MATH 4032 : Handout on Binomial Identities

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In the following n is a positive integer, unless otherwise mentioned. An n-set refers to a set of n elements. We use $[n]$ to denote the set $\{1, 2, \ldots, n\}$.

For $1 \leq k \leq n$, $\binom{n}{k} = \frac{n!}{k!(n-k)!}$, is the number of subsets of size k, which are subsets of an n-set.

For a (finite) set S, we use 2^S to denote the set of all subsets of S; this is sometimes referred to as the *powerset* of S. Note: $|2^S| = 2^{|S|}$.

1.

$$\binom{n}{k} = \binom{n}{n-k},$$

since there is a 1-1 correspondence between k-sets and $(n-k)$-sets.

2. For $k \geq 1$,

$$k\binom{n}{k} = n\binom{n-1}{k-1}.$$

LHS: choose a subset of size k and then choose a "leader" for the subset.

RHS: choose a leader first and then the remaining $k-1$ elements to form the k-set.

3.

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

RHS: k-sets can be counted according to whether they contain a "distinguished" element or not; accordingly, one obtains the two terms on the RHS.

4.

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n,$$

is the total number of subsets of an n-set.

5.

$$1 + 2 + \cdots + n = \binom{n+1}{2} = \frac{(n+1)n}{2}.$$

Suppose every pair of people at a gathering of size $n+1$ shakes hands once.

RHS: The number of handshakes is the number of 2-sets (or pairs).

LHS: Imagine the people arriving one at a time to the gathering; each person shakes the hands of all the people *already present* at the gathering.

6.

$$\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1}.$$

Proof 1. LHS: choose a subset and a "leader" for the set.

RHS: First choose an element, who will be the leader, then the rest of the subset (of any size) from the rest of the $n-1$ elements.

Proof 2. Using the identity **(2)** above, we may rewrite:

$$\sum_{k=0}^{n} k\binom{n}{k} = \sum_{k=1}^{n} k\binom{n}{k} = \sum_{k=1}^{n} n\binom{n-1}{k-1} = n\sum_{j=0}^{n-1}\binom{n-1}{j} = n2^{n-1},$$

by the identity **(4)**.

Proof 3. (Probabilistic). Divide both sides by 2^n and interpret probabilistically – choose a subset, uniformly at random, from an n-set and estimate the size of the chosen set:
LHS: By definition, the average size of the random subset equals $\frac{1}{2^n}\sum_{k=0}^{n} k\binom{n}{k}$.
RHS: It makes intuitive sense that the average size should be $n/2$, which is the RHS. Formally, we write as follows:

An equivalent way of choosing a subset uniformly at random is by flipping a *fair* coin for each element and including it in the subset, if and only if the coin flip comes up Heads; the coin tosses are (naturally) all mutually independent. Note that this is equivalent to choosing any of the 2^n subsets with equal probability, since the probability of a subset S of size k equals: $(1/2)^k \times (1/2)^{n-k} = 1/2^n$, since the coin tosses for the elements in S have to come up Heads and the rest of the tosses have to come up Tails, and the tosses are all independent!

Computing the average size in this version of the experiment is "trivial", by the *linearity* property of expectation: let R be the random subset and let $I_j := 1$, if the coin toss for element j came up Heads. Then $E[I_j] = \Pr(j \in R) = 1/2$. And by linearity,

$$E[|R|] = E[\sum_j I_j] = \sum_j E[I_j] = n/2\,.$$

(We typically refer to such 0-1 "random variables" I_j as the indicator r.v.s since they assume the value 1 when an event happens and are otherwise 0; in the present case, the event being that the jth coin flip is a Head.

7.

$$\sum_{k=0}^{n}\binom{n}{k}\binom{n}{n-k} = \binom{2n}{n}.$$

HW exercise!

8. The binomial theorem :

$$(x+y)^n = \sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k}\,.$$

LHS: $(x+y)\cdot(x+y)\cdots(x+y)$, a product n times.
RHS: Each term in the expansion has some number k of x's and the rest $n-k$ of them y's. The number of times such a term appears is $\binom{n}{k}$, since the choices for the x's can be indexed by k-sets correspoinding to the k positions out of the n-fold product, the rest being y's.
Remark 1. Plug in $x = y = 1$ in the binomial theorem to get the identity **(2)**.

Remark 2. Choose $y = 1$, differentiate with respect to x to get:

$$n(1+x)^{n-1} = \frac{d}{dx}\Big(\sum_{k=1}^{n} \binom{n}{k} x^k\Big) = \sum_{k=1}^{n} k\binom{n}{k} x^{k-1}.$$

Now plug in $x = 1$ to get the identity bf (6).

10. *The number of subsets of even size equals the number of subsets of odd size.*

Use $x = 1$ and $y = -1$ in the binomial theorem to get the identity $(1-1)^n = 0$, yielding:

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots \mp \binom{n}{n-1} \pm \binom{n}{n} = 0,$$

or

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots$$

11. Let $n \equiv 0 \pmod 8$. Then compute the number of subsets (of $[n]$) of size $0 \pmod 4$.

Solution. Note that the even-size subsets are of size either $0 \pmod 4$ or $2 \pmod 4$. Suppose the number of sets of the first type is A and of the second type is B. Then by **(10)** above, $\mathbf{A + B = 2^{n-1}}$. So if we can figure out $A - B$, then we can solve for A (and B).

Consider (yes, mind the dangerous word) the binomial expansion of $(1+i)^n$, where $i = \sqrt{-1}$, and write it as : (Real part) + $i\times$ (Imaginary part):

$$(1+i)^n = \Big(\binom{n}{0} + \binom{n}{2}i^2 + \binom{n}{4}i^4 + \cdots\Big) + i\Big(\binom{n}{1} + \binom{n}{3}i^2 + \binom{n}{5}i^4 + \cdots\Big),$$

which simplifies as

$$\Big(\binom{n}{0} - \binom{n}{2} + \binom{n}{4} - \cdots\Big) + i\Big(\binom{n}{1} - \binom{n}{3} + \binom{n}{5} - \cdots\Big).$$

Now the real part above is precisely $A - B$. On the other hand,

$$(1+i)^n = [(1+i)^2]^{n/2} = [2i]^{n/2} = 2^{n/2}(i^2)^{n/4} = 2^{n/2},$$

since n is a multiple of 8 (so $n/4$ is an even integer).

Hence $\mathbf{A - B = 2^{n/2}}$. Solving for A gives $A = \frac{1}{2}[2^{n-1} + 2^{n/2}]$.

12. *The product of k consecutive integers is always divisible by $k!$.*

Proof. Let the given k consecutive integers be written as a product, $(n+k)(n+k-1)\cdots(n+1)$. Note that the RHS below is an integer, since the LHS is:

$$\binom{n+k}{k} = \frac{(n+k)(n+k-1)\cdots(n+1)}{k!}.$$