MATH 4032 Combinatorial Analysis (Spring'13) – Test 1

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Office Hours: **Wed. Fri. 1:00-2:00pm, Thurs. 2:00–3:00pm**

Name: ________________________________ **Score**: _______ out of 30 + 5 (bonus) in class.

1. (5 pts) Give a combinatorial proof of the identity

$$\binom{r}{r} + \binom{r+1}{r} + \binom{r+2}{r} + \cdots + \binom{n}{r} = \binom{n+1}{r+1}.$$

Hint 1: Think of choosing the largest element in the subset first, then the rest of the elements.
Hint 2: Feel free to ignore Hint 1 and think independently.

Solution.

2. (5 pts) There are 20 chairs in a row in a room and 6 students are to take their final exam in the room. In how may ways can the examiner seat the students making sure that there is at least one empty chair between every consecutive pair of students?

Hint. Visualize the solutions and see what other problem this reminds you of.

Solution.

3. (5+3+2 = 10 pts). For $n \geq 1$, let $s(n)$ denote the number of permutations of a set of n elements having the property that all their cycles have length 1 or 2. For example, $s(1) = 1$, $s(2) = 2$, and $s(3) = 4$.

(i) Write a recurrence relation for $s(n)$ for $n \geq 3$.

(ii) Check your answer for $n = 4$, by enumerating the corresponding permutations for $n = 4$.

(iii) Show that $s(n)$ is even for all $n > 1$.

Solution.

4. (10pts). Recall that a tournament is an oriented complete graph, and that we may think of it as a round robin tournament on the set V of n players with no ties – depending on whether x beats y or the otherway around, there is a directed edge from one to the other. A tournament has the property P_k if for every set of k players, there is someone else who beats them all, i.e., if for any subset $S \subseteq V$ of k players, there exists a player $y \notin S$ such that y beats every $x \in S$.

Prove that if $\binom{n}{k}(1-2^{-k})^{n-k} < 1$, then there exists a tournament on n players that has the property P_k.

(**Bonus : 5 pts**) Determine for what n, as a function of k, the condition above holds.

Solution.

You may bring the solutions to the last two below to class this Wednesday:

5. (10 pts) We say a subset A of a set S of distinct positive integers is *sum-avoiding* if the sum of any distinct pair of integers from A is *not contained in* S. That is, $\forall a, b \in A$, we need $a + b \notin S$. Show that any set of n distinct positive integers contains a sum-avoiding subset of size at least $\log n$.

Hint: No need for any probability; but you may use the bound on the independence number of a graph we saw in class : $\alpha(G) \geq \sum_v \frac{1}{d(v)+1}$ *, where* $d(v)$ *is the degree of a vertex* v*. Also recall that* $\sum_{i=1}^{n} 1/i$ *diverges logarithmically in* n*.*

6. *(10 pts) Recall the Stirling numbers of the first kind* $s(n, k)$ *and the second kind* $S(n, k)$*, and the identities we proved in class, relating these to the polynomials* x^n *and* $x_{(n)}$*. Prove that for* $n \geq m \geq 1$*,*

$$\sum_{k=m}^{n} S(n, k)s(k, m) = I_{\{n=m\}} ,$$

where $I_{\{n=m\}} = 1$ *if* $n = m$*, and is zero otherwise.*