MATH 4032 Combinatorial Analysis (Spring'13) – Homework 3

Instructor : Prasad Tetali, office: Skiles 234, email: tetali@math.gatech.edu
Office Hours: **Wed. Fri. 1:00-2:00pm, Thurs. 2:00–3:00pm**

Due: Wednesday, March 6, 2013

Note: These are from Chapter 10 of the van Lint-Wilson book.

1. How many positive integers less than 1000 have no factor between 1 and 10?
(*Hint*: Use inclusion-exclusion with four E_i's: with E_i denoting the integers between 1 and 1000 divisible by i, for $i = 2, 3, 5, 7$.)

2. How many monic polynomials in x of degree n are there in $\mathcal{F}_p[x]$ that do not take on the value 0 for $x \in \mathcal{F}_p$? (Recall that $\mathcal{F}_p$ may be thought of as consisting of integers, $0, 1, \ldots, p$, with addition and multiplication modulo p.)
(*Hint*: Use inclusion-exclusion and the fact that for a polynomial $f(x)$, if $f(i) = 0$, then $(x - i)$ is a factor of $f(x)$.)

3. One of the most famous functions in complex analysis is the so-called Riemann ζ-function $\zeta(s) := \sum_{n=1}^{\infty} n^{-s}$, defined in the complex plane with the real part $\Re(s) > 1$. Prove that

$$1/\zeta(s) = \sum_{n=1}^{\infty} \mu(n) n^{-s}\,.$$

(*Hint*: Multiply $\sum_{n=1}^{\infty} a_n n^{-s}$ and $\sum_{m=1}^{\infty} b_m m^{-s}$. Determine the coefficient of k^{-s} and use the theorem about $\sum_{d|n} \mu(d)$.)

4. Count the number of permutations $x_1, x_2, \ldots, x_{2n}$ of the integers 1 to $2n$ such that $x_i + x_{i+1} \neq 2n + 1$ for $i = 1, 2, \ldots, 2n - 1$.
(*Hint*: Use the general theorem on inclusion-exclusion.)

5. Prove that for $0 \leq k \leq n$,

$$\sum_{i=0}^{k} \binom{k}{i} d_{n-i} = \sum_{j=0}^{n-k} (-1)^j \binom{n-k}{j} (n-j)!$$

Recall that d_n denotes the number of derangements on n distinct objects.
(*Hint*: Count permutations of integers $1, 2, \ldots, n$ that fix none of $1, 2, \ldots, n - k$.)

6. Prove the following direction of the Möbius inversion theorem: Suppose f and g satisfy the relation $g(n) = \sum_{d|n} \mu(d) f(n/d)$, then show that $f(n) = \sum_{d|n} g(d)$.