MATH 4032 Combinatorial Analysis (Spring'13) – Homework 1

Instructor : Prasad Tetali, office: Skiles 234, email: tetali@math.gatech.edu
Office Hours: **Wed. Fri. 1:00-2:00pm, Thurs. 2:00–3:00pm**

Due: next Friday in class

1. Let H be a 3-uniform hypergraph on a set V of 9 elements: $\{u_1, u_2, \ldots, u_9\}$.

(i) (*Trivial*). How many Red-Blue colorings are there of V with precisely 5 elements that are Red and 4 elements that are Blue?

(ii) Suppose we choose a *random* coloring *uniformly* from the set of all (5,4) colorings from Part (i) – that is, any of the colorings in Part (i) has equal probability of being chosen. Then what is the probability that a fixed hyperedge (such as $\{u_1, u_4, u_5\}$, say) is monochromatic?

2. Let $\mathcal{F}$ be a family of sets with the property that $A, B \in \mathcal{F}$ implies that $|A \cap B| \neq 1$. Then show that $\mathcal{F}$ is 2-colorable. (That is, show that the underlying points can be colored Red or Blue so that no set in $\mathcal{F}$ is monochromatic.) You may assume that every set has at least two points, to avoid degeneracy.

(*Hint*: No need for probability! Just consider coloring points one by one...)

3. Prove that $R(4, 3) \leq 10$ from first principles, without using any theorem proved in class. (Hint: use an argument similar to the one used to show $R(3, 3) \leq 6$.)

4. Find the RHS of the following identity – it should not have any summations – and find a combinatorial proof of the resulting identity: $\sum_{k=1}^{n} \binom{n}{k}\binom{n}{n-k} =??$

5. Suppose $n \equiv 1 \pmod 8$. Show that the number of subsets of an n-element set, whose size is 0 (mod 4) is $2^{n-2} + 2^{(n-3)/2}$.

6. Let n be any positive integer. Prove that if $a_1, a_2, \ldots, a_n$ are n (not necessarily distinct) integers, then there is a subset of the integers whose sum is divisible by n. (*Hint*: Show that there exist $1 \leq i \leq j \leq n$ such that $\sum_{k=i}^{j} a_k$ is divisible by n.)

Optional problems.

O1. Let $G = (V, E)$ be a graph. Let $d(x)$ denote the degree of x, for each $x \in V$. Explain why the following holds:

$$\sum_{x \in V} d(x)^2 = \sum_{\{x,y\} \in E} (d(x) + d(y)) .$$

O2. Give a combinatorial proof of the identity

$$\binom{r}{r} + \binom{r+1}{r} + \binom{r+2}{r} + \cdots + \binom{n}{r} = \binom{n+1}{r+1} .$$

Unresolved problems.

1. The best known bounds : $\sqrt{2} \leq R(k, k)^{1/k} \leq 4$. Improvements?

2. $R(3, 10) =?$, $R(4, 6) =?$, and $R(5, 5) =?$.