2 + |t - s|)^{1/2}.$$

This is the metric in which the diameter $\text{diam}(Q)$ and the distances $\text{dist}((x, t), \partial_p Q)$, $\text{dist}(Q', \partial_p Q)$ are measured. We will use the abbreviations $d(Q)$ for $\text{diam}(Q)$ and $d(\Omega)$ for $\text{diam}(\Omega)$. In particular we have $d(Q_r) = \sqrt{5}r$, and $d(B_r) = 2r$. The multiple uses of “ d ” are distinguished by context. The notation $Q' \subset\subset Q$ means $Q' \subset Q$ and $\text{dist}(Q', \partial_p Q) > 0$.

Function Classes and Norms

Let, for the moment, Q be an arbitrary subset of $\mathbb{R}^n \times \mathbb{R}$. We will use the function classes $C(\bar{Q})$, $L^p(Q)$, $C^\kappa(Q)$, $0 < \kappa \leq 1$, $C^{1+\alpha}(Q)$, and $W^{2,1,p}(Q)$ together with their local versions. Concerning $C(\bar{Q})$, $L^p(Q)$, these are normed by

$$\|u\|_{L^\infty(Q)} \quad \text{and} \quad \|u\|_{L^p(Q)} = \left(\int_Q |f(x, t)|^p dx dt \right)^{\frac{1}{p}} = \left(\int_Q |f|^p \right)^{\frac{1}{p}}$$

if $p < \infty$, and no more needs be said. The other spaces are defined and normed as follows:

$C^\kappa(Q)$: the space of those $u: Q \rightarrow \mathbb{R}$ such that

$$\|u\|_{C^\kappa(Q)} := \|u\|_{L^\infty(Q)} + \sup_{\substack{(x,t),(y,s) \in Q \\ (y,s) \neq (x,t)}} \frac{|u(x,t) - u(y,s)|}{d^\kappa((x,t), (y,s))} < \infty.$$

$C^{1+\alpha}(Q)$: the space of those u whose spatial gradient $Du(x,t)$ exists classically for every $(x,t) \in Q$ and that satisfy

$$\|u\|_{C^{1+\alpha}(Q)} = \|u\|_{L^\infty(Q)} + \|Du\|_{L^\infty(Q)} + \sup_{\substack{(x,t),(y,s) \in Q \\ (y,s) \neq (x,t)}} \frac{|u(y,s) - u(x,t) - \langle Du(x,t), y - x \rangle|}{d^{1+\alpha}((x,t), (y,s))} < \infty.$$

It is easy to see that if $u \in C^{1+\alpha}(Q)$ then (each component of) $Du \in C^\alpha(Q)$ and u is Hölder continuous with exponent $(1+\alpha)/2$ in t .

$W^{2,1,p}(Q)$: the space of those u which satisfy $u, Du, D^2u, u_t \in L^p(Q)$ in the sense of distributions normed by:

$$\|u\|_{W^{2,1,p}(Q)} = \left(\|u\|_{L^p(Q)}^p + \|u_t\|_{L^p(Q)}^p + \|Du\|_{L^p(Q)}^p + \|D^2u\|_{L^p(Q)}^p \right)^{\frac{1}{p}}.$$

Remark 1.1. In this paper it will always be the case that $(n+2)/2 < p$. This guarantees that $W_{\text{loc}}^{2,1,p}(Q)$ is imbedded in $C(Q)$ (see [2], Theorem 10.4) and if $u \in W_{\text{loc}}^{2,1,p}(Q)$ then u is “twice parabolically differentiable a.e.” (see the appendix of [9]). By saying $Du, D^2u \in L^p(Q)$ we mean that all components of Du and all entries of D^2u are in $L^p(Q)$.

$H^{\frac{1}{2},p}(Q)$: the Nikol’skii space (see [1], page 225) of those u such that

$$\|u\|_{H^{\frac{1}{2},p}(Q)} = \|u\|_{L^p(Q)} + \sup_{\substack{h \in \mathbb{R}^{n+1} \\ 0 < |h| < \eta}} \left(\int_{Q^\eta} \frac{|u(z+h) - u(z)|^p}{|h|^{\frac{p}{2}}} dz \right)^{\frac{1}{p}} < +\infty,$$

where $|h|$ is the usual Euclidean norm of $h \in \mathbb{R}^{n+1}$, and

$$Q^\eta = \{z \in Q: \text{distance}(z, \partial Q) \geq 2\eta\}$$

(the distance being the Euclidean distance in $\mathbb{R}^{n+1}$).

$H^{1,\frac{1}{2},p}(Q)$: the parabolic Nikol'skii space of those $u \in H^{\frac{1}{2},p}(Q)$ which satisfy $Du \in L^p(Q)$ normed by

$$\|u\|_{H^{1,\frac{1}{2},p}(Q)} = \|u\|_{H^{\frac{1}{2},p}(Q)} + \|Du\|_{L^p(Q)}.$$

Strong and L^p -viscosity Notions of Solutions, etc.

We assume that F satisfies (SC) and begin by defining subsolutions. For each choice of “regularity” class $\mathcal{R} = C(Q)$ or $\mathcal{R} = L^p_{\text{loc}}(Q)$ and $f \in \mathcal{R}$, we say that u is an $\mathcal{R}$ -viscosity subsolution of

$$(PE) \quad u_t + F(x, t, u, Du, D^2u) = f(x, t)$$

in Q provided that u is continuous and if φ is a “test function” whose derivatives (in the sense of distributions) satisfy $\varphi_t, D\varphi, D^2\varphi \in \mathcal{R}$ and $(\hat{x}, \hat{t}) \in \Omega \times (0, T)$ is a point such that

$$(1.3) \quad \varphi_t(x, t) + F(x, t, u(x, t), D\varphi(x, t), D^2\varphi(x, t)) \geq f(x, t) + \delta$$

for some $\delta > 0$ and almost all (x, t) in a Euclidean neighborhood of $(\hat{x}, \hat{t})$ in $\mathbb{R}^{n+1}$, then $u - \varphi$ cannot have a local maximum at $(\hat{x}, \hat{t})$. Equivalently, if φ has the regularity above and $u - \varphi$ has a maximum at $(\hat{x}, \hat{t})$, then

$$(1.4) \quad \text{essential} \liminf_{(x,t) \rightarrow (\hat{x}, \hat{t})} \left(\varphi_t(x, t) + F(x, t, u(x, t), D\varphi(x, t), D^2\varphi(x, t)) - f(x, t) \right) \leq 0.$$

It turns out that it doesn't matter whether we use parabolic or Euclidean neighborhoods of $(\hat{x}, \hat{t}) \in \mathbb{R}^{n+1}$ in this definition. The two notions are equivalent, see Lemma 1.4 below. In particular, in (1.4) one can require that $t \uparrow \hat{t}$. Throughout this work we will usually deal with parabolic neighborhoods.

As is standard, an $\mathcal{R}$ -viscosity supersolution then is defined by replacing “maximum” by “minimum”, reversing the inequality (1.4), and replacing $\liminf$ by $\limsup$, while an $\mathcal{R}$ -viscosity solution is a function which is both an $\mathcal{R}$ -viscosity subsolution and an $\mathcal{R}$ -viscosity supersolution.

We will say “ u is an $\mathcal{R}$ -viscosity solution of $u_t + F(x, t, u, Du, D^2u) \leq f$ ” and mean “ u is an $\mathcal{R}$ -viscosity subsolution of $u_t + F(x, t, u, Du, D^2u) = f$ ”.

Finally, we abbreviate the classes $L^p_{\text{loc}}(Q)$, $C(Q)$ to simply L^p , C in this context. Thus we speak of “ L^p -viscosity subsolutions”, etc. Our notation varies a bit from that of Wang [30] in this regard.

Remark 1.2. The L^p -viscosity definitions just given “make sense” without restrictions on $(n + 2)/2 < p$ (recall Remark 1.1), without requiring F to satisfy (SC), and for arbitrary f . However, we will always have $f \in L^p_{\text{loc}}(Q)$ and further restrictions on p are needed later. Similar comments hold for C -viscosity notions; they are not useful unless F, f are continuous. In this event, C -viscosity notions are the same as the classical viscosity solutions concepts (e.g., [7]), except that we are only concerned about continuous functions u here.

Remark 1.3. The test functions φ used in the L^p -viscosity notions were assumed to lie in $W^{2,1,p}_{\text{loc}}(Q)$. However, it is evident that the notions are local and do not change if φ in (1.3) is merely required to be in $W^{2,1,p}$ in some neighborhood of $(\hat{x}, \hat{t})$.

Strong subsolutions, supersolutions and solutions are used here for $\mathcal{R} = L^p_{\text{loc}}(Q)$, which corresponds to the class $W^{2,1,p}_{\text{loc}}(Q)$ of functions u with $u_t, Du, D^2u \in L^p_{\text{loc}}(Q)$. That is, u is an L^p -strong subsolution of (PE) if $u \in W^{2,1,p}_{\text{loc}}(Q)$ and

$$u_t(x, t) + F(x, t, u(x, t), Du(x, t), D^2u(x, t)) \leq f(x, t)$$

holds a.e., etc.

As mentioned above, throughout this paper we will usually work with parabolic neighborhoods. To justify this approach, now we will show that “two-sided” viscosity notions which involve full neighborhoods of $(\hat{x}, \hat{t})$ in $\mathbb{R}^{n+1}$ are equivalent to “one-sided” notions that use parabolic neighborhoods $Q_r(\hat{x}, \hat{t})$. It is clear that one-sided implies two-sided.

Lemma 1.4. *Let F satisfy (SC), $(n + 2)/2 < p$ and $f \in L^p_{\text{loc}}(Q)$. Let u be an L^p -viscosity subsolution of $u_t + F(x, t, u, Du, D^2u) = f$ in Q . Let $(\hat{x}, \hat{t}) \in Q$, $0 < r, \delta$, $Q_r(\hat{x}, \hat{t}) \subset\subset Q$, and $\varphi \in W^{2,1,p}_{\text{loc}}(Q)$ be a strong solution of $\varphi_t + F(x, t, u(x, t), D\varphi, D^2\varphi) \geq f + \delta$ in $Q_r(\hat{x}, \hat{t})$. Then $u - \varphi$ does not have a local maximum at $(\hat{x}, \hat{t})$ relative to $Q_r(\hat{x}, \hat{t})$.*

Proof. We sketch a proof by contradiction. Assume that $\varphi, r, \delta, (\hat{x}, \hat{t})$ are as in the statement but that $u - \varphi$ has a maximum at $(\hat{x}, \hat{t})$ relative to $Q_r(\hat{x}, \hat{t})$. By Remark 1.1 φ is continuous in the closure of $Q_r(\hat{x}, \hat{t})$. Let $\tilde{\varphi}(x, t) =$

$\varphi(x, t) + \nu(|x - \hat{x}|^2 + (\hat{t} - t))$, where $\nu > 0$ is to be chosen. Owing to the properties (SC), as soon as ν is sufficiently small $\tilde{\varphi}_t + F(x, t, u(x, t), D\tilde{\varphi}, D^2\tilde{\varphi}) \geq f + \delta/2$ a.e. in $Q_r(\hat{x}, \hat{t})$ while $(\hat{x}, \hat{t})$ is a *strict* maximum of $u - \tilde{\varphi}$ relative to the closure of $Q_r(\hat{x}, \hat{t})$. We thus simply assume that the same obtains for $u - \varphi$. Then put $\tilde{\varphi}(x, t) = \varphi(x, t) + \nu/(\hat{t} - t)$, where again $0 < \nu$. Clearly $\tilde{\varphi}_t + F(x, t, u(x, t), D\tilde{\varphi}, D^2\tilde{\varphi}) \geq \varphi_t + F(x, t, u(x, t), D\varphi, D^2\varphi) \geq f(x, t) + \delta$ in $Q_r(\hat{x}, \hat{t})$ while $u - \tilde{\varphi}$ has a maximum relative to the closure of $Q_r(\hat{x}, \hat{t})$ at some point $(\tilde{x}, \tilde{t})$ with $\tilde{t} < \hat{t}$. Due to the strictness, $(\tilde{x}, \tilde{t}) \rightarrow (\hat{x}, \hat{t})$ as $\nu \downarrow 0$, but then the assumption that u is a (two-sided) L^p -viscosity subsolution of (PE) in Q is violated at $(\tilde{x}, \tilde{t})$ when ν is sufficiently small to guarantee that $(\tilde{x}, \tilde{t})$ is interior to $Q_r(\hat{x}, \hat{t})$. $\square$

Remark 1.5. The reduction to strict extrema and the derivation of “one-sided” from “two-sided” above follow standard arguments, see for instance Proposition I.2 in Lions [24]. Note in particular that $\hat{t} = T$ is allowed above, while the two-sided L^p -viscosity notions do not impose any condition at T .

Remark 1.6. Lemma 1.4 is formulated for subsolutions. However, results for subsolutions always imply symmetrical results for supersolutions for the following reason: u is an L^p -viscosity solution of $u_t + F(x, t, u, Du, D^2u) \leq f$ if and only if $v = -u$ is an L^p -viscosity solution of $v_t + \tilde{F}(x, t, v, Dv, D^2v) \geq -f$ where $\tilde{F}(x, t, r, p, X) = -F(x, t, -r, -p, -X)$. Moreover, F satisfies (SC) if and only if $\tilde{F}$ satisfies (SC). This remark is used throughout this work, not always by explicit reference.

Contact Sets

The following notation is required. If $w: Q \rightarrow \mathbb{R}$, then

$$\begin{aligned} ,^+(w, Q) &= \{(x, t) \in Q : w(x, t) \geq \sup_{\partial_p Q} w, \text{ and } \exists p \in \mathbb{R}^n \text{ such that} \\ &\quad w(y, s) \leq w(x, t) + \langle p, y - x \rangle \text{ for } (y, s) \in Q, s \leq t\}. \end{aligned}$$

$,^+(w, Q)$ will be called the “upper contact set” of w . Moreover, if $\kappa > 0$, then

$$\begin{aligned} ,^+_{\kappa}(w, Q) &= \{(x, t) \in Q : w(x, t) \geq \sup_{\partial_p Q} w \text{ and } \exists p \in \mathbb{R}^n, |p| \leq \kappa, \text{ such that} \\ &\quad w(y, s) \leq w(x, t) + \langle p, y - x \rangle \text{ for } (y, s) \in Q, s \leq t\}. \end{aligned}$$

Section 2. Preliminaries on Maximum Principles and Consistency

In this section we first present various results of “maximum principle” type. After this some observations are made concerning the “consistency” of the notions: that is, strong solutions are viscosity solutions and C -viscosity solutions are L^p -viscosity solutions when both notions are appropriate. Other results needed for these purposes and later are also given.

We first formulate the fundamental parabolic “maximum principle” in the form needed here. Recall that $Q = \Omega \times (0, T]$. For $u: Q \rightarrow \mathbb{R}$ we set

$$\{u > k\} = \{(x, t) \in Q: u(x, t) > k\}.$$

Theorem 2.1. *Let $f \in L^{n+1}(Q)$ and let $u \in C(\overline{Q})$ be an L^{n+1} -viscosity solution of*

$$(2.1) \quad u_t + \mathcal{P}^-(D^2u) - \gamma|Du| \leq f \quad \text{on } \{u > \sup_{\partial_p Q} u\}.$$

Then for $0 < \kappa \leq (\sup_Q u - \sup_{\partial_p Q} u)/d(\Omega)$

$$(2.2) \quad \kappa \leq Cd(\Omega)^{-\frac{1}{n+1}} \|f^+\|_{L^{n+1}(\Gamma_\kappa^+(u, Q))},$$

where $C = C(1/\lambda, n, |Q|\gamma^{n+1}/d(\Omega))$ depends only on the indicated quantities and is bounded for bounded arguments. In particular,

$$(2.3) \quad \sup_Q u \leq \sup_{\partial_p Q} u + d(\Omega)^{\frac{n}{n+1}} \cdot C \cdot \|f^+\|_{L^{n+1}(\Gamma^+(u, Q))}.$$

This result is essentially Theorem 1.1 of [9] as augmented by the remarks following the formal statement. While only the consequence (2.3) of (2.2) is stated in [9], the proof in fact relies on (2.2) via the results of Tso [29] invoked in the proof. It is an extension of a sequence of results by Krylov, Tso, Wang and others [20], [29], [26], [25], [30], [12] as explained in [9].

Remark 2.2. In case when $f \in C(Q)$ the conclusion of Theorem 2.1 is valid for a C -viscosity solution $u \in C(\overline{Q})$ of (2.1), and is in fact considerably easier

to prove, see Lemma 1.2 in [9]. We will refer to this result as classical viscosity solution maximum principle.

Remark 2.3. It follows from Remark 1.6 that if u is an L^{n+1} -viscosity solution of $u_t + \mathcal{P}^+(D^2u) + \gamma|Du| \geq f$, then $v = -u$ is an L^{n+1} -viscosity solution of $v_t + \mathcal{P}^-(D^2v) - \gamma|Dv| \leq -f$; in particular (2.3) implies that an L^{n+1} -viscosity solution u of $u_t + \mathcal{P}^+(D^2u) + \gamma|Du| \geq f$ on $\{u < \inf_{\partial_p Q} u\}$, will satisfy

$$\inf_Q u \geq \inf_{\partial_p Q} u - d(\Omega)^{\frac{n}{n+1}} \cdot C \cdot \|f^-\|_{L^{n+1}(\Gamma^+(-u, Q))}.$$

The following result is proved in [9], Section 3. It will be referred to as the “generalized maximum principle”, highlighting that the integral in (2.4) is over Q and not the contact set.

Theorem 2.4. *There exists $p_0 = p_0(n, \lambda, \Lambda, \gamma d(Q)) < n+1$ with the following property: If $f \in L^p(Q)$ and $u \in C(\overline{Q})$ is an L^p -strong solution of*

$$u_t + \mathcal{P}^-(D^2u) - \gamma|Du| \leq f \quad \text{in } Q,$$

then for $p_0 < p$

$$(2.4) \quad u \leq \sup_{\partial_p Q} u + C \left(\int_Q (f^+)^p \right)^{\frac{1}{p}}.$$

Here $C = C(\lambda, \Lambda, n, p, \gamma d(Q))$.

Hereafter p_0 will occur in the formulation of results via an assumption

$$p_0 < p;$$

p_0 always means the p_0 of Theorem 2.4. We note that necessarily $p_0 > (n+2)/2$ (as $W_{\text{loc}}^{2,1,p}(Q)$ is not embedded in $C(Q)$ if $p \leq (n+2)/2$).

Remark 2.5. If $Q = Q_r$, $r \leq 1$, and $p > p_0(n, \lambda, \Lambda, \gamma\sqrt{5})$, then by parabolic scaling to the unit cylinder - $v(x, t) = u(rx, r^2t)$ solves $v_t + \mathcal{P}^-(D^2v) - r\gamma|Dv| \leq r^2f(rx, rt)$ - we have

$$u \leq \sup_{\partial_p Q_r} u + Cr^{2-\frac{n+2}{p}} \left(\int_{Q_r} (f^+)^p \right)^{\frac{1}{p}}.$$

The next result differs from Theorem 2.4 in being formulated for more general equations and in that it applies to L^p -viscosity subsolutions rather than to L^p -strong solutions. We point out here that the dependence of p_0 on $\gamma d(Q)$ complicates the bookkeeping and we pay attention to it in what follows. In the statements of all theorems the exponent p_0 will always depend on the γ and $d(Q)$ which appear there and so it may change from one theorem to another.

Proposition 2.6. *Let F satisfy (SC), $k > 0$, $p > p_0(n, \lambda, \Lambda, k)$, $f \in L^p(Q)$, $u \in C(\overline{Q})$, and $\gamma d(Q) \leq k$. If u is an L^p -viscosity solution of $u_t + F(x, t, u, Du, D^2u) \leq f$ in Q , then*

$$(2.5) \quad \sup_Q u^+ \leq \sup_{\partial_p Q} u^+ + Cd(Q)^{2-\frac{n+2}{p}} \|f^+\|_{L^p(Q)},$$

where $C = C(\lambda, \Lambda, n, p, k)$ is from Theorem 2.4.

In proving Proposition 2.6 we use the two following results. The first is elementary and collects some obvious consequences of (SC) and the definitions.

Lemma 2.7. *Let F satisfy (SC) and $f, g \in L^p_{\text{loc}}(Q)$. Then:*

(i) *If u is an L^p -viscosity solution of $u_t + F(x, t, u, Du, D^2u) \leq f$ in Q and φ is an L^p -strong solution of $\varphi_t + \mathcal{P}^-(D^2\varphi) - \gamma|D\varphi| - \omega(\varphi^-) \geq g$ in Q , then $w = u - \varphi$ is an L^p -viscosity solution of $w_t + F(x, t, w, Dw, D^2w) \leq f - g$ in Q .*

(ii) *If u is an L^p -viscosity supersolution of $u_t + F(x, t, u, Du, D^2u) \geq f$ in Q and φ is an L^p -strong solution of $\varphi_t + \mathcal{P}^+(D^2\varphi) + \gamma|D\varphi| + \omega(\varphi^+) \leq g$ in Q , then $w = u - \varphi$ is an L^p -viscosity solution of $w_t + F(x, t, w, Dw, D^2w) \geq f - g$ in Q .*

Proof. Statement (i) follows from observing that the formal consequence

$$(2.6) \quad \begin{aligned} & (u - \varphi)_t + F(x, t, (u - \varphi), D(u - \varphi), D^2(u - \varphi)) \\ & \leq u_t + F(x, t, u, Du, D^2u) - (\varphi_t + \mathcal{P}^-(D^2\varphi) - \gamma|D\varphi| - \omega(\varphi^-)) \end{aligned}$$

of (SC) is correct in the sense of viscosity solutions under the conditions of the lemma. The second statement is equivalent to the first via Remark 1.6. $\square$

The second tool we need is Theorem 2.1 of [9], a fundamental existence result for extremal equations. It is proved in [9] using the results of Wang [30] and Escauriaza [13]; see [9] in this regard. It will be used often, a typical example being its role in the proof of Proposition 2.6.

Theorem 2.8. *Let Ω satisfy a uniform exterior cone condition, $p > p_0$, $f \in L^p(Q)$, and $\psi \in C(\partial_p Q)$. Then there exists $u \in C(\overline{Q}) \cap W_{\text{loc}}^{2,1,p}(Q)$ such that $u = \psi$ on $\partial_p Q$ and*

$$(2.7) \quad u_t(x, t) + \mathcal{P}^-(D^2 u(x, t)) - \gamma |Du(x, t)| = f(x, t) \quad \text{a.e. in } Q.$$

Moreover, for $Q' \subset\subset Q$

$$\|u\|_{W^{2,1,p}(Q')} \leq C(\|\psi\|_{L^\infty(\partial_p Q)} + \|f\|_{L^p(Q)}),$$

where $C = C(n, p, \gamma, \lambda, \Lambda, d(\Omega), T, \text{dist}(Q', \partial_p Q))$.

We return to the proof of Proposition 2.6.

Proof of Proposition 2.6. First of all, by approximation from the interior, we can assume without loss of generality that Ω satisfies a uniform exterior cone condition. Next we notice that if u is an L^p -viscosity solution of $u_t + F(x, t, u, Du, D^2 u) \leq f$ then by (SC) u is an L^p -viscosity solution of

$$u_t + \mathcal{P}^-(D^2 u) - \gamma |Du| + F(x, t, u, 0, 0) \leq f(x, t).$$

Since $v \equiv 0$ is an L^p -viscosity solution of

$$v_t + \mathcal{P}^-(D^2 v) - \gamma |Dv| + F(x, t, v, 0, 0) = 0$$

it is immediate from the definitions that $w = u^+ = \max(u, 0)$ is an L^p -viscosity solution of

$$w_t + \mathcal{P}^-(D^2 w) - \gamma |Dw| + F(x, t, w, 0, 0) \leq f^+(x, t),$$

and thus, since $F(x, t, w(x, t), 0, 0) \geq 0$, is also an L^p -viscosity solution of

$$w_t + \mathcal{P}^-(D^2 w) - \gamma |Dw| \leq f^+(x, t).$$

We now solve, via Theorem 2.8,

$$v_t + \mathcal{P}^-(D^2 v) - \gamma |Dv| = f^+(x, t) \quad \text{in } Q, \quad v = u^+ \quad \text{on } \partial_p Q.$$

Then, according to Lemma 2.7, $z = w - v$ is an L^p -viscosity solution of

$$z_t + \mathcal{P}^-(D^2z) - \gamma|Dz| \leq 0.$$

Since the equation is continuous, “ L^p -viscosity” implies the usual notion of viscosity solution ([7]). By the classical viscosity maximum principle, see Remark 2.2, $z \leq 0$. The result now follows from (2.4) applied to v . $\square$

We end this section with the results on the compatibility of various notions of solutions. In the next result we show that if F, f are continuous, then “ C -viscosity” implies “ L^p -viscosity”; the converse is trivial and was used above.

Lemma 2.9. *Let $p_0 < p$, $f \in C(Q)$ and F be continuous in all its arguments and satisfy (SC). Then C -viscosity (sub-,super-) solutions of $u_t + F(x, t, u, Du, D^2u) = f$ in Q are also L^p -viscosity (sub-,super-) solutions of $u_t + F(x, t, u, Du, D^2u) = f$ in Q .*

Proof. Assume u is a C -viscosity subsolution of $u_t + F(x, t, u, Du, D^2u) \leq f$, $\varphi \in W_{\text{loc}}^{2,1,p}$ and $u - \varphi$ has a strict maximum at $(\hat{x}, \hat{t}) \in Q$ relative to the closure of a small parabolic neighborhood $Q_r(\hat{x}, \hat{t})$ of $(\hat{x}, \hat{t})$, while $\varphi_t + F(x, t, u, D\varphi, D^2\varphi) \geq f + \delta$ near $(\hat{x}, \hat{t})$ for some $\delta > 0$ (see Remark 1.5). We seek a contradiction.

By the device in the proof of Lemma 1.2 of [9], we may assume that u is an L^∞ -strong solution of $u_t + F(x, t, u, Du, D^2u) \leq f + \delta/2$ in $Q_r(\hat{x}, \hat{t})$ while $w = u - \varphi$ is greater at $(\hat{x}, \hat{t})$ than on $\partial_p Q_r(\hat{x}, \hat{t})$ for any fixed (but small) r . But w is an L^p -strong solution of

$$\begin{aligned} w_t + \mathcal{P}^-(D^2w) - \gamma|Dw| \\ \leq u_t + F(x, t, u, Du, D^2u) - (\varphi_t + F(x, t, u, D\varphi, D^2\varphi)) \leq -\frac{\delta}{2} \end{aligned}$$

in a parabolic neighborhood of $(\hat{x}, \hat{t})$. Theorem 2.4 then implies that $w(\hat{x}, \hat{t}) \leq \sup_{\partial_p Q_r(\hat{x}, \hat{t})} w$, a contradiction. $\square$

Putting $\varphi = u - v$, and assuming

$$u_t + F(x, t, u, Du, D^2u) \leq f \quad \text{and} \quad v_t + F(x, t, v, Dv, D^2v) \geq g \quad \text{in } Q$$

(in some sense) in the formal expression (2.6) used to prove Lemma 2.7 yields

$$\begin{aligned} (u - v)_t + \mathcal{P}^-(D^2(u - v)) - \gamma|D(u - v)| \\ \leq u_t + F(x, t, u, Du, D^2u) - (v_t + F(x, t, v, Dv, D^2v)) \leq f - g. \end{aligned}$$

It is immediate from the definitions that this is in fact correct in the L^p -viscosity sense if the relation for u holds in the L^p -viscosity sense and that for v holds in the L^p -strong sense. When $f \leq g$, we conclude from the classical viscosity solution maximum principle that if $u - v$ is continuous on $\overline{Q}$, it attains its maximum on $\partial_p Q$. Combining this with the analogous statement for an L^p -viscosity supersolution and an L^p -strong subsolution, we deduce the next result.

Proposition 2.10. *Let F satisfy (SC), $1 \leq p$, $f \in L^p(Q)$, and $u, v \in C(\overline{Q})$. Let $w = u$ be an L^p -viscosity solution of $w_t + F(x, t, w, Dw, D^2w) = f$ and $w = v$ be an L^p -strong solution of the same equation. If $u = v$ on $\partial_p Q$, then $u = v$ in Q .*

Note that in Proposition 2.10 we did not require $p > p_0$, but that it is not quite a uniqueness theorem in general, for the notions of solution do not coincide. The next result takes care of this provided $p > p_0$. It is proved in exactly the same way as Lemma 2.9, except that there is no need to regularize u .

Proposition 2.11. *Let F satisfy (SC), let $p > p_0$ and $f \in L^p_{\text{loc}}(Q)$. If $u \in W^{2,1,p}_{\text{loc}}(Q)$ is a strong subsolution (supersolution) of $u_t + F(x, t, u, Du, D^2u) = f$ on Q , then u is an L^p -viscosity subsolution (respectively, supersolution) of $u_t + F(x, t, u, Du, D^2u) = f$ in Q .*

Combining Propositions 2.10 and 2.11 we have a uniqueness result.

Corollary 2.12. *Let F satisfy (SC), $\psi \in C(\partial_p Q)$, $f \in L^p(Q)$ and $p > p_0$. Let $v \in C(\overline{Q})$ be an L^p -strong solution of $v_t + F(x, t, v, Dv, D^2v) = f$ in Q and $v = \psi$ on $\partial_p Q$. Then v is the unique solution of the problem: $u \in C(\overline{Q})$, u is an L^p -viscosity solution of $u_t + F(x, t, u, Du, D^2u) = f$ in Q and $u = \psi$ on $\partial_p Q$.*

The converse to Proposition 2.11, stating that an L^p -viscosity subsolution u of

$$u_t + F(x, t, u, Du, D^2u) \leq f$$

which happens to satisfy $u \in W_{\text{loc}}^{2,1,p}$ also satisfies the inequality in a pointwise sense, is correct but lies deeper and is proved in the next section.

Section 3. Pointwise Properties of Viscosity Solutions

We turn now to pointwise properties of L^p -viscosity solutions. Some results in this direction can be found in [30]. For the developments we require pointwise notions of “parabolically twice (sub, super) differentiable”. Given $w: Q \rightarrow \mathbb{R}$ and $(x, t) \in Q$, we say that w is parabolically twice differentiable at $(x, t) \in Q$ and write $w_t(x, t) = a$, $Dw(x, t) = p$, $D^2w(x, t) = X$ in the pointwise sense provided that $a \in \mathbb{R}$, $p \in \mathbb{R}^n$, $X \in \mathcal{S}(n)$ and

$$(3.1) \quad w(y, s) = w(x, t) + a(s - t) + \langle p, y - x \rangle + \frac{1}{2} \langle X(y - x), y - x \rangle + o(|t - s| + |y - x|^2).$$

This relation is to hold as $y \rightarrow x$ and $s \uparrow t$; note the “one-sided”

character of the definition in time and that there is at most one triple a, p, X for which (3.1) holds.

As a matter of fact, it is shown in [17] that for a continuous function $w: Q \rightarrow \mathbb{R}$ which is parabolically twice differentiable (in the sense above) a.e. in Q – as are L^p -viscosity solutions, see Proposition 3.2 – the restriction $s \leq t$ in (3.1) can be removed for almost all (x, t) , and thus a.e. “one-sided” in t can be replaced by a.e. “two-sided” in t . However, it will be more convenient to deal with one-sided notions here.

We are also interested in inequality versions of (3.1) and let $\mathcal{D}^{2,1,+}w(x, t)$ be the (possibly empty) subset of $\mathbb{R} \times \mathbb{R}^n \times \mathcal{S}(n)$ defined as follows: $(a, p, X) \in \mathcal{D}^{2,1,+}w(x, t)$ if

$$(3.2) \quad w(y, s) \leq w(x, t) + a(s - t) + \langle p, y - x \rangle + \frac{1}{2} \langle X(y - x), y - x \rangle + o(|t - s| + |y - x|^2),$$

where the “little o ” has the same meaning as above. Similarly, $\mathcal{D}^{2,1,-}w(x, t)$ is defined by reversing the inequality in (3.2), which also amounts to $\mathcal{D}^{2,1,-}w(x, t) = -\mathcal{D}^{2,1,+}(-w)(x, t)$.

One says that w is parabolically twice superdifferentiable (respectively, subdifferentiable) at $(x, t) \in Q$ if $\mathcal{D}^{2,1,+}w(x, t)$ (respectively, $\mathcal{D}^{2,1,-}w(x, t)$) is nonempty.

The next result states that at almost every point at which an L^p -viscosity subsolution (etc.) is parabolically twice superdifferentiable the corresponding subsolution (etc.) inequality holds in a pointwise sense. After this is proved, another result is established to the effect that subsolutions are in fact parabolically twice superdifferentiable a.e. In the next Section, the corresponding pointwise interpretations are shown to be equivalent to the L^p -viscosity notions.

Proposition 3.1. *Let F satisfy (SC), let $p > p_0$, and let $f \in L^p_{\text{loc}}(Q)$. Then there is a null set $\mathcal{N}' \subset Q$ such that if u is an L^p -viscosity solution of $u_t + F(x, t, u, Du, D^2u) \leq f$ (respectively, $u_t + F(x, t, u, Du, D^2u) \geq f$) in Q and $(x, t) \in Q \setminus \mathcal{N}'$ and $(a, p, X) \in \mathcal{D}^{2,1,+}u(x, t)$ (respectively, $(a, p, X) \in \mathcal{D}^{2,1,-}u(x, t)$), then*

$$a + F(x, t, u(x, t), p, X) \leq f(x, t)$$

(respectively,

$$a + F(x, t, u(x, t), p, X) \geq f(x, t)).$$

Proof. We treat the subsolution case. We will write $G(x, t, r, p, X)$ for $F(x, t, r, p, X) - f(x, t)$. Let $\mathcal{L}(r, p, X)$ be the set of L^p -Lebesgue points of $G(x, t, r, p, X)$, defined using parabolic cylinders, i.e. those points $(x, t) \in Q$ such that

$$\lim_{\rho \downarrow 0} \frac{1}{\rho^{n+2}} \int_{Q_\rho(x, t)} |G(y, s, r, p, X) - G(x, t, r, p, X)|^p dy ds = 0.$$

Let $\mathcal{C} \subset \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}(n)$ be a countable dense set. By a version of Lebesgue's theorem (see [9], Remark 2.2), $\mathcal{L}(r, p, X)$ has full measure for fixed (r, p, X) and hence so does

$$\mathcal{E} = \bigcap_{(r, p, X) \in \mathcal{C}} \mathcal{L}(r, p, X).$$

Since $G(x, t, r, p, X)$ is continuous in (r, p, X) uniformly in x, t , in fact $\mathcal{E} \subset \mathcal{L}(r, p, X)$ for every (r, p, X) .

We claim that if $(\hat{x}, \hat{t}) \in \mathcal{E}$ and $(a, p, X) \in \mathcal{D}^{2,+}u(\hat{x}, \hat{t})$, then

$$a + G(\hat{x}, \hat{t}, u(\hat{x}, \hat{t}), p, X) \leq 0.$$

Without loss of generality we assume that $(\hat{x}, \hat{t}) = (0, 0)$. Suppose, contrary to our claim, that

$$(3.3) \quad 0 < \delta < a + G(0, 0, u(0, 0), p, X).$$

Since

$$u(x, t) \leq u(0, 0) + at + \langle p, x \rangle + \frac{1}{2} \langle Xx, x \rangle + o(|x|^2 + |t|),$$

by continuity we may replace X by $X + 2\eta I$ and a by $a - \eta$ for small $\eta > 0$, maintain (3.3) while achieving

$$(3.4) \quad u(x, t) \leq u(0, 0) + at + \langle p, x \rangle + \frac{1}{2} \langle Xx, x \rangle - \frac{\eta}{2}(|x|^2 + |t|)$$

for $0 < |x| \leq r$, $-r^2 < t < 0$ and small $r > 0$. We now solve, via Theorem 2.8,

$$\psi_t + \mathcal{P}^-(D^2\psi) - \gamma|D\psi| = \alpha(r) + G(0, 0, u(0, 0), p, X) - G(x, t, u(0, 0), p, X)$$

in Q_r subject to $\psi = 0$ on $\partial_p Q_r$, where $\alpha(r) = \gamma\|X\|r + \omega(\sigma(\sqrt{2}r))$; here σ is a modulus of continuity of u . Then $\varphi(x, t) = u(0, 0) + at + \langle p, x \rangle + (1/2)\langle Xx, x \rangle + \psi(x, t)$ is a strong solution of

$$\varphi_t + G(x, t, u(x, t), D\varphi, D^2\varphi) \geq a + G(0, 0, u(0, 0), p, X) > 0$$

in Q_r . Moreover, the generalized maximum principle (in the guise of Remark 2.5) yields

$$\begin{aligned} \|\psi\|_{L^\infty(Q_r)} &\leq Cr^{2-\frac{n+2}{p}} \left(r^{\frac{n+2}{p}} \alpha(r) + r^{\frac{n+2}{p}} \left(\frac{1}{r^{n+2}} \int_{Q_r} |G(0, 0, u(0, 0), p, X) \right. \right. \\ &\quad \left. \left. - G(x, t, u(0, 0), p, X)|^p dxdt \right)^{1/p} \right). \end{aligned}$$

Using the Lebesgue point assumption, we conclude that

$$\|\psi\|_{L^\infty(Q_r)} = o(r^2).$$

Since $u - \varphi \leq -\eta r^2/2$ on $\partial_p Q_r$ while $u(0, 0) - \varphi(0, 0) = -\psi(0, 0) = o(r^2)$, $u - \varphi$ has a maximum in Q_r for small r , yielding a contradiction with the assumption that u is a subsolution (recall Lemma 1.4). Thus the lemma is proved with $\mathcal{N}' = Q \setminus \mathcal{E}$. $\square$

The previous result states that pointwise relations are satisfied by sub- and supersolutions at most points where they are twice parabolically sub- and superdifferentiable. The next result states that the latter occurs at almost every point.

Proposition 3.2. *Let F satisfy (SC), $p > p_0$, $f \in L^p_{\text{loc}}(Q)$, and u be an L^p -viscosity solution of $u_t + F(x, t, u, Du, D^2u) \leq f$ (respectively, $u_t + F(x, t, u, Du, D^2u) \geq f$) in Q . Then u is parabolically twice superdifferentiable (respectively, subdifferentiable) a.e. in Q . In particular, if u is an L^p -viscosity solution of $u_t + F(x, t, u, Du, D^2u) = f$, then u is parabolically twice differentiable a.e. and the pointwise derivatives satisfy the equation a.e.*

Proof. We focus on the subsolution case. The proof is similar to the proof of the corresponding result in the elliptic case, see [6] and references therein. Solving an extremal inequality and using differentiability properties of functions in $W^{2,1,p}$ (see the appendix of [9]) it is enough to show superdifferentiability in the case of u satisfying $u_t + \mathcal{P}^-(D^2u) - \gamma|Du| \leq 0$. To this end we fix $(y, s) \in Q$, $0 < R < 1$ such that $Q_R(y, s) \subset Q$. Given $k \geq 1$, define

$$u_k(x, t) = u(x, t) - \frac{k}{2} (|x - y|^2 - R^2) (R^2 + t - s).$$

Then u_k is an L^p -viscosity solution of

$$(u_k)_t + \mathcal{P}^-(D^2u_k) - \gamma|Du_k| \leq k \left(n\Lambda R^2 + R^3\gamma + \frac{R^2}{2} \right)$$

in $Q_R(y, s)$. We may assume that $u_k \geq 0$ and using (2.3) of Theorem 2.1 we conclude

$$\begin{aligned} u_k(x, t) &\leq \sup_{\partial_p Q_R(y, s)} u + CR^{\frac{n}{n+1}} k R^2 | \cdot |^+ (u_k, Q_R(y, s)) |^{\frac{1}{n+1}} \\ &\leq \sup_{Q_R(y, s)} u + CR^{\frac{n}{n+1}} k R^2 | \cdot |^+ (u_k, Q_R(y, s)) |^{\frac{1}{n+1}} \end{aligned}$$

on $Q_R(y, s)$ for some constant C independent of k and $R \leq 1$. Since $\inf_{Q_R(y, s)} u + (k/2)R^4 \leq \sup_{Q_R(y, s)} u_k$, we conclude that

$$(3.5) \quad \inf_{Q_R(y, s)} u + \frac{k}{2} R^4 \leq \sup_{Q_R(y, s)} u + CR^{\frac{n}{n+1}} k R^2 | \cdot |^+ (u_k, Q_R(y, s)) |^{\frac{1}{n+1}}.$$

We will say that $(x, t) \in E^{2,1,+}(w)$ provided that $(x, t) \in Q$ and $\mathcal{D}^{2,1,+}w(x, t) \neq \emptyset$, i.e. w is parabolically twice superdifferentiable at (x, t) . $E^{2,1,-}(w)$ is defined analogously, by replacing $\mathcal{D}^{2,1,+}$ by $\mathcal{D}^{2,1,-}$.

We obviously have $E^{2,1,+}(u_k) = E^{2,1,+}(u)$ and ${}^{+}(u_k, Q_R(y, s)) \subset E^{2,1,+}(u_k)$. Letting $k \rightarrow \infty$ in (3.5) and using $|\mathcal{O}|$ to denote the Lebesgue measure of $\mathcal{O}$, we obtain $\frac{R^2}{2} \leq R^{\frac{n}{n+1}} C |E^{2,1,+}(u) \cap Q_R(y, s)|^{\frac{1}{n+1}}$ or

$$\left(\frac{1}{2C}\right)^{n+1} \leq \frac{|E^{2,1,+}(u) \cap Q_R(y, s)|}{R^{n+2}}$$

for every $(y, s) \in Q$ and $0 < R < 1$ such that $Q_R(y, s) \subset Q$. This implies (see Remark 2.2 in [9]) that the Lebesgue density $E^{2,1,+}(u)$ is positive and hence 1 at almost every point of Q and we conclude that $|E^{2,1,+}(u)| = |Q|$.

In exactly the same manner one shows that $|E^{2,1,-}(u)| = |Q|$. The final assertion of the theorem now follows from the parabolic generalized Rademacher-Stepanov Theorem (see Theorem 3 in [17]). The equation is satisfied pointwise by Proposition 3.1. $\square$

Corollary 3.3. *Under the assumptions of Proposition 3.1, if $u \in W_{\text{loc}}^{2,1,p}(Q)$ is an L^p -viscosity subsolution (supersolution) of $u_t + F(x, t, u, Du, D^2u) = f$ in Q , then it is an L^p -strong subsolution (respectively, supersolution) of $u_t + F(x, t, u, Du, D^2u) = f$ in Q .*

Section 4. Pointwise Characterizations of Viscosity Solutions

From the results of the previous section, we know that if F satisfies (SC), $p_0 < p$, $f \in L^p(Q)$, and u is an L^p -viscosity solution of

$$u_t + F(x, t, u, Du, D^2u) \leq f,$$

then u is parabolically twice superdifferentiable almost everywhere, and that for almost all (x, t)

$$(4.1) \quad a + F(x, t, u(x, t), p, X) \leq f(x, t) \quad \text{for } (a, p, X) \in \mathcal{D}^{2,1,+}u(x, t).$$

Corresponding statements hold for supersolutions and solutions. The issues treated in this section revolve around the converse, that is the extent to which (4.1) implies that u is a subsolution. The simplest instance of the results we will

prove is this: if u is continuous and for some b, B the function $u(x, t) - bt + B|x|^2$ is nonincreasing in t and convex in x , then u is a C -viscosity solution of $u_t - \Delta u \leq 0$ if and only if the relation holds a.e. (note that u is necessarily parabolically twice differentiable a.e. [21]). (In this linear example, the viscosity notions are equivalent to the sense of distributions and the assertion is well known.)

However, before we take up issues directly related to the preceding remarks, in order to keep the readers' interest we formulate the result used frequently in the next sections:

Proposition 4.1. *Let F, G satisfy (SC), $p > p_0$, $f \in L^p_{\text{loc}}(Q)$ and u be an L^p -viscosity solution of $u_t + F(x, t, u, Du, D^2u) = f(x, t)$ in Q . Let g be defined pointwise a.e. in Q by*

$$g(x, t) = u_t(x, t) + G(x, t, u(x, t), Du(x, t), D^2u(x, t)),$$

where $u_t(x, t), Du(x, t), D^2u(x, t)$ are understood in the pointwise sense of (3.1). If $g \in L^p_{\text{loc}}(Q)$, then u is an L^p -viscosity solution of $u_t + G(x, t, u, Du, D^2u) = g(x, t)$ in Q .

It will take us a while to come to the proof of Proposition 4.1. We begin with stronger assumptions and treat the case $p \geq n + 1$.

Definition 4.2. *A function $u \in C(Q)$ satisfies the pointwise maximum principle in Q if for every $\varphi \in W^{2,1,n+1}_{\text{loc}}(Q)$ and strict local maximum point $(\hat{x}, \hat{t}) \in Q$ of $u - \varphi$ there exists $r_0 > 0$ such that for every $r < r_0$ and every $\rho > 0$ the set $\mathcal{O}_{r,\rho}$ of those $(x, t) \in Q_r(\hat{x}, \hat{t})$ for which there exists $(a, p, X) \in \mathcal{D}^{2,1,+}u(x, t)$ such that*

$$X - D^2\varphi(x, t) \leq 0, \quad |p - D\varphi(x, t)| \leq \rho, \quad a - \varphi_t(x, t) \geq 0$$

has positive measure.

The definition is wordy, but intuitive: we have on a set of positive measure the sort of relations we would expect if u and φ were smooth. Tso [29] notes that it is a consequence of the maximum principle that $W^{2,1,n+1}_{\text{loc}}$ functions satisfy the pointwise maximum principle. We next define the notion of “pointwise a.e.” solutions, etc., of our equations and then note that for functions

satisfying the pointwise maximum principle, pointwise a.e. and L^{n+1} -viscosity are equivalent.

Definition 4.3. *Let F satisfy (SC) and $f \in L^{n+1}(Q)$. Then $u \in C(Q)$ is a pointwise a.e. viscosity subsolution of $u_t + F(x, t, u, Du, D^2u) = f$ in Q if for almost all $(x, t) \in Q$*

$$(4.2) \quad a + F(x, t, u(x, t), p, X) \leq f(x, t) \quad \text{for every } (a, p, X) \in \mathcal{D}^{2,1,+}_u(x, t).$$

Moreover, u is a pointwise a.e. viscosity supersolution if (4.2) holds with “ $\geq$ ” in place of “ $\leq$ ” and $\mathcal{D}^{2,1,-}$ in place of $\mathcal{D}^{2,1,+}$, and u is a pointwise a.e. viscosity solution if it is both a pointwise a.e. viscosity subsolution and a pointwise a.e. viscosity supersolution.

Remark 4.4. We have two options in defining pointwise a.e. viscosity solutions - using the conjunction of the sub- and supersolution notions as above or using only pointwise data from points of parabolic twice differentiability. It is easy to see the notions coincide for functions which are parabolically twice differentiable a.e.

Proposition 4.5. *Let F satisfy (SC), and let $f \in L^{n+1}(Q)$. Then $u \in C(Q)$ is an L^{n+1} -viscosity subsolution of $u_t + F(x, t, u, Du, D^2u) = f$ in Q if and only if u satisfies the pointwise maximum principle in Q and is a pointwise a.e. viscosity subsolution of $u_t + F(x, t, u, Du, D^2u) \leq f$ in Q .*

The proof of Proposition 4.5 consists of a straightforward extension of the arguments given in the proofs of Proposition 3.2 and Theorem 3.3 in [8] and is therefore omitted.

The maximum principle available for $n+1 \leq p$ guarantees that L^p -viscosity subsolutions of $u_t + F(x, t, u, Du, D^2u) \leq f$ satisfy the pointwise maximum principle, but the generalized maximum principle applicable for $p_0 < p < n+1$ does not. Thus there is no direct analogue of Proposition 4.5 in the general case $p_0 < p$. Nonetheless, the regularity inherited by solutions of $u_t + F(x, t, u, Du, D^2u) = f$ in the general case allows one to use pointwise characterizations in other equations of the general class. This is the point of Proposition 4.1.

In turn Proposition 4.1 is an immediate consequence of the next lemma, which addresses the independence of the equation via extremal equations. While Proposition 4.1 has an elegant formulation, the lemma is not elegant.

Lemma 4.6. *Let F satisfy (SC), $p > p_0$, $f \in L^p_{\text{loc}}(Q)$ and $u \in C(Q)$ be parabolically twice differentiable a.e. in Q . Then u is an L^p -viscosity subsolution (respectively, supersolution) of $u_t + F(x, t, u, Du, D^2u) = f$ in Q if and only if it is a pointwise a.e. viscosity subsolution (supersolution, respectively) of the same equation, and whenever $\varphi \in W^{2,1,p}_{\text{loc}}(Q)$ and $u - \varphi$ has a local maximum (respectively, minimum) at $(\hat{x}, \hat{t})$ then*

$$(4.3) \quad \text{ess} \limsup_{\substack{(x,t) \rightarrow (\hat{x}, \hat{t}) \\ t \leq \hat{t}}} \left((u - \varphi)_t(x, t) + \mathcal{P}^-(D^2(u - \varphi)(x, t)) - \gamma |D(u - \varphi)(x, t)| \right) \geq 0$$

(respectively,

$$\text{ess} \liminf_{\substack{(x,t) \rightarrow (\hat{x}, \hat{t}) \\ t \leq \hat{t}}} \left((u - \varphi)_t(x, t) + \mathcal{P}^+(D^2(u - \varphi)(x, t)) + \gamma |D(u - \varphi)(x, t)| \right) \leq 0).$$

Proof. The proof is presented for subsolutions. Suppose u is an L^p -viscosity solution of $u_t + F(x, t, u, Du, D^2u) \leq f$, $\varphi \in W^{2,1,p}_{\text{loc}}(Q)$ and $u - \varphi$ has a local maximum at $(\hat{x}, \hat{t})$ which we may assume to be strict. By an analogue of Lemma 2.7 $u - \varphi$ is an L^p -viscosity solution of

$$v_t + \mathcal{P}^-(D^2v) - \gamma |Dv| \leq h(x, t),$$

where $h(x, t) = (f(x, t) - \varphi_t(x, t) - F(x, t, u(x, t), D\varphi(x, t), D^2\varphi(x, t)))^+ \in L^p_{\text{loc}}(Q)$. Let $h_k(x, t) = \min(h(x, t), k)$, and let $w_k \in W^{2,1,p}_{\text{loc}}(Q_r(\hat{x}, \hat{t})) \cap C(\overline{Q}_r(\hat{x}, \hat{t}))$ be the solution of

$$w_k + \mathcal{P}^-(D^2w_k) - \gamma |Dw_k| = h(x, t) - h_k(x, t) \text{ in } Q_r(\hat{x}, \hat{t}), \quad w_k = 0 \text{ on } \partial_p Q_r(\hat{x}, \hat{t}).$$

provided by Theorem 2.8. By the generalized maximum principle

$$(4.4) \quad \|w_k\|_{L^\infty(Q_r(\hat{x}, \hat{t}))} \rightarrow 0 \text{ as } k \rightarrow \infty;$$

moreover $\xi_k = u - \varphi - w_k$ is an L^p -viscosity solution of

$$(\xi_k)_t + \mathcal{P}^-(D^2\xi_k) - \gamma |D\xi_k| \leq h_k(x, t) \quad \text{in } Q_r(\hat{x}, \hat{t}).$$

By (4.4), there exist local maxima $(x_k, t_k) \rightarrow (\hat{x}, \hat{t})$ of ξ_k such that $\xi_k(x_k, t_k) \rightarrow u(\hat{x}, \hat{t}) - \varphi(\hat{x}, \hat{t})$. However $h_k \in L_{\text{loc}}^{n+1}(Q)$ (in fact, it is bounded) and so by Lemma 4.5 ξ_k satisfies the pointwise maximum principle. Perturbing to produce a strict maximum at (x_k, t_k) if necessary, we conclude from the pointwise maximum principle that

$$\text{ess} \limsup_{\substack{(x,t) \rightarrow (x_k, t_k) \\ t \leq t_k}} \left((\xi_k)_t(x, t) + \mathcal{P}^-(D^2 \xi_k(x, t)) - \gamma |D \xi_k(x, t)| \right) \geq 0.$$

We now notice that

$$\begin{aligned} & (u - \varphi)_t(x, t) + \mathcal{P}^-(D^2(u - \varphi)(x, t)) - \gamma |D(u - \varphi)(x, t)| \\ &= ((\xi_k)_t + (w_k)_t)(x, t) + \mathcal{P}^-(D^2(\xi_k + w_k)(x, t)) - \gamma |D(\xi_k + w_k)(x, t)| \\ &\geq (\xi_k)_t(x, t) + \mathcal{P}^-(D^2 \xi_k(x, t)) - \gamma |D \xi_k(x, t)| + h(x, t) - h_k(x, t) \\ &\geq (\xi_k)_t(x, t) + \mathcal{P}^-(D^2 \xi_k(x, t)) - \gamma |D \xi_k(x, t)| \end{aligned}$$

and (4.3) follows.

In the other direction, if (4.3) holds,

$$u_t(x, t) + F(x, t, u(x, t), Du(x, t), D^2 u(x, t)) \leq f(x, t)$$

a.e. and u is not an L^p -viscosity subsolution of $u_t + F(x, t, u, Du, D^2 u) = f$, then there exist $\varphi \in W_{\text{loc}}^{2,1,p}(Q)$ and $\epsilon > 0$ such that $u - \varphi$ has a local maximum at $(\hat{x}, \hat{t})$ and

$$\varphi_t(x, t) + F(x, t, u(x, t), D\varphi(x, t), D^2 \varphi(x, t)) > f(x, t) + \epsilon$$

in a neighborhood of $(\hat{x}, \hat{t})$. But then using (SC) we obtain

$$(u_t - \varphi_t)(x, t) + \mathcal{P}^-(D^2(u - \varphi)(x, t)) - \gamma |D(u - \varphi)(x, t)| < -\epsilon$$

in a neighborhood of $(\hat{x}, \hat{t})$, which contradicts (4.3). $\square$

Section 5. Continuity Tools: Harnack Inequality and Hölder Continuity

In this section we collect two of the basic ingredients used in the next section to study the regularity of solutions of equations $u_t + F(x, t, u, Du, D^2 u) = f$ under conditions on the (x, t) dependence of $F(x, t, r, p, X)$. One such result

is the Hölder continuity of solutions u of the general equation. We need this in a greater generality than is treated in Wang [30] and so present the outline of the proof. The Harnack inequality and Hölder continuity for $p < n$ for elliptic equations are treated in Cabré [3], Escauriaza [13] and Fok [15]. The second result is the interior $C^{1+\alpha}$ regularity for the equation $u_t + F(D^2u) = 0$ in which there is no x, t, u, Du dependence. ($C^{1+\alpha}$ is defined in Section 1; note that $u \in C^{1+\alpha}$ does not require that u_t be Hölder continuous.) This latter result we formulate, but take as known. It is proved in Wang [30] and Krylov [20] for classical solutions. Using the Hölder continuity, the proof resembles the elliptic case treated in [5]. In fact, a stronger result is true under the assumptions of Lemma 5.4; namely u_t is also Hölder continuous (see [31]). However we will not need the stronger result here. The reader who is willing to do so may note the statements of these results and then pass directly to the more novel material of the next sections.

Suppose that u is an L^p -viscosity solution of $u_t + F(x, t, u, Du, D^2u) = f$; it follows from (SC) that u is an L^{n+1} -viscosity solution of both

$$u_t + \mathcal{P}^-(D^2u) - \gamma|Du| \leq g \quad \text{and} \quad g \leq u_t + \mathcal{P}^+(D^2u) + \gamma|Du|,$$

where $g(x, t) = f(x, t) - F(x, t, u(x, t), 0, 0)$. This explains the point of the next lemma.

Lemma 5.1. *Let $0 < \gamma_0$, $r \leq 1$, $0 \leq r\gamma \leq \gamma_0$, $p > p_0(n, \lambda, \Lambda, \gamma\sqrt{5}(2r))$, and $f \in L^p(Q_{2r})$. Then there exist constants $\kappa = \kappa(n, p, \lambda, \Lambda, \gamma_0)$ and $K_1 = K_1(n, p, \lambda, \Lambda, \gamma_0)$ such that every L^p -viscosity solution u of both*

$$(5.1) \quad u_t + \mathcal{P}^-(D^2u) - \gamma|Du| \leq f \quad \text{and} \quad u_t + \mathcal{P}^+(D^2u) + \gamma|Du| \geq -f$$

in Q_{2r} satisfies

$$(5.2) \quad \|u\|_{C^\kappa(Q_r)} \leq K_1 r^{-\kappa} (\|u\|_{L^\infty(Q_{2r})} + r^{2-\frac{n+2}{p}} \|f\|_{L^p(Q_{2r})}).$$

We first comment on the formulation. If u is as in the lemma and $U(x, t) = u(rx, r^2t)$, then U satisfies inequalities corresponding to (5.1) in Q_2 . The γ parameter for the equation for U is $r\gamma$, and in order to deduce the general estimate (5.2) from the special case $r = 1$ applied to U without unduly restricting γ itself, one must merely control the size of $r\gamma$, hence the use of γ_0 . Lemma

5.1 for $r = 1$ follows in a standard way from a suitable Harnack inequality to which we now turn. It will be convenient here only to use the notation

$$Q_R(a) = B_R \times (a - R^2, a];$$

note the labeling of the “top” of the cylinder. Several times below choices of constants are made in order to be explicit and set up manipulations to follow. These choices are not significant - the first is the “ $\sqrt{3}$ ” below. We regard Lemma 5.1 as proved once the Harnack inequality is proved.

Lemma 5.2. *Let $0 < \gamma_0$, $r \leq 1$, $0 \leq r\gamma \leq \gamma_0$, $p > p_0(n, \lambda, \Lambda, \gamma\sqrt{5}2r)$, and $f \in L^p(Q_{2r})$. There is a constant $C = C(n, p, \lambda, \Lambda, \gamma_0)$ such that if $0 \leq u$ satisfies (5.1) in Q_{2r} then*

$$(5.3) \quad \sup_{Q_{\frac{r}{\sqrt{3}}}(-\frac{2r^2}{3})} u \leq C \left(\inf_{Q_{\frac{r}{\sqrt{3}}}(0)} u + r^{2-\frac{n+2}{p}} \|f\|_{L^p(Q_{2r})} \right).$$

Proof. The result follows quickly from the weak Harnack inequality and local maximum principle, proved in Wang [30] for the case in which f is continuous and $p = n + 1$, coupled with our Theorem 2.8. The scaled and simplified versions of Wang’s results we need follow. All constants below depend on $n, p, \lambda, \Lambda, \gamma_0$ in addition to other displayed dependencies.

Local Maximum Principle: If $q > 0$ there exists a constant C_q such that if w is a C -viscosity solution of $w_t + \mathcal{P}^-(D^2w) - \gamma|Dw| \leq 0$ in $Q_{\frac{2R}{\sqrt{3}}}(a)$ and $R \leq 1$, then

$$\sup_{Q_R(a)} w \leq C_q \left(\frac{1}{|Q_{\frac{2R}{\sqrt{3}}}(a)|} \int_{Q_{\frac{2R}{\sqrt{3}}}(a)} (w^+)^q \right)^{\frac{1}{q}}.$$

Weak Harnack Inequality: There exists $\delta > 0$ such that if $R \leq 1$ and $v \geq 0$ is a C -viscosity solution of $v_t + \mathcal{P}^+(D^2v) + \gamma|Dv| \geq 0$ in $Q_{2R}(0)$, then

$$\left(\frac{1}{|Q_R(-\frac{3R^2}{2})|} \int_{Q_R(-\frac{3R^2}{2})} v^\delta \right)^{\frac{1}{\delta}} \leq C \left(\inf_{Q_R(0)} v \right).$$

To proceed, let $\varphi, \psi \in W_{\text{loc}}^{2,1,p}(Q_{2r}) \cap C(\overline{Q}_{2r})$ solve (via Theorem 2.8), respectively,

$$(5.4) \quad \varphi_t + \mathcal{P}^-(D^2\varphi) - \gamma|D\varphi| = f \quad \text{in } Q_{2r}, \quad \varphi = 0 \quad \text{on } \partial_p Q_{2r}$$

and

$$(5.5) \quad \psi_t + \mathcal{P}^+(D^2\psi) + \gamma|D\psi| = f \quad \text{in } Q_{2r}, \quad \psi = 0 \quad \text{on } \partial_p Q_{2r}.$$

By Remark 2.5 we have

$$(5.6) \quad \|\varphi\|_{L^\infty(Q_{2r})} + \|\psi\|_{L^\infty(Q_{2r})} \leq Cr^{2-\frac{n+2}{p}} \|f\|_{L^p(Q_{2r})} =: k$$

where the last relation defines k . Then $v = u - \psi + k \geq 0$ is an L^p -viscosity solution of $v_t + \mathcal{P}^+(D^2v) + \gamma|Dv| \geq 0$ in Q_{2r} , and $w = u - \varphi$ is an L^p -viscosity subsolution of $w_t + \mathcal{P}^-(D^2w) - \gamma|Dw| \leq 0$ in Q_{2r} . The estimates which follow use $0 \leq v$ and in sequence: the local maximum principle with $q = \delta$, $w = u - \varphi$, $R = r/\sqrt{3}$, and $a = -2r^2/3$, $-\varphi \leq -\psi + k$ (owing to the definition of k), the weak Harnack inequality with $v = u - \psi + k$ and $R = 2r/3$, and $1/\sqrt{3} \leq 2/3$:

$$\begin{aligned} \sup_{Q_{\frac{r}{\sqrt{3}}}\left(-\frac{2r^2}{3}\right)} (u - \varphi) &\leq C_1 \left(\frac{1}{|Q_{\frac{2r}{3}}\left(-\frac{2r^2}{3}\right)|} \int_{Q_{\frac{2r}{3}}\left(-\frac{2r^2}{3}\right)} ((u - \varphi)^+)^{\delta} \right)^{\frac{1}{\delta}} \\ &\leq C_1 \left(\frac{1}{|Q_{\frac{2r}{3}}\left(-\frac{2r^2}{3}\right)|} \int_{Q_{\frac{2r}{3}}\left(-\frac{2r^2}{3}\right)} (u - \psi + k)^{\delta} \right)^{\frac{1}{\delta}} \\ &\leq C_2 \left(\inf_{Q_{\frac{2r}{3}}(0)} (u - \psi + k) \right) \\ &\leq C_2 \left(\inf_{Q_{\frac{r}{\sqrt{3}}}(0)} (u - \psi + k) \right). \end{aligned}$$

The result now follows from (5.6). $\square$

The following result follows immediately from Lemma 5.1 and the remarks which precede it.

Proposition 5.3. *Let $0 < \gamma_0$, $r \leq 1$, $0 \leq r\gamma \leq \gamma_0$, $p > p_0(n, \lambda, \Lambda, \gamma\sqrt{5}(2r))$, and $f \in L^p(Q_{2r})$. Then every L^p -viscosity solution u of*

$$u_t + F(x, t, u, Du, D^2u) = f$$

in $Q_{2r}(x, t)$ satisfies

$$(5.7) \quad \begin{aligned} \|u\|_{C^\kappa(Q_r(x,t))} &\leq K_1 r^{-\kappa} \left(\|u\|_{L^\infty(Q_{2r}(x,t))} \right. \\ &\quad \left. + r^{2-\frac{n+2}{p}} \|f\|_{L^p(Q_{2r}(x,t))} + r^2 \omega \left(\|u\|_{L^\infty(Q_{2r}(x,t))} \right) \right) \end{aligned}$$

where K_1, κ are from Lemma 5.1.

The last result is the Hölder continuity of the gradient, about which we say no more than was said in the comments introducing this section.

Proposition 5.4. *Let u be a C -viscosity solution (or equivalently an L^p -viscosity solution) of $u_t + F(D^2u) = 0$ in Q_1 , where F is uniformly elliptic. Then there exist $\bar{\alpha} = \bar{\alpha}(n, \lambda, \Lambda)$ and $K_2 = K_2(n, \lambda, \Lambda)$ such that*

$$(5.8) \quad \|u\|_{C^{1+\bar{\alpha}}(Q_{\frac{1}{2}})} \leq K_2 \|u\|_{L^\infty(Q_1)}.$$

Section 6. Continuity With Respect to the Equation

Suppose we are presented with a sequence of L^p -viscosity solutions u_m of a sequence of equations

$$(u_m)_t + F_m(x, t, u_m, Du_m, D^2u_m) = f_m(x, t)$$

arising perhaps from an approximation process in an attempt to solve a “limit problem” $u_t + F(x, t, u, Du, D^2u) = f(x, t)$. Suppose the F_m ’s and F satisfy (SC). It follows from Lemma 5.1 that if the u_m ’s are uniformly locally bounded, then they are locally uniformly equi-Hölder continuous and we may assume $u_m \rightarrow u$ locally uniformly for some continuous u . Does u solve the limit problem? The first result below asserts that this is so in considerable generality. It is proved by arguments parallel to those used to establish Theorem 3.8 in [6] (given the technology already developed herein) and hence the proof is left to the interested reader. Convergence results for strong solutions are treated in [20].

Theorem 6.1. *Let F_m, F satisfy (SC) for $m = 1, 2, \dots$, let $p > p_0$, let $f, f_m \in L^p(Q)$ for $m = 1, 2, \dots$, and let u_m be L^p -viscosity subsolutions (supersolutions) of*

$$(6.1) \quad (u_m)_t + F_m(x, t, u_m, Du_m, D^2u_m) = f_m \quad \text{in } Q$$

for $m = 1, 2, \dots$. Assume that $u_m \rightarrow u$ locally uniformly as $m \rightarrow \infty$. Assume, moreover, that for $Q_r(x_0, t_0) \subset\subset Q$ and $\varphi \in W^{2,1,p}(Q_r(x_0, t_0))$

$$g_m(x, t) = F_m(x, t, u_m(x, t), D\varphi(x, t), D^2\varphi(x, t)) - f_m(x, t),$$

and $g(x, t) = F(x, t, u(x, t), D\varphi(x, t), D^2\varphi(x, t)) - f(x, t)$ satisfy

$$(6.2) \quad \|(g - g_m)^+\|_{L^p(Q_r(x_0, t_0))} \rightarrow 0 \quad \left(\|(g - g_m)^-\|_{L^p(Q_r(x_0, t_0))} \rightarrow 0 \right).$$

Then u is an L^p -viscosity subsolution (respectively, supersolution) of $u_t + F(x, t, u, Du, D^2u) = f$. Moreover, if F and f are continuous, then u is a C -viscosity subsolution (respectively supersolution) if (6.2) holds for $\varphi_t, D\varphi, D^2\varphi \in C(Q)$.

The consequences of Theorem 6.1 include existence theorems. Basic results of this sort are developed in Section 4 of [10]. However, let us provide one example here, using different arguments from [10] and highlighting a uniqueness result not presented there: for arbitrary Ω satisfying a uniform exterior cone condition, arbitrary $\psi \in C(\partial_p Q)$ and arbitrary $F(r, p, X)$ (independent of x, t) satisfying (SC) and $f \in L^p(Q)$ for some $p_0 < p$ there is a unique L^p -viscosity solution $u \in C(\bar{Q})$ of

$$(6.3) \quad u_t + F(u, Du, D^2u) = f \quad \text{in } Q \quad \text{and} \quad u = \psi \quad \text{on } \partial_p Q.$$

Theorem 2.1 of [10] asserts that (6.3) has a C -viscosity solution if $f \in C(\bar{Q})$; we begin our argument assuming this existence and couple it with the following lemma:

Lemma 6.2. *Let F satisfy (SC), let $f, g \in L^p(Q)$, $p_0 < p$ and $u, v \in C(\bar{Q})$ be L^p -viscosity solutions of $u_t + F(u, Du, D^2u) = f$ and $v_t + F(v, Dv, D^2v) = g$ in Q . Then there is a constant C independent of f, g such that*

$$\|u - v\|_{L^\infty(Q)} \leq \|u - v\|_{L^\infty(\partial_p Q)} + C\|f - g\|_{L^p(Q)}.$$

Proof. Approximating Ω from the inside by regular domains we can assume without loss of generality that Ω satisfies a uniform exterior cone condition. Let g_j be smooth and such that $\|g_j - g\|_{L^p(Q)} \rightarrow 0$ as $j \rightarrow \infty$. We find (via Theorem 2.1 of [10] together with Theorem 2.8) $\varphi_j, \psi_j \in W_{\text{loc}}^{2,1,p}(Q) \cap C(\bar{Q})$ that solve

$$(\varphi_j)_t + \mathcal{P}^-(D^2\varphi_j) - \gamma|D\varphi_j| - \omega(\varphi_j^-) = f - g_j \text{ in } Q, \quad \varphi_j = \|u - v\|_{L^\infty(\partial_p Q)} \text{ on } \partial_p Q$$

and

$$(\psi_j)_t + \mathcal{P}^+(D^2\psi_j) + \gamma|D\psi_j| + \omega(\psi_j^+) = g - g_j \text{ in } Q, \quad \psi_j = 0 \quad \text{on } \partial_p Q.$$

Then, by Lemma 2.7, $w_j = u - \varphi_j$ is an L^p -viscosity solution of

$$(6.4) \quad (w_j)_t + F(w_j, Dw_j, D^2 w_j) \leq g_j \text{ in } Q, \quad w_j = u - \|u - v\|_{L^\infty(\partial_p Q)} \text{ on } \partial_p Q$$

and $z_j = v - \psi_j$ is an L^p -viscosity solution of

$$(6.5) \quad (z_j)_t + F(z_j, Dz_j, D^2 z_j) \geq g_j \text{ in } Q, \quad z_j = v \text{ on } \partial_p Q.$$

Since $w_j \leq z_j$ on $\partial_p Q$, by comparison (see [16]) from (6.4) and (6.5) we obtain $w_j \leq z_j$ in Q . Finally, using Proposition 2.6 as well, we have

$$u - v \leq \varphi_j - \psi_j \leq \|u - v\|_{L^\infty(\partial_p Q)} + C(\|f - g_j\|_{L^p(Q)} + \|g - g_j\|_{L^p(Q)})$$

and we conclude by letting $j \rightarrow \infty$ and then applying a symmetrical estimate.

□

We turn now to an application of Theorem 6.1 which will play a crucial role in the next section. For simplicity, we will assume that $F(x, t, r, p, X)$ is independent of r and write $F(x, t, p, X)$, etc. Fix (y, s) . The result states that the difference between solutions v and ξ of Cauchy-Dirichlet problems in $Q_1(y, s)$ for $v_t + F(x, t, Dv, D^2 v) = f(x, t)$ and for $\xi_t + F(y, s, 0, D^2 \xi) = 0$ can be made small by making the difference between the equations small – that is, f small in L^p , γ small (so that $F(y, s, p, X) \approx F(y, s, 0, X)$), and $F(x, t, 0, X) \approx F(y, s, 0, X)$. This latter condition will be quantified by smallness requirements on the parabolic variant

$$(6.6) \quad \beta((y, s), (x, t)) = \sup_{X \in \mathcal{S}(n)} \frac{|F(x, t, 0, X) - F(y, s, 0, X)|}{1 + \|X\|}$$

of Caffarelli's ([4]) measure of continuity of the principal coefficients (speaking informally). Of course, the quantity β “makes sense” only a.e. in $(y, s), (x, t)$, but we will not always note this in discussion. We will integrate over (x, t) and may think of (y, s) as in the “universal” Lebesgue set constructed in the proof of Proposition 3.1. One easily sees that $\beta \leq 2\Lambda$ is bounded.

The following lemma is a version of Lemma 1.4 in [31]; we provide a complete proof. The first results of this sort are due to Caffarelli (see Lemmas 7.9 and 8.2 of [5]), while the elliptic version in the corresponding technical generality is found in [28]. The function β was defined differently in [31] so that the results there apply to equations of quasilinear type. Therefore Lemma 1.4 and

Theorem 1.3 in [31] do not apply as stated to Hamilton-Jacobi-Bellman-Isaacs equations. Since these equations are our main motivation we redefined β to handle this case.

Lemma 6.3. *Let F satisfy (SC) in $Q_1(y, s)$, $K > 0$, $p > \lim_{k \downarrow 0} p_0(n, \lambda, \Lambda, k)$, $f \in L^p(Q_1(y, s))$ and $\psi \in C(\partial_p Q_1(y, s))$ have ρ as a modulus of continuity. Let $\|\psi\|_{L^\infty(\partial_p Q_1(y, s))} \leq K$. Then for every $\epsilon > 0$ there exists $0 < \delta = \delta(\epsilon, n, p, \lambda, \Lambda, \rho, K)$ such that: If β is given by (6.6),*

$$(6.7) \quad \max \left\{ \left(\int_{Q_1(y, s)} \beta((y, s), (x, t))^p dx dt \right)^{\frac{1}{p}}, \|f\|_{L^p(Q_1(y, s))}, \gamma \right\} \leq \delta$$

and $v, \xi \in C(\overline{Q_1(y, s)})$ are L^p -viscosity solutions of

$$(6.8) \quad v_t + F(x, t, Dv, D^2 v) = f(x, t) \quad \text{and} \quad \xi_t + F(y, s, 0, D^2 \xi) = 0 \quad \text{in } Q_1(y, s)$$

satisfying

$$(6.9) \quad v = \xi = \psi \quad \text{on } \partial_p Q_1(y, s)$$

then

$$(6.10) \quad \|v - \xi\|_{L^\infty(Q_1(y, s))} \leq \epsilon.$$

Proof of Lemma 6.3. We will assume that $(y, s) = (0, 0)$. If the claim is not satisfied then there exist $\epsilon > 0$ and sequences F_j satisfying (SC) with $\gamma = \gamma_j$ and f_j such that (with obvious notation)

$$\|\beta_j\|_{L^p(Q_1)}, \|f_j\|_{L^p(Q_1)}, \gamma_j \rightarrow 0 \quad \text{as } j \rightarrow \infty,$$

and L^p -viscosity solutions $v_j, \xi_j \in C(\overline{Q_1})$ of (6.8) (with F_j, f_j in place of F, f) such that (6.9) holds with $\psi = \psi_j$ having a common modulus ρ and $|\psi_j| \leq K$, but

$$(6.11) \quad \|v_j - \xi_j\|_{L^\infty(Q_1)} > \epsilon.$$

To derive a contradiction, we first claim that v_j and ξ_j lie in a precompact subset of $C(\overline{Q_1})$. This assertion follows directly from a slight generalization of Theorem 4.5 of [10] to allow f_j to be bounded in L^p rather than L^{n+1}

(the proof is the same in this case). Note that $p > p_0(n, \lambda, \Lambda, \gamma_j d(Q_1(y, s)))$ for large j by the assumption on p and $\gamma_j \rightarrow 0$. We can thus assume that $v_j \rightarrow v_\infty$ and $\xi_j \rightarrow \xi_\infty$ in $C(\overline{Q_1})$ as $j \rightarrow \infty$. We will show that $v_\infty = \xi_\infty$ (after selecting further subsequences as necessary) by showing that they solve the same problem for which uniqueness is known. This contradicts (6.11) and then the result is proved.

From (SC) it follows that $F_j(0, 0, 0, X) \rightarrow F_\infty(X)$ (along a subsequence if necessary) for some $F_\infty : \mathcal{S}(n) \rightarrow \mathbb{R}$ satisfying (SC), and the convergence is locally uniform in X . Put $\psi_\infty = v_\infty|_{\partial_p Q} = \xi_\infty|_{\partial_p Q}$ and note that by standard viscosity solutions theory (e.g., [7]) ξ_∞ is a C -viscosity solution of

$$(6.12) \quad (\xi_\infty)_t + F_\infty(D^2 \xi_\infty) = 0 \quad \text{in } Q_1, \quad \xi_\infty = \psi_\infty \quad \text{on } \partial_p Q_1.$$

Moreover, the solution of (6.12) is unique.

It remains to show that v_∞ also solves (6.12). To this end, take $\varphi \in C^{2,1}(Q_1)$ and estimate

$$\begin{aligned} & |F_j(x, t, D\varphi(x, t), D^2\varphi(x, t)) - f_j(x, t) - F_\infty(D^2\varphi(x, t))| \\ & \leq \gamma_j |D\varphi(x, t)| + \beta_j((0, 0), (x, t)) (|D^2\varphi(x, t)| + 1) + |f_j(x, t)| \\ & \quad + |F_j(0, 0, D^2\varphi(x, t)) - F_\infty(D^2\varphi(x, t))|. \end{aligned}$$

Since the L^p -norm of the right hand side goes to 0 as $j \rightarrow \infty$, the conclusion follows from the final assertions of Lemma 6.1. Hence $v_\infty = \xi_\infty$. $\square$

Remark 6.4. As Lemma 6.3 is used in conjunction with scaling and “renormalization”, we make some relevant observations here. We may assume $(y, s) = (0, 0)$. If v is a solution of $v_t + F(x, t, Dv, D^2v) = f$ in Q_η , $N > 0$ and

$$(6.13) \quad w(x, t) = \frac{v(\eta x, \eta^2 t)}{N},$$

then w is a solution of

$$(6.14) \quad w_t + \tilde{F}(x, t, Dw, D^2w) = \tilde{f}(x, t) = \frac{\eta^2}{N} f(\eta x, \eta^2 t) \quad \text{in } Q_1,$$

where

$$(6.15) \quad \tilde{F}(x, t, p, X) = \frac{\eta^2}{N} F\left(\eta x, \eta^2 t, \frac{Np}{\eta}, \frac{NX}{\eta^2}\right).$$

$\tilde{F}$ enjoys the same parameters λ, Λ as does F , while the parameters $\tilde{\beta}, \tilde{\gamma}$ corresponding to $\tilde{F}$ are controlled by those for F by the relations

$$(6.16) \quad \tilde{\beta}((0, 0), (x, t)) \leq \max\left(1, \frac{\eta^2}{N}\right) \beta((0, 0), (\eta x, \eta^2 t)), \quad \tilde{\gamma} = \eta \gamma.$$

In particular, we note that

$$\left(\frac{1}{r^{n+2}} \int_{Q_r} |\tilde{f}(x, t)|^p dx dt\right)^{\frac{1}{p}} = \frac{\eta^2}{N} \left(\frac{1}{(\eta r)^{n+2}} \int_{Q_{\eta r}} |f(x, t)|^p dx dt\right)^{\frac{1}{p}}$$

and

$$\begin{aligned} \frac{1}{r^{n+2}} \int_{Q_r} |\tilde{\beta}((0, 0), (x, t))|^p dx dt \\ \leq \left(\max\left(1, \frac{\eta^2}{N}\right)\right)^p \frac{1}{(\eta r)^{n+2}} \int_{Q_{\eta r}} |\beta((0, 0), (x, t))|^p dx dt. \end{aligned}$$

Roughly speaking, we see that γ can be made small, averages of $|f|^p$ can be made small (if finite) and assumptions involving averages of β^p remain valid upon taking $\eta^2/N \leq 1$.

Remark 6.5. There are some difficulties in the literature as regards the treatment of the factors $\max(1, \eta^2/N)$ arising above in estimating the “new” β . We explain these later in the sequel. For this purpose, we note now that in [5] an alternative to the β employed here is in use that for gradient dependent equations would be

$$(6.17) \quad \hat{\beta}((y, s), (x, t)) = \sup_{X \in \mathcal{S}(n)} \frac{|F(x, t, 0, X) - F(y, s, 0, X)|}{\|X\|}.$$

A point of interest is that this quantity is invariant under rescaling and renormalization in such a way that the factors $\max(1, \eta^2/N)$ do not appear when estimating averages of the transformed $\hat{\beta}^p$.

Section 7. Gradient Estimates

The main result of this section is Theorem 7.3 below concerning the regularity of solutions of $u_t + F(x, t, u, Du, D^2u) = f(x, t)$ when F satisfies (SC) along with smallness conditions on the function β of (6.6).

According to the theorem below, when the condition on β is satisfied, $f \in L^p(Q)$ and $p_0 < p \leq n + 2$, then $|Du|^q$ (with Du understood in the sense of distributions) is locally integrable for suitable q and u belongs to a corresponding fractional Sobolev space with respect to t . In the event $n+2 < p$, Du is locally Hölder continuous while u is locally Hölder continuous in t (with better exponents than those provided by Lemma 5.1). The ranges for the parameters q and the Hölder exponents are given fairly explicitly. In the next section, these results are used to obtain existence theorems.

The smallness condition on β is parameterized by the exponent p and a number θ and asks that for some $r_0 > 0$:

(7.1)

$$\left(\frac{1}{r^{n+2}} \int_{Q_r(y,s)} \beta((y,s), (x,t))^p dx dt \right)^{\frac{1}{p}} \leq \theta \quad \text{for } r \leq \min(r_0, \text{dist}((y,s), \partial Q)).$$

In the course of discussion, this condition may be imposed at a single point $(y,s) \in Q$ or for almost all $(y,s) \in Q$.

Remark 7.1. For results below to hold, θ will have to be sufficiently small. However, the meaning of “sufficiently small” is determined by the nonconstructive argument in the proof of Lemma 6.3, so that in practice one will need to be able to verify (7.1) for all θ . The basic situation in which this can be achieved is: $\beta((y,s), (x,t)) \rightarrow 0$ as $(x,t) \rightarrow (y,s)$. One can also perturb from this situation - e.g., $F(x,t,X) = -\text{trace}(X) - \nu \text{trace}(A(x,t)X)$ will satisfy (7.1) if ν is small and (y,s) is a point such that $r^{-(n+2)} \int_{Q_r(y,s)} |A(x,t) - A(y,s)|^p dx dt$ is bounded.

Remark 7.2. In the developments, the ω appearing in the structure conditions will complicate calculations and statements in such a way that we prefer to remove it at this time. To do this, note that by Proposition 4.1 an L^p -viscosity solution of $u_t + F(x,t,u,Du,D^2u) = f(x,t)$ is also an L^p -viscosity solution of

$$u_t + F(x,t,0,Du,D^2u) = f(x,t) + g(x,t),$$

where

$$g(x,t) = F(x,t,0,Du(x,t),D^2u(x,t)) - F(x,t,u(x,t),Du(x,t),D^2u(x,t))$$

satisfies

$$|g(x,t)| \leq \omega(|u(x,t)|).$$

In consequence, $f + g$ belongs to $L^p(Q)$ if $f \in L^p(Q)$.

Hereafter we use the following notation: if $1 \leq p \leq n + 2$ then

$$(7.2) \quad p^* = \begin{cases} \frac{p(n+2)}{n+2-p} & \text{if } p < n+2 \\ \infty & \text{if } p = n+2. \end{cases}$$

Moreover, $\bar{\alpha} = \bar{\alpha}(n, \lambda, \Lambda)$ is from Proposition 5.4.

Theorem 7.3. *Let F satisfy (SC) and $f \in L^p(Q)$. Let u be an L^p -viscosity solution of $u_t + F(x, t, Du, D^2u) = f(x, t)$ in Q . Then:*

(i) *If $p_0 < p \leq n + 2$ there exists $\theta = \theta(n, p, \lambda, \Lambda, \gamma)$ such that if (7.1) is satisfied for some $r_0 > 0$ and almost all $(y, s) \in Q$, then $u \in H_{\text{loc}}^{1, \frac{1}{2}, q}(Q)$ for every $q < p^*$, and for every $Q' \subset\subset Q$ we have*

$$(7.3) \quad \|u\|_{H^{1, \frac{1}{2}, q}(Q')} \leq C \left(\|u\|_{L^\infty(Q)} + \|f\|_{L^p(Q)} + \sqrt{\|u\|_{L^\infty(Q)} + \|f\|_{L^p(Q)}} \right),$$

where $C = C(r_0, n, p, q, \lambda, \Lambda, \gamma, d(\Omega), T, \text{dist}(Q', \partial_p Q))$.

(ii) *Let $p > n + 2$, $\bar{\alpha} = \bar{\alpha}(n, \lambda, \Lambda)$ be determined by Lemma 5.1, $0 < \alpha < \bar{\alpha}$ also satisfy $\alpha \leq 1 - (n + 2)/p$. Then there exists $\theta = \theta(n, p, \lambda, \Lambda, \gamma, \alpha)$ such that if (7.1) is satisfied for some $r_0 > 0$ and almost all $(y, s) \in Q$, then $u \in C_{\text{loc}}^{1+\alpha}(Q)$, i.e. $Du \in C_{\text{loc}}^\alpha(Q)$, and for every $Q' \subset\subset Q$ we have*

$$(7.4) \quad \|u\|_{C^{1+\alpha}(Q')} \leq C \left(\|u\|_{L^\infty(Q)} + \|f\|_{L^p(Q)} + \left(\|u\|_{L^\infty(Q)} + \|f\|_{L^p(Q)} \right)^{\frac{1-\alpha}{2}} \right),$$

where $C = C(r_0, n, p, \lambda, \Lambda, \gamma, \alpha, \bar{\alpha}, d(\Omega), T, \text{dist}(Q', \partial_p Q))$.

The proof of Theorem 7.3 is long. Essential use is made of the approximation result Lemma 6.3 and the basic lemma immediately below which provides a preliminary estimate on Du . The proof of Theorem 7.3 is completed after the lemma. Regarding the point (y, s) in the statement, recall the comments following (6.6).

Lemma 7.4. *Let F satisfy (SC), $0 < r_0 \leq 2$, $f \in L^p(Q_{r_0}(y, s))$, and let $p > p_0(n, \lambda, \Lambda, \gamma\sqrt{5}r_0)$. Let u be an L^p -viscosity solution of $u_t + F(x, t, Du, D^2u) =$*

$f(x, t)$ in $Q_{r_0}(y, s)$. Let $\bar{\alpha} = \bar{\alpha}(n, \lambda, \Lambda)$ be determined by Lemma 5.1, and let $0 < \alpha < \bar{\alpha}$. Then there exist constants $\theta = \theta(n, p, \lambda, \Lambda, \gamma, \alpha)$ and $C = C(r_0, n, p, \lambda, \Lambda, \gamma, \alpha)$ and $\sigma_0 = \sigma_0(r_0, n, p, \lambda, \Lambda, \gamma, \alpha) > 0$ such that if

$$(7.5) \quad \left(\frac{1}{r^{n+2}} \int_{Q_r(y, s)} \beta((y, s), (x, t))^p dx dt \right)^{\frac{1}{p}} \leq \theta \quad \text{for } r \leq r_0$$

and

$$(7.6) \quad L = \|u\|_{L^\infty(Q_{r_0}(y, s))} + \sup_{r \leq r_0} \left\{ r^{1-\alpha} \left(\frac{1}{r^{n+2}} \int_{Q_r(y, s)} |f(x, t)|^p dx dt \right)^{\frac{1}{p}} \right\} < \infty,$$

then $Du(y, s)$ exists and satisfies

$$(7.7) \quad |Du(y, s)| \leq C (L + \sqrt{L}).$$

Moreover, for all $(x, t) \in Q_{\sigma_0}$

$$(7.8) \quad \frac{|u(y+x, s+t) - u(y, s) - \langle Du(y, s), x \rangle|}{(|x|^2 + |t|)^{\frac{1+\alpha}{2}}} \leq C \left(L + L^{\frac{1-\alpha}{2}} \right),$$

$$(7.9) \quad \frac{|u(y+x, s) - u(y, s)|}{|x|} \leq C (L + \sqrt{L}),$$

$$(7.10) \quad \frac{|u(y, s+t) - u(y, s)|}{|t|^{\frac{1}{2}}} \leq C (L + \sqrt{L}).$$

Proof of Lemma 7.4. We reiterate the meaning of $\bar{\alpha}$ while reintroducing the constants

$$(7.11) \quad \begin{aligned} K_1 &= K_1(n, p, \lambda, \Lambda, \gamma), \quad \kappa = \kappa(n, p, \lambda, \Lambda, \gamma), \\ K_2 &= K_2(n, \lambda, \Lambda), \quad \bar{\alpha} = \bar{\alpha}(n, \lambda, \Lambda) \end{aligned}$$

of Lemma 5.1 and Proposition 5.4. We are about to present a host of parameters and restrictions upon them. This corresponds to the technical nature of the proof to follow. At this juncture the reader should note the definitions and that the restrictions on the parameters are consistent – they can be satisfied. The roles played will only become evident in the course of the proof.

We define a modulus ρ and restrict $0 < \alpha, \mu$ by:

$$(7.12) \quad \rho(s) = 2K_1 s^\kappa, \quad \alpha < \bar{\alpha}, \quad 9K_2 \mu^{(\bar{\alpha}-\alpha)} \leq 1, \quad \mu < \frac{1}{4}.$$

Here $\alpha < \bar{\alpha}$ will remain fixed and then μ can be chosen so that the remaining two conditions are satisfied. After this, μ will remain fixed and can be thought of as a function of α, λ, Λ . Set

$$(7.13) \quad \epsilon = K_2 \mu^{1+\bar{\alpha}}, \quad M = K_2 \sum_{i=0}^{\infty} \mu^{i\alpha} \quad \text{and} \quad \theta = \delta = \delta(\epsilon, n, p, \lambda, \Lambda, \rho, 1),$$

where $\delta(\epsilon, n, p, \lambda, \Lambda, \rho, 1)$ is provided by Lemma 6.3.

Without loss of generality we take $(y, s) = (0, 0)$ hereafter. We will use θ above in the proof, that is we now assume that

$$(7.14) \quad \left(\frac{1}{r^{n+2}} \int_{Q_r} \beta((0, 0), (x, t))^p dx dt \right)^{\frac{1}{p}} \leq \delta \quad \text{for} \quad r \leq r_0.$$

We may also assume

$$(7.15) \quad \delta \leq \frac{1}{2}.$$

Let V_n be the volume of the unit ball in $\mathbb{R}^n$. We make an initial reduction, the point of which is to reduce to the situation in which

$$(7.16) \quad \begin{cases} \text{(i)} & |u| \leq 1 \quad \text{in} \quad Q_2 \\ \text{(ii)} & \gamma \leq \frac{\delta}{8M(V_n)^{1/p}} \\ \text{(iii)} & \sup_{r \leq 1} \left(\frac{1}{r^{n+2}} \int_{Q_r} \beta((0, 0), (x, t))^p dx dt \right)^{\frac{1}{p}} \leq \theta = \delta \\ \text{(iv)} & \sup_{r \leq 2} r^{1-\alpha} \left(\frac{1}{r^{n+2}} \int_{Q_r} |f(x, t)|^p dx dt \right)^{\frac{1}{p}} \leq \frac{\delta}{4}. \end{cases}$$

First, we let

$$(7.17) \quad N = \|u\|_{L^\infty(Q_{r_0})} + \sup_{r \leq r_0} \left\{ r^{1-\alpha} \left(\frac{1}{r^{n+2}} \int_{Q_r} |f(x, t)|^p dx dt \right)^{\frac{1}{p}} \right\} \delta^{-1}.$$

N differs from L of (7.6) only in the factor δ^{-1} in the second term, so $N < \infty$.

Choose σ_0 to be the largest number σ such that

$$(7.18) \quad \sigma \gamma \leq \frac{\delta}{8M(V_n)^{1/p}}, \quad \sigma^{1+\alpha} \leq \frac{1}{4}, \quad 2\sigma \leq r_0.$$

For

$$(7.19) \quad \sigma = \min(\sigma_0, \sqrt{N})$$

we rescale and renormalize, putting

$$(7.20) \quad \tilde{u}(x, t) = \frac{u(\sigma x, \sigma^2 t)}{N}$$

as in Remark 6.4. In this case, $w = \tilde{u}$ is a solution of (6.14) (with $\eta = \sigma$) in Q_2 (via $2\sigma \leq r_0$) and clearly $\tilde{u}$ satisfies (7.16) (i) in place of u . The relation (7.16) (ii) holds for $\tilde{\gamma} = \sigma\gamma$ in place of γ in view of the first condition in (7.18). Let us note here that also $\tilde{\gamma} \leq \gamma$ since $\sigma < 1$. The reader may recall this point when K_1 and κ of (7.11) are used below via applications of Lemma 5.1; the use will be seen to be legitimate. A similar point will concern the condition “ $p > p_0$ ” - this condition is preserved upon changes of scale, renormalization and passage to subcylinders.

Next, for every $r \leq 2$

$$(7.21) \quad \begin{aligned} & r^{1-\alpha} \left(\frac{1}{r^{n+2}} \int_{Q_r} |\tilde{f}(x, t)|^p dx dt \right)^{\frac{1}{p}} \\ &= r^{1-\alpha} \frac{\sigma^2}{N} \left(\frac{1}{r^{n+2}} \int_{Q_r} |f(\sigma x, \sigma^2 t)|^p dx \right)^{\frac{1}{p}} \\ &\leq \frac{\sigma^{1+\alpha}}{N} (\sigma r)^{1-\alpha} \left(\frac{1}{(\sigma r)^{n+2}} \int_{Q_{\sigma r}} |f(x, t)|^p dx dt \right)^{\frac{1}{p}} \leq \sigma^{1+\alpha} \delta \leq \frac{\delta}{4}, \end{aligned}$$

where (7.18) is used at the end. Thus (7.16) (iv) is satisfied with $\tilde{f}$ in place of f . Also for $0 < r \leq 1$ we have $\sigma r \leq r_0$ and so, using $\sigma^2 \leq N$ from (7.19),

$$\begin{aligned} & \left(\frac{1}{r^{n+2}} \int_{Q_r} |\tilde{\beta}((0, 0), (x, t))|^p dx dt \right)^{\frac{1}{p}} \\ &\leq \left(\frac{1}{(\sigma r)^{n+2}} \int_{Q_{\sigma r}} |\beta((0, 0), (x, t))|^p dx dt \right)^{\frac{1}{p}} \leq \delta \end{aligned}$$

by (7.14). Thus $\tilde{\beta}$ satisfies (7.16) (iii).

Hereafter σ satisfying (7.19) will be fixed and we will consider the function $\tilde{u}$ in Q_2 , drop the tildes and simply assume that (7.16) holds for the original F, f, u, β, γ .

To continue, we define linear functions l_k of x for $k = 1, 2, \dots$ by

$$(7.22) \quad l_k(x) = a_k + \langle b_k, x \rangle = \mathcal{L}(\xi_k)(x) := \xi_k(0, 0) + \langle D\xi_k(0, 0), x \rangle$$

by means of the solutions ξ_k of

$$(7.23) \quad \begin{cases} (\xi_k)_t + F(0, 0, 0, D^2 \xi_k) = 0 & \text{in } Q_{\mu^{k-1}} \\ \xi_k = u & \text{on } \partial_p Q_{\mu^{k-1}}. \end{cases}$$

In (7.22) “ $\mathcal{L}$ ” is by definition the operator of linearization in x at $(0, 0)$. Initializing via $l_{-1} = l_0 \equiv 0$ we will inductively verify that for $k = 0, 1, \dots$:

$$\begin{aligned} (a) \quad & \|u - l_k\|_{L^\infty(Q_{\mu^k})} \leq \mu^{k(1+\alpha)}, \\ (b) \quad & |a_k - a_{k-1}| + \mu^{k-1}|b_k - b_{k-1}| \leq K_2 \mu^{(k-1)(1+\alpha)}, \\ (c) \quad & |(u - l_k)(\mu^k x, \mu^{2k} t) - (u - l_k)(\mu^k z, \mu^{2k} \tau)| \leq 2K_1 \mu^{k(1+\alpha)} (|x - z|^2 + |t - \tau|)^{\frac{\kappa}{2}} \end{aligned}$$

for all $(x, t), (z, \tau) \in Q_1$. Notice that (7.13) and (b) for $k = 0, 1, \dots, j$ imply that

$$(7.24) \quad |b_j| \leq M.$$

By (7.16) (i), (a) is satisfied for $k = 0$ while (b) is trivially satisfied for $k = 0$. We note that (7.16) (iv) implies

$$(7.25) \quad \left(\int_{Q_r} |f(x, t)|^p dx dt \right)^{\frac{1}{p}} \leq r^{\left(\alpha - 1 + \frac{n+2}{p}\right)} \frac{\delta}{4}.$$

Note that

$$u_t = \mathcal{P}^-(D^2 u) - \gamma |Du| \leq |f| \quad \text{and} \quad -|f| \leq u_t + \mathcal{P}^+(D^2 u) + \gamma |Du|$$

in Q_2 and (7.25) with $r = 2$ yields

$$\|f\|_{L^p(Q_2)} \leq 2^{\left(\alpha - 1 + \frac{n+2}{p}\right)} \frac{\delta}{4} \leq 2^{1+\alpha} \frac{\delta}{4} \leq \delta$$

since $p > (n + 2)/2$ always holds. Thus Lemma 5.1 yields

$$\|u\|_{C^\kappa(Q_1)} \leq K_1(1 + \delta)$$

and hence (c) for $k = 0$ as we have taken $\delta \leq 1/2$.

Assume now that (a), (b), and (c) hold for $k \leq j$. We will prove that they hold for $k = j + 1$. First let us note that $w(x, t) = u(x, t) - l_j(x)$ is an L^p -viscosity solution of

$$(7.26) \quad w_t + F(x, t, Dw, D^2 w) = f(x, t) + g(x, t),$$

where

$$g(x, t) = F(x, t, Dw(x, t), D^2w(x, t)) - F(x, t, Dw(x, t) + b_j, D^2w(x, t))$$

satisfies

$$(7.27) \quad |g(x, t)| \leq \gamma |b_j| \leq \frac{\delta}{8M(V_n)^{1/p}} M \leq \frac{\delta}{8(V_n)^{1/p}}.$$

The above used (7.16) (ii) and (7.24). Now put

$$v(x, t) = \frac{w(\mu^j x, \mu^{2j} t)}{\mu^{j(1+\alpha)}} = \frac{u(\mu^j x, \mu^{2j} t) - l_j(\mu^j x)}{\mu^{j(1+\alpha)}}.$$

As in Remark 6.4, we see that the function v is an L^p -viscosity solution of

$$v_t + F_j(x, t, Dv, D^2v) = f_j(x, t) + g_j(x, t)$$

in Q_2 , where

$$F_j(x, t, p, X) = \mu^{j(1-\alpha)} F\left(\mu^j x, \mu^{2j} t, \mu^{j\alpha} p, \frac{1}{\mu^{j(1-\alpha)}} X\right),$$

$$f_j(x, t) = \mu^{j(1-\alpha)} f(\mu^j x, \mu^{2j} t),$$

$$g_j(x, t) = \mu^{j(1-\alpha)} g(\mu^j x, \mu^{2j} t).$$

By (7.16) (iv)

$$\|f_j\|_{L^p(Q_1)} = \mu^{j(1-\alpha)} \left(\frac{1}{\mu^{j(n+2)}} \int_{Q_{\mu^j}} |f(x, t)| dx dt \right)^{\frac{1}{p}} \leq \frac{\delta}{4},$$

and this and (7.27) imply that

$$(7.28) \quad \begin{aligned} \|f_j + g_j\|_{L^p(Q_1)} &\leq \|f_j\|_{L^p(Q_1)} + \|g_j\|_{L^p(Q_1)} \\ &\leq \frac{\delta}{4} + \mu^{j(1-\alpha)} (V_n)^{1/p} \frac{\delta}{8(V_n)^{1/p}} < \frac{\delta}{4} + \frac{\delta}{8} < \delta. \end{aligned}$$

We also have $\beta_{F_j}((0, 0), (x, t)) \leq \beta((0, 0), (\mu^j x, \mu^{2j} t))$. Moreover, by (a), $\|v\|_{L^\infty(Q_1)} \leq 1$ and by (c) the ρ of (7.12) is a modulus of continuity for $v|_{\partial_p Q_1}$.

Let ξ_{j+1} be defined via (7.23), (7.22). Then

$$\xi(x, t) = \frac{\xi_{j+1}(\mu^j x, \mu^{2j} t) - l_j(\mu^j x)}{\mu^{j(1+\alpha)}}$$

is the C -viscosity solution of

$$\begin{cases} \xi_t + F_j(0, 0, 0, D^2\xi) = 0 & \text{in } Q_1 \\ \xi = v & \text{on } \partial_p Q_1. \end{cases}$$

In view of the above and Lemma 6.3 with $F = F_j$, $\psi = v$, ϵ , and δ (see (7.13)) we have $|v - \xi| \leq \epsilon$ on Q_1 or

$$(7.29) \quad |u(x, t) - \xi_{j+1}(x, t)| \leq K_2 \mu^{1+\bar{\alpha}} \mu^{j(1+\alpha)} \quad \text{for } (x, t) \in Q_{\mu^j}.$$

The maximum principle yields $\|\xi\|_{L^\infty(Q_1)} \leq \|v\|_{L^\infty(Q_1)} \leq 1$, and therefore we obtain from Proposition 5.4 that

$$(7.30) \quad \|\xi\|_{C^{1+\bar{\alpha}}(Q_{1/2})} \leq K_2.$$

In particular,

$$|\xi(x, t) - \mathcal{L}(\xi)(x)| \leq K_2(|x|^2 + |t|)^{\frac{1+\bar{\alpha}}{2}} \quad \text{for } (x, t) \in Q_{1/2}.$$

Noting

$$\xi(x, t) - \mathcal{L}(\xi)(x) = \frac{\xi_{j+1}(\mu^j x, \mu^{2j} t) - \mathcal{L}(\xi_{j+1})(\mu^j x)}{\mu^{j(1+\alpha)}} = \frac{\xi_{j+1}(\mu^j x, \mu^{2j} t) - l_{j+1}(\mu^j x)}{\mu^{j(1+\alpha)}},$$

this implies

$$(7.31) \quad |\xi_{j+1}(x, t) - l_{j+1}(x)| \leq K_2 8^{(1+\bar{\alpha})/2} \mu^{1+\bar{\alpha}} \mu^{j(1+\alpha)} \quad \text{for } (x, t) \in Q_{2\mu^{j+1}}.$$

Combining (7.29) and the estimate above, we find

$$(7.32) \quad \begin{aligned} |u(x, t) - l_{j+1}(x)| &\leq |u(x, t) - \xi_{j+1}(x, t)| + |\xi_{j+1}(x, t) - l_{j+1}(x)| \\ &\leq K_2 \mu^{j(1+\alpha)} \mu^{(1+\bar{\alpha})} \left(8^{(1+\bar{\alpha})/2} + 1 \right) \\ &\leq 9K_2 \mu^{(j+1)(1+\alpha)} \mu^{(\bar{\alpha}-\alpha)} \leq \mu^{(j+1)(1+\alpha)} \end{aligned}$$

on $Q_{2\mu^{j+1}}$ (at the end, we used (7.12)); in particular (a) holds for $k = j + 1$.

Also

$$|\xi(0, 0)| + |D\xi(0, 0)| \leq K_2$$

amounts to (b) for $k = j + 1$.

Now $w = u(x, t) - l_{j+1}(x)$ is a solution of (7.26) with b_{j+1} in place of b_j and (7.27) still holds for the same reasons as before. Thus by Lemma 5.1, (7.25) and (7.32)

$$\begin{aligned}
& \|w\|_{C^\kappa(Q_{\mu^{j+1}})} \\
& \leq \mu^{-\kappa(j+1)} K_1 \left(\mu^{(j+1)(2-\frac{n+2}{p})} \left(\|f\|_{L^p(Q_{2\mu^{j+1}})} + (2\mu^{(j+1)})^{\frac{n+2}{p}} \frac{\delta}{8} \right) + \mu^{(j+1)(1+\alpha)} \right) \\
& \leq \mu^{-\kappa(j+1)} K_1 \left(\mu^{(j+1)(2-\frac{n+2}{p})} \left((2\mu^{(j+1)})^{(\alpha-1+\frac{n+2}{p})} \frac{\delta}{4} \right. \right. \\
& \quad \left. \left. + (2\mu^{(j+1)})^{\frac{n+2}{p}} \frac{\delta}{8} \right) + \mu^{(j+1)(1+\alpha)} \right) \\
& \leq \mu^{-\kappa(j+1)} \mu^{(j+1)(1+\alpha)} K_1 \left(2^{(1+\alpha)} \frac{\delta}{4} + \mu^{(j+1)(1-\alpha)} 2^2 \frac{\delta}{8} + 1 \right) \\
& \leq \mu^{-\kappa(j+1)} \mu^{(j+1)(1+\alpha)} K_1 \left(\delta + \frac{\delta}{2} + 1 \right) \\
& \leq \mu^{-\kappa(j+1)} \mu^{(j+1)(1+\alpha)} 2K_1,
\end{aligned}$$

which implies (c).

As in [9] (see also [5]), the result follows upon noting that (b) implies $a_k, b_k \rightarrow a, b$, where

$$|a - a_k| + \mu^k |b - b_k| \leq \mu^{k(1+\alpha)} 2M,$$

and so, by (a),

$$|u(x, t) - l(x)| \leq |u(x, t) - l_k(x)| + |l_k(x) - l(x)| \leq (1 + 2M) \mu^{k(1+\alpha)}$$

on Q_{μ^k} , where $l(x) = a + \langle b, x \rangle$. For $(x, t) \in Q_{\mu^k} \setminus Q_{\mu^{k+1}}$ we deduce

$$(7.33) \quad \frac{|u(x, t) - l(x)|}{(|x|^2 + |t|)^{\frac{1+\alpha}{2}}} \leq \frac{(1 + 2M)}{\mu^{(1+\alpha)}},$$

which then holds for $(x, t) \in Q_1$.

At this point we recall that u has been standing for the $\tilde{u}$ of (7.20) in the above discussion. From the above $D\tilde{u}(0, 0) = b$ exists and $|D\tilde{u}(0, 0)| \leq M$. Next we multiply through by N and return to the original variables to conclude that

$$(7.34) \quad |Du(y, s)| = \frac{N}{\sigma} |D\tilde{u}(0, 0)| \leq \frac{N}{\sigma} M$$

and, from (7.33),

$$(7.35) \quad \frac{|u(y+x, s+t) - u(y, s) - \langle Du(y, s), x \rangle|}{(|x|^2 + |t|)^{\frac{1+\alpha}{2}}} \leq \frac{(1+2M)N}{(\mu\sigma)^{(1+\alpha)}}$$

for $(x, t) \in Q_\sigma$, with σ from (7.19). Now (7.7) follows from (7.17) and (7.34). Moreover, (7.35) yields (7.8) for all $(x, t) \in Q_\sigma$. If $|x| \leq \sigma$ then

$$\begin{aligned} \frac{|u(y+x, s) - u(y, s)|}{|x|} &\leq \frac{|u(y+x, s) - u(y, s) - \langle Du(y, s), x \rangle|}{|x|^{1+\alpha}} |x|^\alpha + |Du(y, s)| \\ &\leq \frac{(1+2M)N}{\mu^{(1+\alpha)}\sigma} + M \frac{N}{\sigma} \end{aligned}$$

and we have (7.9) for all such x . Similarly, if $-\sigma^2 < t < 0$ then by (7.35)

$$\frac{|u(y, s+t) - u(y, s)|}{|t|^{\frac{1}{2}}} \leq \frac{(1+2M)N}{\mu^{(1+\alpha)}\sigma^{1+\alpha}} |t|^{\frac{\alpha}{2}} \leq \frac{(1+2M)N}{\mu^{(1+\alpha)}\sigma}.$$

Thus (7.8), (7.9) and (7.10) hold for all $(x, t) \in Q_\sigma$.

If $(x, t) \in Q_{\sigma_0} \setminus Q_\sigma$ then $|x|^2 + t > N$ and the left hand side of (7.8) can be estimated by

$$2\|u\|_{L^\infty(Q_{\sigma_0})} N^{-\frac{1+\alpha}{2}} + |Du(y, s)| N^{-\frac{\alpha}{2}} \leq 2N^{\frac{1-\alpha}{2}} + \frac{M}{\sigma} N^{1-\frac{\alpha}{2}}$$

(where we used (7.17) and (7.34)), proving (7.8) for all $(x, t) \in Q_{\sigma_0}$.

Finally, (7.9) for $|x| \in (\sigma, \sigma_0)$ and (7.10) for $t \in (-\sigma_0^2, -\sigma^2)$ follow easily since then both expressions are dominated by $2N/\sigma$.

Of course, the dependencies of C, σ_0 include all parameters and that of θ all but r_0 , so these assertions follow with little checking. $\square$

Proof of Theorem 7.3. (i) We argue as in [28] with several refinements. Assume that

$$(7.36) \quad 2r_0 \leq \text{dist}((z, \tau), \partial_p Q) \quad \text{for } (z, \tau) \in Q'.$$

Lemma 7.4 will be applied with α close to 0 together with the following fact: If $f \in L^p(Q)$ and $\mathcal{M}(f)(y, s)$ is defined for $(y, s) \in Q'$ by

$$\begin{aligned} (7.37) \quad \mathcal{M}(f)(y, s) &= \sup_{r \leq r_0} \left(r^{(1-\alpha)} \left\{ \frac{1}{r^{n+2}} \int_{Q_r(y, s)} |f(x, t)|^p dx dt \right\}^{\frac{1}{p}} \right) \\ &= \sup_{r \leq r_0} \left(r^{(1-\alpha)p} \frac{1}{r^{n+2}} \int_{Q_r(y, s)} |f(x, t)|^p dx dt \right)^{\frac{1}{p}}, \end{aligned}$$

where $\alpha < 1$, then for all $(z, \tau) \in Q'$

$$(7.38) \quad \|\mathcal{M}(f)\|_{L^q(Q_{r_0}(z, \tau))} \leq (r_0)^{(1-\alpha)+(n+2)\left(\frac{1}{q}-\frac{1}{p}\right)} C_{q, \alpha, p} \|f\|_{L^p(Q_{2r_0}(z, \tau))}$$

for

$$(7.39) \quad 1 \leq q < \frac{(n+2)p}{n+2-(1-\alpha)p}.$$

See [9], Section 4 in this regard. Given $q < p^* = (n+2)p/(n+2-p)$, α may be chosen sufficiently small so that the restriction on q above is satisfied.

Suppose that $(y, s), (z, \tau) \in Q'$. We regard the $L = L(y, s)$ of (7.6) as a function of (y, s) , where r_0 is as above, but in so doing, also replace the $\|u\|_{L^\infty(Q_{r_0}(y, s))}$ by $\|u\|_{L^\infty(Q)}$ and note that

$$L(y, s) \leq \|u\|_{L^\infty(Q)} + \mathcal{M}(f)(y, s).$$

Since (7.9) holds for all $|x| \leq \sigma_0$, we have for all such x

$$\begin{aligned} \left(\int_{Q_{r_0}(z, \tau)} \frac{|u(y+x, s) - u(y, s)|^q}{|x|^q} dy ds \right)^{\frac{1}{q}} &\leq C \|L + \sqrt{L}\|_{L^q(Q_{r_0}(z, \tau))} \\ &\leq C \left((V_n r_0^{n+2})^{1/q} \left(\|u\|_{L^\infty(Q)} + \sqrt{\|u\|_{L^\infty(Q)}} \right) \right. \\ &\quad \left. + \|\mathcal{M}(f)\|_{L^q(Q_{r_0}(z, \tau))} + \|\sqrt{\mathcal{M}(f)}\|_{L^q(Q_{r_0}(z, \tau))} \right). \end{aligned}$$

Now using Jensen's inequality and (7.38) we get $Du \in L^q(Q_{r_0}(z, \tau))$ and

$$\begin{aligned} \left(\int_{Q_{r_0}(z, \tau)} |Du(y, s)|^q dy ds \right)^{\frac{1}{q}} \\ \leq C \left(\|u\|_{L^\infty(Q)} + \sqrt{\|u\|_{L^\infty(Q)}} + \|f\|_{L^p(Q_{2r_0}(z, \tau))} + \sqrt{\|f\|_{L^p(Q_{2r_0}(z, \tau))}} \right), \end{aligned}$$

where C now depends on r_0 as well.

To conclude the proof of (7.3) we notice that for every $(y, s) \in Q_{r_0}(z, \tau)$, (7.10) holds for all $-\sigma_0^2 < t < 0$ and arguing as above we obtain that for every $q < p^*$

$$\begin{aligned} \left(\int_{Q_{r_0}(z, \tau)} \frac{|u(y, s+t) - u(y, s)|^q}{|t|^{\frac{q}{2}}} dy ds \right)^{\frac{1}{q}} \\ \leq C_1 \left(\|u\|_{L^\infty(Q)} + \sqrt{\|u\|_{L^\infty(Q)}} + \|f\|_{L^p(Q_{2r_0}(z, \tau))} + \sqrt{\|f\|_{L^p(Q_{2r_0}(z, \tau))}} \right), \end{aligned}$$

and thus $u \in H^{\frac{1}{2},q}(Q_{r_0}(z, \tau))$. Now the usual covering argument completes the proof.

(ii) We apply Lemma 7.4 with $p = n + 2$ to find that for $(y, s) \in Q'$

$$\sup_{(x,t) \in Q_{\sigma_0}} \left(\frac{|u(y+x, s+t) - u(y, s) - \langle Du(y, s), x \rangle|}{(|x|^2 + |t|)^{\frac{1+\alpha}{2}}} \right) \leq C \left(L + L^{\frac{1-\alpha}{2}} \right),$$

where now

$$L = \|u\|_{L^\infty(Q)} + \sup_{r \leq r_0} \left\{ r^{1-\alpha} \left(\frac{1}{r^{n+2}} \int_{Q_r(y,s)} |f(x, t)|^{n+2} dx dt \right)^{\frac{1}{n+2}} \right\}.$$

Using the Hölder inequality (with the $p > n + 2$ of the current theorem) we obtain

(7.40)

$$\begin{aligned} & \sup_{r \leq r_0} \left\{ r^{(1-\alpha)} \left(\frac{1}{r^{n+2}} \int_{Q_r(y,s)} |f(x, t)|^{n+2} dx dt \right)^{\frac{1}{n+2}} \right\} \\ & \leq C \sup_{r \leq r_0} \left\{ r^{-\alpha + \frac{p-n-2}{p}} \left(\int_{Q_r(y,s)} |f(x, t)|^p dx dt \right)^{\frac{1}{p}} \right\} \leq C \|f\|_{L^p(Q)} \end{aligned}$$

if $\alpha \leq 1 - \frac{n+2}{p}$. It follows from (7.8) that for every $(y, s) \in Q'$ and $|x| \leq \sigma_0/2$, $-\sigma_0^2/4 < t \leq 0$

$$\begin{aligned} & |u(y+x, s+t) - u(y, s) - \langle Du(y, s), x \rangle| \\ & \leq C \left(\|u\|_{L^\infty(Q)} + \|f\|_{L^p(Q)} + \left(\|u\|_{L^\infty(Q)} + \|f\|_{L^p(Q)} \right)^{\frac{1-\alpha}{2}} \right) (|x|^2 + |t|)^{\frac{1+\alpha}{2}} \end{aligned}$$

and from (7.7) that

$$\begin{aligned} |Du(y, s)| & \leq C \left(\|u\|_{L^\infty(Q)} + \|f\|_{L^p(Q)} + \left(\|u\|_{L^\infty(Q)} + \|f\|_{L^p(Q)} \right)^{\frac{1}{2}} \right) \\ & \leq C \left(\|u\|_{L^\infty(Q)} + \|f\|_{L^p(Q)} + \left(\|u\|_{L^\infty(Q)} + \|f\|_{L^p(Q)} \right)^{\frac{1-\alpha}{2}} \right), \end{aligned}$$

where at the end we used $(1-\alpha)/2 \leq 1/2$. When $(x, t) \in Q'$ and $|x| \geq \sigma/2$ or $|t| > \sigma^2/4$ we notice that

$$\begin{aligned} & |u(y+x, s+t) - u(y, s) - \langle Du(y, s), x \rangle| \\ & \leq \left(2^{2+\alpha} \sigma^{-1-\alpha} \|u\|_{L^\infty(Q)} + 2^\alpha \sigma^{-\alpha} \|Du\|_{L^\infty(Q')} \right) (|x|^2 + |t|)^{\frac{1+\alpha}{2}} \end{aligned}$$

and use the estimate just above. This yields (7.4). $\square$

Remark 7.5. Since by [1], page 226, $H^{\frac{1}{2},q} \hookrightarrow W^{\tau,q}$ (the usual fractional Sobolev space) for every $\tau < 1/2$ we obtain that $u \in W_{\text{loc}}^{1,\tau,q}(Q)$ for all $\tau < 1/2, q < p^*$. $W_{\text{loc}}^{1,\tau,q}(Q)$ means the space of $v \in W_{\text{loc}}^{\tau,q}(Q)$ such that $Dv \in L_{\text{loc}}^q(Q)$.

Remark 7.6. Using Sobolev inclusion $W^{1,\tau,q} \hookrightarrow C^\kappa$, where C^κ is taken with respect to the distance

$$d_\tau((x, t), (y, s)) = \sqrt{|x - y|^2 + |s - t|^{2\tau}}$$

in $\mathbb{R}^{n+1}$ for

$$q > n + \frac{1}{\tau}, \quad \kappa = 1 - \frac{n + \frac{1}{\tau}}{q},$$

we can conclude from (7.4) that for every

$$\kappa < 1 - \frac{(n+2-p)(n+\frac{1}{2})}{p(n+2)}$$

we have

$$|u(x, t) - u(y, s)| \leq C \left(\|u\|_{L^\infty(\partial_p Q)} + \sqrt{\|u\|_{L^\infty(\partial_p Q)}} + \|f\|_{L^p(Q)} + \sqrt{\|f\|_{L^p(Q)}} \right) (|x - y|^\kappa + |t - s|^{\kappa\tau})$$

for all $(x, t), (y, s) \in Q'$, where $C = C(r_0, n, p, \lambda, \Lambda, \gamma, \kappa, d(\Omega), T, \text{dist}(Q', \partial_p Q))$.

Remark 7.7. If in (7.1) and (7.5) instead of β defined by (6.6) we use $\hat{\beta}$ of (6.17), then Theorem 7.3 holds with the following estimates in place of (7.3) and (7.4):

$$(7.41) \quad \|u\|_{H^{1,\frac{1}{2},q}(Q')} \leq C \left(\|u\|_{L^\infty(\partial_p Q)} + \|f\|_{L^p(Q)} \right),$$

$$(7.42) \quad \|Du\|_{C^{1+\alpha}(Q')} \leq C \left(\|u\|_{L^\infty(\partial_p Q)} + \|f\|_{L^p(Q)} \right).$$

The proof is the same as above – in fact the proof is simpler since β defined in (6.17) is renormalization invariant, see Remark 6.4 and 6.5. See Remark 8.2 and Example 8.3 below as well.

Section 8. An Existence Theorem

We present a theorem about the solvability of the Dirichlet problem for equations that are convex or concave in D^2u . It is a consequence of the gradient regularity results and $W^{2,1,p}$ estimates for equations with no gradient dependence. The corresponding result for elliptic equations is presented in [28]. The case of Bellman-Isaacs' equations is discussed in the next section.

We begin this section by recalling the $W^{2,1,p}$ regularity estimate of Wang [30] and Escauriaza [13] for equations with no gradient dependence. The results of [13] were proved for smooth solutions. However, similarly as in [30], given the machinery presented here they can be proved for L^p -viscosity solutions. Another way of demonstrating the result below is to repeat the existence proof (see Theorems 4.1, 4.5 and Remark 4.8 of [10]). In this method we smooth out the equation, produce classical solutions, then use the results of [30] and [13] to get the $W^{2,1,p}$ estimate, and finally pass to the limit to obtain the unique L^p -viscosity and strong solution that satisfies the required estimate. We point out that the result is true if the convexity/concavity assumption is replaced by a more general one concerning interior $W^{2,1,\infty}$ estimates for homogeneous equations with fixed (x, t) (see the formulations of Theorems 5.6, 5.7, and Lemma 5.1 of [30]).

Theorem 8.1. *Let F satisfy (SC) and be convex or concave in D^2u . Let $f \in L^p(Q)$, and let $p > p_0$. Let u be an L^p -viscosity solution of $u_t + F(x, t, D^2u) = f(x, t)$ in Q . Then there exists $\theta = \theta(n, p, \lambda, \Lambda, \gamma)$ such that if for almost all $(y, s) \in Q$*

$$(8.1) \quad \left(\frac{1}{r^{n+2}} \int_{Q_r(y,s)} \beta((y, s), (x, t))^p dx dt \right)^{\frac{1}{p}} \leq \theta \quad \text{for } r \leq \min(r_0, \text{dist}((y, s), \partial_p Q))$$

for some $r_0 > 0$ then $u \in W_{\text{loc}}^{2,1,p}(Q)$ is a strong solution and for every $Q' \subset\subset Q$

$$(8.2) \quad \|u\|_{W^{2,1,p}(Q')} \leq C \left(1 + \|u\|_{L^\infty(Q)} + \|f\|_{L^p(Q)} \right),$$

where $C = C(r_0, n, p, \lambda, \Lambda, \gamma, d(\Omega), T, \text{dist}(Q', \partial_p Q))$.

Remark 8.2. The constant term on the right hand side of (8.2) is not present in the analogous estimates in [30] and [13]; however, the proofs of these works

do not seem valid for small values of $\|u\|_{L^\infty(\Omega)}$ and $\|f\|_{L^p(Q)}$ when β is defined by (6.6). The function β is invariant under the scaling $v(x, t) = u(\sqrt{r}x, rt)/r$; however, other scalings are used in the course of proof, whence the difficulty. It is not clear to us at this point if the form (8.2) is best possible and we do not know if the constant term can be dropped. However, we believe that (8.2) is what results if one keeps the books carefully enough in the arguments of [30] and [13]. We note that in [5] the analogous problem is dealt with by using $\hat{\beta}$ of (6.17) in place of β , and if $\hat{\beta}$ is used, then the constant term can indeed be dropped (see Remark 7.7). However, the example below shows that conditions on $\hat{\beta}$ limit the class of equations covered. A corresponding error also appears in the statement of the gradient regularity theorem in [28] (Theorem 2.1). It is the result of the same scaling problem and the use of estimates of [30] and [13]. The statement is correct if the equation is independent of u and if β is replaced by $\hat{\beta}$ of (6.17). For β used in [28], which is that of (6.6), the right hand sides of estimates (2.2) and (2.4) in Theorem 2.1 there (and hence in Corollary 2.2 and Theorem 3.1) should be

$$C \left(1 + \|u\|_{L^\infty(\partial\Omega)} + \|f\|_{L^p(\Omega)} \right).$$

Finally, the subtleties concerning the relations between L^p -viscosity solutions $u \in W_{\text{loc}}^{2,1,p}(Q)$ and strong solutions needed in Theorem 8.1 are resolved in our Proposition 3.2 and Corollary 3.3.

Example 8.3. The reader may skip this remark. The simple “Bellman” operator

$$G(x, X) = \max \{ 2(-\text{trace}(X) + h(x)), -\text{trace}(X) + h(x) \}$$

“renormalized” as usual to

$$F(x, X) = G(x, X) - G(x, 0)$$

provides an operator F which can also be written

$$(8.3) \quad F(x, X) = g(-\text{trace}(X) + h(x)) - g(h(x)),$$

where $g'(r) = 2$ for $r > 0$ and $g'(r) = 1$ for $r < 0$, $g(0) = 0$. F satisfies (SC) with $1 = \lambda < \Lambda = 2$ and $\omega = 0$, $\gamma = 0$. The function h is at our disposal.

Clearly

$$(8.4) \quad \beta(x, y) = \sup_X \frac{|F(x, X) - F(y, X)|}{1 + \|X\|} \leq |h(x) - h(y)|,$$

while

$$(8.5) \quad \begin{aligned} \hat{\beta}(x, y) &= \sup_X \frac{|F(x, X) - F(y, X)|}{\|X\|} \\ &\geq \lim_{s \downarrow 0} \frac{|g(s + h(x)) - g(h(x)) - (g(s + h(y)) - g(h(y)))|}{s} \\ &= |g'(h(x)) - g'(h(y))| \end{aligned}$$

if $h(x) \neq 0 \neq h(y)$.

In view of (8.4)

$$(8.6) \quad \left(\frac{1}{r^{n+2}} \int_{Q_r(y, s)} \beta(x, y)^p dx dt \right)^{\frac{1}{p}} \leq \theta$$

provided that $V_n |h(x) - h(y)|^p \leq \theta^p$ for $|x - y| \leq r$, where V_n is the volume of the unit ball in R^n ; thus if h is uniformly continuous and $0 < \theta$ there is an $r_0 > 0$ such that (8.6) holds for $r \leq r_0$.

In contrast, let $h(x) = x_1$ (for example) and consider $\hat{\beta}$. For $y_1 > 0$ and $r = 2y_1$ we have $|Q_r(y, 0) \cap \{x_1 < 0\}| \geq \alpha r^{n+2}$ for some positive α . Then in view of (8.5) and $|g'(x_1) - g'(y_1)| = 1$ when $y_1 > 0$, $x_1 < 0$, we have

$$(8.7) \quad \left(\frac{1}{r^{n+2}} \int_{Q_r(y, 0)} \hat{\beta}(x, y)^p dx dt \right)^{\frac{1}{p}} > \alpha^{1/p}.$$

Consequently, we cannot make the left hand side of (8.7) small uniformly in y by choosing r small.

In a similar manner, if h is not required to be continuous then even (8.6) cannot be satisfied in general by choosing r small uniformly in y . Choosing $X = (1/n)I$ for example, $\beta(x, y) \geq |g(-1 + h(x)) - g(h(x)) - (g(-1 + h(y)) - g(h(y)))|/2$. If $h(y) = 0$ for $y_1 > 0$, and $h(x) = 10$ for $x_1 < 0$, then $\beta(x, y) \geq 1/2$ for such x, y and we proceed as above.

We see below that Proposition 4.1 easily overcomes these technical difficulties.

Theorem 8.4. *Let F satisfy (SC) and be convex or concave in D^2u . Let $f \in L^p(Q)$, $p > p_0$ and Ω satisfy a uniform exterior cone condition. Let $\psi \in$*

$C(\partial_p Q)$. Then there exists $\theta = \theta(n, p, \lambda, \Lambda, \gamma)$ such that if (8.1) holds for a.e. $(y, s) \in Q$, then there exists a unique L^p -strong solution $u \in W_{\text{loc}}^{2,1,p}(Q) \cap C(\overline{Q})$ of

$$(8.8) \quad u_t + F(x, t, u, Du, D^2u) = f(x, t) \quad \text{in } Q \quad \text{and} \quad u = \psi \quad \text{on } \partial_p Q.$$

Moreover, for $Q' \subset\subset Q$ there is a constant

$$C = C(r_0, n, p, q, \lambda, \Lambda, \gamma, d(\Omega), T, \text{dist}(Q', \partial_p Q))$$

such that

$$(8.9) \quad \|u\|_{W^{2,1,p}(Q')} \leq C \left(1 + \|u\|_{L^\infty(Q)} + \|\omega(|u|)\|_{L^\infty(Q)} + \|f\|_{L^p(Q)} \right).$$

Remark 8.5. We stated Theorem 8.4 for the equations which are convex/concave in D^2u for the sake of simplicity. The result is true under the more general assumption of interior $W^{2,1,\infty}$ -estimates of the corresponding homogeneous problems.

Proof of Theorem 8.4. According to Theorem 4.5 and Remark 4.8 of [10], there is an L^p -viscosity solution u of (8.8). According to Proposition 4.1, u is an L^p -viscosity solution of $u_t + F(x, t, 0, 0, D^2u) = g(x, t)$, where

$$g(x, t) = F(x, t, u(x, t), Du(x, t), D^2u(x, t)) - F(x, t, 0, 0, D^2u) + f(x, t)$$

satisfies

$$|g(x, t)| \leq \omega(|u(x, t)|) + \gamma|Du(x, t)| + |f(x, t)|,$$

provided that $g \in L_{\text{loc}}^p(Q)$. On the other hand, if θ is chosen according to Theorem 7.3, this is so. Indeed, we may choose $q = p$ in (7.3) in the case $p \leq n + 2$ while $Du \in C(Q)$ if $n + 2 < p$. According to Theorem 8.1 u is then a strong solution provided we further choose θ sufficiently small to satisfy that result as well. Finally, L^p strong solutions of (8.8) are unique L^p -viscosity solutions by Corollary 2.12. The asserted estimate is obtained by iterating the estimates of Theorems 7.3 and 8.1. $\square$

Remark 8.6. We did not discuss the implications of Theorem 7.3 for the case $n + 2 < p$ in Theorem 8.4, for what it provides is weaker than what follows

from the embedding of $W_{\text{loc}}^{2,1,p}(Q)$ into $C_{\text{loc}}^{1+\alpha}(Q)$ for $\alpha = 1 - (n + 2)/p$ in that case.

Section 9. Bellman–Isaacs Equations

In this section we apply the theory to equations of Bellman-Isaacs type. Some of the results presented here are new, some are more or less known but we believe that they have never been spelled out in this context. We will consider the following equation:

$$(9.1) \quad u_t + \sup_{\mu} \inf_{\nu} \left(- \text{trace} \left(A^{\mu,\nu}(x, t) D^2 u \right) + \langle B^{\mu,\nu}(x, t), Du \rangle + c^{\mu,\nu}(x, t)u - f^{\mu,\nu}(x, t) \right) = 0$$

subject to the boundary condition

$$(9.2) \quad u = \psi \quad \text{on } \partial_p Q.$$

Here are our standing assumptions. The indices μ, ν range over countable index sets, the bounded domain Ω satisfies a uniform exterior cone condition while $\psi \in C(\partial_p Q)$. The symmetric matrices $A^{\mu,\nu}$, the vectors $B^{\mu,\nu}$, and the functions $c^{\mu,\nu}$ and $f^{\mu,\nu}$ satisfy

$$(9.3) \quad \lambda I \leq A^{\mu,\nu}(x, t) \leq \Lambda I,$$

$$(9.4) \quad |B^{\mu,\nu}(x, t)| \leq \gamma, \quad 0 \leq c^{\mu,\nu}(x, t) \leq \gamma,$$

$$(9.5) \quad g := \sup_{\mu, \nu} |f_{\mu, \nu}| \in L^p(Q), \quad p > p_0.$$

If the $A^{\mu,\nu}$ are independent of ν then the equation is convex in “ $D^2 u$ ”. If in addition all coefficients are uniformly Hölder continuous it follows from Safonov [27] in the elliptic case and Theorem 1.1 of Wang [31] in the parabolic case that equation (9.1) has classical solutions with Hölder continuous derivatives. For other results on the classical theory of Bellman equations see [21], [23], [14], [19]. We will discuss what is known when we do not have classical solutions.

The first theorem deals with equations of Bellman type and is an almost exact version of an analogous result for linear equations (see [22]).

Theorem 9.1. *Suppose that (9.3), (9.4) and (9.5) hold, and that Ω satisfies a uniform exterior cone condition. Suppose that the matrices $A^{\mu,\nu}$ are independent of ν : $A^{\mu,\nu} = A^\mu$, and moreover*

$$(9.6) \quad A^\mu \text{ is uniformly continuous on } \overline{Q}, \text{ uniformly in } \mu.$$

Then there is a unique L^p -strong solution $u \in W_{\text{loc}}^{2,1,p}(Q) \cap C(\overline{Q})$ of (9.1), (9.2). Moreover, for $Q' \subset\subset Q$

$$(9.7) \quad \|u\|_{W^{2,1,p}(Q')} \leq C \left(\|\psi\|_{L^\infty(\partial_p Q)} + \|g\|_{L^p(Q)} \right),$$

where $C = C(n, p, \gamma, \lambda, \Lambda, d(\Omega), T, \text{dist}(Q', \partial_p Q))$.

Proof of Theorem 9.1. Put

$$G(x, t, r, p, X) = \sup_{\mu} \inf_{\nu} \left(-\text{trace}(A^\mu(x, t)X) + \langle B^{\mu,\nu}(x, t), p \rangle + c^{\mu,\nu}(x, t)r - f^{\mu,\nu}(x, t) \right)$$

and $\hat{G}(x, t, r, p, X) = G(x, t, r, p, X) - G(x, t, 0, 0, 0)$, note that $|G(x, t, 0, 0, 0)| \leq g$ and that the equation can be written $u_t + \hat{G}(x, t, u, Du, D^2u) = -G(x, t, 0, 0, 0)$, to which our theory applies as $\hat{G}$ satisfies (SC). Let u be a solution provided by [10]; it will satisfy $|u| \leq \|\psi\|_{L^\infty(\partial_p Q)} + C\|g\|_{L^p(Q)}$ by the generalized maximum principle. It is, however, not the case that we can apply Theorem 8.4 directly to this equation, even if the $B^{\mu,\nu} = 0$ and the $f^{\mu,\nu}$ are uniformly smooth - see Example 8.3. We may rewrite one more time so that the equation is

$$(9.8) \quad u_t + F(x, t, Du, D^2u) = f(x, t),$$

where

$$F(x, t, p, X) = \sup_{\mu} \inf_{\nu} \left(-\text{trace}(A^\mu(x, t)X) + \langle B^{\mu,\nu}(x, t), p \rangle \right)$$

and

$$f(x, t) = -G(x, t, 0, 0, 0) + F(x, t, Du, D^2u) - \hat{G}(x, t, u, Du, D^2u).$$

In view of the standing assumptions, $|f(x, t)| \leq g(x, t) + \gamma|u(x, t)|$. The result now follows as in the proof of Theorem 8.4 upon noting that $\hat{\beta}((y, s), (x, t))$

for F can be made as small as desired by choosing (x, t) sufficiently close to (y, s) owing to the uniform continuity of the A^μ and recalling the estimates (7.41) and (7.42) of Remark 7.7. $\square$

In Theorems 9.2 and 9.3 below $\bar{\alpha}(n, \lambda, \Lambda)$ is the constant from Proposition 5.4.

Theorem 9.2. *Suppose that (9.3), (9.4) and (9.5) hold, Ω satisfies a uniform exterior cone condition, and that the $A^{\mu, \nu}$ are uniformly continuous on Q , uniformly in μ and ν . Then:*

(i) *Suppose that $p > n + 2$. There is an L^p -viscosity solution u of (9.1)-(9.2) such that $u \in C_{\text{loc}}^{1+\alpha}(Q) \cap C(\overline{Q})$ for every α satisfying $\alpha < \bar{\alpha}(n, \lambda, \Lambda)$ and $\alpha \leq 1 - \frac{n+2}{p}$, and moreover on every $Q' \subset\subset Q$ u satisfies*

$$(9.9) \quad \|Du\|_{C^{1+\alpha}(Q')} \leq C \left(\|\psi\|_{L^\infty(\partial_p Q)} + \|g\|_{L^p(Q)} \right).$$

(ii) *Suppose that $p_0 < p \leq n + 2$ and let p^* be as in (7.2). Then there is an L^p -viscosity solution of (9.1)-(9.2) $u \in H_{\text{loc}}^{1, \frac{1}{2}, q}(Q) \cap C(\overline{Q})$ for all $q \leq p^*$ such that on every $Q' \subset\subset Q$ u satisfies*

$$(9.10) \quad \|u\|_{H^{1, \frac{1}{2}, q}(Q')} \leq C \left(\|\psi\|_{L^\infty(\partial_p Q)} + \|g\|_{L^p(Q)} \right),$$

where $C = C(n, p, \gamma, \lambda, \Lambda, d(\Omega), T, \text{dist}(Q', \partial_p Q))$.

Moreover, every solution of (9.1)-(9.2) satisfies estimates (9.9) or (9.10).

Proof of Theorem 9.2. The proof again reduces (9.1) to (9.8) and then uses smallness of $\hat{\beta}$ and the estimates (7.41) and (7.42) to obtain the claims. $\square$

Theorem 9.3. *Suppose that (9.3), (9.4) and (9.5) hold, and that Ω satisfies a uniform exterior cone condition. Suppose also that $A^{\mu, \nu}$, $B^{\mu, \nu}$, $c^{\mu, \nu}$ and $f^{\mu, \nu}$ are all equicontinuous and equibounded on $\overline{Q}$. Then there exist $\underline{u}, \overline{u} \in C(\overline{Q})$ that are respectively the minimal and the maximal C -viscosity solutions of (9.1) - (9.2). Every C -viscosity solution u of (9.1)-(9.2) is in $C_{\text{loc}}^{1+\alpha}(Q) \cap C(\overline{Q})$ for all $\alpha < \bar{\alpha}(n, \lambda, \Lambda)$. If*

$$|A^{\mu, \nu}(x, t) - A^{\mu, \nu}(y, t)| \leq C|x - y|^\theta, \quad (x, t), (y, t) \in Q$$

for some $\theta > \frac{1-\bar{\alpha}}{2-\bar{\alpha}}$ then the solution is unique.

Proof of Theorem 9.3. The existence of $\underline{u}, \bar{u}$ is proved in [10], Theorem 2.1. The fact that $u \in C_{\text{loc}}^{1+\alpha}(Q)$ for all $\alpha < \bar{\alpha}(n, \lambda, \Lambda)$ follows from Theorem 9.2(a). The proof of uniqueness parallels that of an analogous elliptic result given in [10] whose proof is sketched in Section 1 there. $\square$

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