MATH 4032 Combinatorial Analysis (Spring'13) – Homework 2

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Office Hours: **Wed. Fri. 1:00-2:00pm, Thurs. 2:00–3:00pm**

Due: next Wednesday in class

Note: The first two problems are from Chapter 4 of the van Lint-Wilson book; the next two are from Chapter 18 of the book by Jukna.

1. If a simple graph on n vertices has m edges, then it has at least $\frac{m}{3n}(4m - n^2)$ triangles.
(*Hint*: First note that the number of triangles on an edge xy with end-vertices x and y is at least $d(x) + d(y) - n$. Then sum this over all edges xy.)

2. A $3 \times 3 \times 3$ cube of cheese is divided into 27 small cubes of size $1 \times 1 \times 1$. A mouse eats one small cube each day and an adjacent small cube (sharing a face) the next day. Can the mouse eat the *center* small cube on the last day?
(*Hint*: The graph is bipartite.)

3. Prove a lower bound for the general van der Waerden's function $W(r, k)$.
(*Hint*: Modify the proof of the theorem covered in class to the case of more than two colors.)

4. Consider the family of all pairs (A, B) of disjoint k-element subsets of $[n] := \{1, 2, \ldots, n\}$. A set Y *separates* the pair (A, B) if $A \subset Y$ and $B \cap Y = \phi$. Prove that there exist $\ell = 2k4^k \ln n$ sets such that every pair (A, B) is separated by at least one of them.
(*Hint*: Choose ℓ sets uniformly and independently at random from the 2^n possible subsets of $[n]$. Estimate the probability that none of these ℓ sets separates a given pair (A, B), and ...)

5. Find the number of ways to distribute 20 identical soup cans to 4 different families, if the 1st family gets at least one soup can, the 2nd gets at most two, and the third gets at least three.

6. (a) Let h_n denote the number of regions into which a convex polygon region with $n + 2$ sides is divided by its diagonals, assuming no three diagonals have a common point. Define $h_0 = 0$. Show that

$$h_n = h_{n-1} + \binom{n+1}{3} + n, \quad (n \geq 1)\,.$$

(b) Then determine the generating function and from it obtain a formula for h_n.

7. Show that the Ramsey number $R(3, 4)$ is bigger than 8. That is, show that there is a way to 2-color the edges of K_8 so that there is no Red K_3 nor a blue K_4.

Optional problems.
O1. For $0 \leq l \leq k \leq n$, let $T(n, k, l)$ be (the Turán number) the smallest number of l-sets of an n-set X such that every k-set of X contains at least one of these l-sets. Show that $T(n, k, l) \geq \binom{n}{l} / \binom{k}{l}$.
Hint: Explain why the following identity is true; you might want to use it in proving the above:

$$\binom{n}{k}\binom{k}{l} = \binom{n}{l}\binom{n-l}{k-l}.$$

O2. Solve the following recurrence using the generating function approach:

$$h_n = 2h_{n-1} - h_{n-2}, \text{ for } n \geq 1, \text{ and } h_0 = 1, \; h_1 = 2.$$