MATH 4032 (Spring'13) – Supplementary Problems II

Instructor : Prasad Tetali, office: Skiles 234, email: tetali@math.gatech.edu
Office Hours: **Wed. Fri. 1:00-2:00pm, Thurs. 2:00–3:00pm**

For Practice Only: no need to submit

I. (Exercise 19.2 from S. Jukna's book). Let $\mathcal{F}$ be a k-uniform k-regular family, i.e., each set has k points and each point belongs to k sets. Let $k \geq 10$. Show that there exists a 2-coloring of points that leaves no set of $\mathcal{F}$ monochromatic.

II. (Exercise 5.11 from S. Jukna's book). Let $t < n/2$ and let $\mathcal{F}$ be a family of subsets of an n-element set X. Suppose that: (i) each member of $\mathcal{F}$ has size at most t, and (ii) $\mathcal{F}$ is an antichain. Let $\mathcal{F}_t$ be the family of all those t-element subsets of X, which contain at least one member of $\mathcal{F}$. Prove that then $|\mathcal{F}| \leq |\mathcal{F}_t|$.

Hint: Use Proposition 5.7 to extend each member of $\mathcal{F}$ to a unique member in the family $\mathcal{F}_t$. Recall that Proposition 5.7 asserts that for $k \leq (n-1)/2$ there is a matching between level k and level $k+1$ in the Boolean lattice, of size equalling the size of level k.

III. (Exercise 5.12 from S. Jukna's book). Let A be a 0-1 matrix with m 1's. Let s be the maximal number of 1s in a row or column of A, and suppose that A has no square $rtimesr$ all-1-sub-matrix. Use the König-Egerváry theorem to show that we then need at least $m/(sr)$ all-1 (not necessarily square) sub-matrices to cover all 1s in A.

Hint: There are at least m/s independent 1s, and at most r of them can be covered by one all-1 sub-matrix.

IV. (Exercise 6.2 from S. Jukna's book). Take s pairwise disjoint $(k-1)$-element sets $V_1, V_2, \ldots, V_s$ and consider the family

$$\mathcal{F} = \{S : |S| = s \text{ and } |S \cap V_i| = 1 \text{ for all } i = 1, \ldots, s\}.$$

This family has $(k-1)^s$ sets. Show that it has no sunflower with k petals.

V. (Exercise 6.4 from S. Jukna's book). Argue as in the proof of the sunflower lemma to show that any set of more than $2(k-1)^2$ edges either contains a matching of size k or a star of size k. [Recall that a *star* of size k is a set of k edges incident to one vertex.]

VI. (Exercise 7.2 from S. Jukna's book). Let $\mathcal{F}$ be an intersecting family of subsets of an n-element set X. Show that there is an intersecting family $\mathcal{F}' \supseteq \mathcal{F}$ such that $|\mathcal{F}| = 2^{n-1}$.

Hint: Show that for any set A such that neither A nor $\bar{A}$ belongs to $\mathcal{F}$, exactly one of A and $\bar{A}$ can be added to $\mathcal{F}$.

VII. (Exercise 7.4 from S. Jukna's book). The upper bound $\binom{n-1}{k-1}$ given by Erdös-Ko-Rado theorem is achieved by the families of sets containing a fixed element. Show that for $n = 2k$ there are other families achieving this bound.

Hint: Include one set out of every pair of sets formed by a k-element set and its complement.

Reminder. Test 2 on Wednesday, April 3rd, in class. OPEN NOTES, but no textbooks allowed. Please review all material covered in class since Test 1.