Counting Antichains and Linear Extensions in Generalizations of the Boolean Lattice

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1 Introduction

In 1897, Dedekind [6] posed the problem of estimating the number of antichains in the Boolean lattice; in particular, he asked whether the logarithm of the number is asymptotic to the size of the middle layer of the n-dimensional Boolean lattice $\mathfrak{B}_n$. Although Kleitman confirmed the truth of this conjecture in 1969 [13], enumerating antichains in $\mathfrak{B}_n$ has continued to generate interest in the mathematical and computer science communities ([14], [17], [11], ...), culminating in the works of Korshunov [15] and Sapozhenko [19] who found sharp estimates on the actual number of antichains (rather than producing results at the logarithmic level). Note the order of these results: although asymptotics for the number of antichains was known in 1980 subsequent research provided estimates which were *less* accurate. Although language barriers may have contributed to this progression of results, (some of the seminal papers have not been translated from their original Russian), there are other relevant factors. The proofs of these sharp estimates are very complicated and involve intense case analysis. Counting antichains is a problem well suited to modern entropy-based enumeration techniques, since information about local properties, such as vertex degrees, can be translated into a global property of the poset. Entropy based enumeration proofs are typically beautiful and succinct, with the tradeoff that the results are for the logarithm of the number of objects one is trying to count.

A similarly compelling question was raised by Stanley (see [20]) and others independently: How many linear extensions of the Boolean Lattice can be formed?

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This problem is also well suited to an entropic result, as individual vertex degrees contain much of the information needed to answer this global question. Brightwell-Tetali [5] (improving on an earlier result of Sha-Kleitman [20]) obtained accurate (up to *second order*) asymptotics for the logarithm of the number of linear extensions of $\mathfrak{B}_n$. Not only have these questions benefitted from investigation from the perspective of entropy based enumeration, these compelling problems have motivated the development of new entropy techniques.

In the present work we consider these questions for two natural generalizations of the Boolean lattice: the Cartesian product of n chains with fixed length t denoted by $[t]^n$, and the poset of partially defined functions, $F_{n,k}$; we define these posets precisely in the following section. Each of these generalizations sacrifice some of the 'nice' properties of the Boolean Lattice which are used extensively in the proofs of the results above, in particular symmetry about a central rank and a single parameter which describing the poset. In [2] and [1] Alekseev gives sharp estimates for the number of antichains in $[t]^n$ but, inspired by the elegant entropy approaches for Dedekind's original problem by Pippenger [17] and Kahn [11], our first main contribution is the use of entropy methods to obtain accurate (up to first order) asymptotics of the logarithm of the number of antichains of both these posets. We also provide the first results for the number of linear extensions for these posets by providing accurate first order term asymptotics for the logarithm of these quantities.

We hope our work inspires more research in this direction in ultimately yielding much sharper estimates as well as additional refinements of entropic enumeration techniques. In the next section, we state our results in precise terms and provide some basic information about the posets under consideration.

1.1 Definitions and Statements of Results

Let P be a partially ordered set under the relation $\prec$.

Definition 1.1 *An antichain is a subset of P which is pairwise incomparable. We use the notation $a(P)$ to represent the number of antichains in P.*

Definition 1.2 *A linear extension of P is a total ordering $(T, <)$ on the ground set of P so that if $x \prec y$ in P, then $x < y$ in T. We also say that the ordering T "preserves" the relationship of $\prec$. We let $L(P)$ represent the number of linear extensions of a poset P.*

For further background on posets we refer the reader to [22] and [7]. Next, we define a standard generalization of the Boolean lattice, then state our results for that poset:

Definition 1.3 *For $n, t \in \mathbb{N}$, $t \geq 1$, the chain product poset is the set of all n-tuples, $x = (x_1, x_2, \ldots, x_n)$, where $0 \leq x_i \leq t-1$ for $i = 1, \ldots, n$ together with the relation $\prec$ defined so that for $y = (y_1, y_2, \ldots, y_n)$, $x \prec y$ if and only if $x_i \leq y_i$ for all $1 \leq i \leq n$. We denote this poset with the notation $[t]^n$.*

From this definition we can quickly see that the Boolean lattice $\mathfrak{B}_n$ is isomorphic to $[2]^n$. We note that the number of elements in $[t]^n$ is t^n, further motivating our naming convention. This is a ranked poset, with rank function $\mathrm{rank}(x) = \sum_{i=1}^{n} x_i$. For all values of t, $[t]^n$ is a lattice with unique maximal and minimal elements. However for $t > 2$ the poset is not bi-regular as the up- and down-degrees of a vertex do not depend only on its rank. The up-degree of a vertex is equal to the number of positions where it takes a value less than n, and the down-degree is equal to its number of nonzero entries.

Let $N(t, n)$ be the size of the middle layer of $[t]^n$, i.e. the number of elements with rank $\lfloor n(t-1)/2 \rfloor$ (when the context of both n and t is clear we refer to $N(t, n)$ simply as N). It was recently shown by Mattner and Roos [16] that

$$N(t, n) = t^n \sqrt{\frac{6}{\pi(t^2-1)n}}\,(1 + o(1))\,. \tag{1}$$

Our main technical contribution is the first part of the following theorem. In this theorem, as in the rest of the paper, we use log to represent the binary logarithm $\log_2$, and ln to represent the natural logarithm.

Theorem 1.4 *For integers t, n such that $1 < t < n$:*

$$N(t, n) \leq \log(a([t]^n)) \leq N(t, n)\left(1 + \frac{11t^2 \log t\ (\log n)^{\frac{3}{2}}}{n^{\frac{1}{4}}}\right),$$

$$\log(N(t, n)) - C + o(1) \leq \frac{\log L([t]^n)}{t^n} \leq \log(N(t, n))\,,$$

for a suitable constant $C > 0$.

In the first estimate from the theorem, the lower bound follows from the hereditary nature of antichains. The additional term in the upper bound is $o(1)$, as long as $t = O(n^{1/8-\epsilon})$, for any $\epsilon > 0$. In this regime, we know that almost all antichains are subsets of this middle layer. This restriction on t's growth with respect to n seems likely to be an artifact of our proof technique, and we do not believe this to be the only range of t where a similar result holds. We adapt a proof technique of Pippenger [17] to derive the first upper bound, whereas the second upper bound in the theorem follows from using known general bounds for ranked posets satisfying the so-called LYM property [5].

We also prove estimates, accurate to first order, for the analogous enumeration problems for another generalization of the Boolean lattice:

Definition 1.5 *For $n, k \in \mathbb{N}$ with $k \geq 1$, the poset of partially defined functions is the set of partially defined functions from $\{1, 2, \ldots n\}$ to $\{1, 2, \ldots, k\}$ where $f \preceq g$ if and only if g is an extension of f. We use the notation $F_{n,k}$ to represent this poset.*

We use $\mathcal{D}(f)$ to denote the domain of a function f. Recall that g is an extension of f if and only if $\mathcal{D}(f) \subseteq \mathcal{D}(g)$ and $f(x) = g(x)$, for all $x \in \mathcal{D}(f)$.

Often it is helpful to think of an element in $f \in F_{n,k}$ being represented by n-tuple $(x_1, x_2, \ldots, x_n)$ where $x_i = f(i)$ whenever $f(i)$ is defined and 0 otherwise. In this way it is easy to see both that $|F_{n,k}| = (k+1)^n$ and that for every integer $n \geq 1$, the poset $F_{n,1}$ is isomorphic to the Boolean lattice $\mathfrak{B}_n$.

$F_{n,k}$ is a ranked poset where the rank of each element is determined by the number of elements in the domain where it is defined. Letting P_i denote the set of elements of rank i, we see that $|P_i| = k^i\binom{n}{i}$. This expression is derived from choosing which i positions are defined and then assigning one of k possible values for each. Maximizing in terms of i, we can see that the mode occurs at the $\left\lfloor \frac{kn}{k+1} \right\rfloor^{th}$ level set. When n and k are both clear from context, we refer to this quantity as $i_{\max}$. Thus we have

$$|P_{i_{\max}}| = k^{i_{\max}} \binom{n}{i_{\max}} = k^{\left\lfloor \frac{kn}{k+1} \right\rfloor} \binom{n}{\left\lfloor \frac{kn}{k+1} \right\rfloor}. \tag{2}$$

Similar to $\mathfrak{B}_n$, $F_{n,k}$ is a graded poset: there is a unique minimum element, and all maximal elements have the same rank. Additionally, to satisfy the definition of a graded poset, the up- and down-degrees of an element must be determined by the level set it belongs to. If f is a member of level i it is defined in exactly i coordinates, with the remaining $n-i$ coordinates left undefined. The down-degree d_i of level i is equal to i for each level. To select a downward neighbor of f, we just need to pick any of the i coordinates where it is undefined and replace the value there with the 'undefined' character. The up-degree of level i for $i = 0, \ldots, n-1$ is $u_i = k(n-i)$: to find an upward neighbor of f, we first choose one of the $n-i$ coordinates which is undefined in f and assign it one of the k allowed values. Before we state our result about the number of linear extensions of $F_{n,k}$, it is helpful to first define a quantity which counts a certain kind of linear extension. Let

$$\mathcal{L}_{n,k} = \frac{1}{(k+1)^n} \log\left(\prod_{i=0}^{n} \left(\binom{n}{i} k^i \right)! \right).$$

Theorem 1.6 *For positive integers k and n, with $i_{\max}$ and $P_{i_{\max}}$ defined as above, we have:*

$$|P_{i_{\max}}| \ \leq \log(a(F_{n,k})) \leq \ |P_{i_{\max}}|\Big(1+O\Big(\frac{\log n}{i_{\max}}\Big)\Big)\,,$$

$$\mathcal{L}_{n,k} \ \leq \ \frac{\log(L(F_{n,k}))}{|F_{n,k}|} \leq \mathcal{L}_{n,k} + O\Big(\frac{\log n}{n}\Big)\,.$$

Once again the lower bounds are straightforward. The first upper bound adapts a proof idea of Kahn [11]. The second upper bound is a straightforward corollary of a general upper bound on the number of linear extensions of any ranked poset with degree regularity, proved by Brightwell-Tetali [5].

It is worth noting that these results give us bounds for the number of antichains and linear extensions for another family of posets.

Definition 1.7 *The cubical poset of order n is formed by taking the set of lower dimensional faces of $\mathfrak{B}_n$ (not including the empty set $\emptyset$) ordered by inclusion. We use the notation Q_n to denote this poset.*

We can think of Q_n as the set of n-tuples taking values in 0,1,2 ordered by $\vec{b} \leq \vec{c}$ if and only if $c_i = 2$ or $b_i = c_i$ for all $i \in [n]$. Given a set $I \in [n]$, for $i \in I$ we fix values $a_i = \alpha_i \in \{0,1\}$. Letting a_i range over 0 and 1 for all $i \notin I$, determines a face of $\mathfrak{B}_n$. $F_{n,2}$ is isomorphic to the dual of the cubic poset, Q_n. Since both the number of antichains and number of linear extensions are preserved by taking the dual of a poset, letting $k = 2$ in 1.6 we find bounds for $a(Q_n)$ and $L(Q_n)$.

2 Dedekind's Problem on Generalized Boolean Lattices

2.1 Number of Antichains in $[t]^n$

An easy lower bound for the number of antichains contained in $[t]^n$, utilizes the quantity $N = N(t,n)$ from equation (1), as each subset of the middle layer is in itself an antichain. We show, using an information theoretic technique, that 2^N asymptotically approximates the number of antichains in $[t]^n$. More precisely,

Theorem 2.1 *Let $a([t]^n)$ be the number of antichains in $[t]^n$, then for $n \geq 4$, $t = o(n^\epsilon)$ with $0 < \epsilon \leq \frac{1}{8}$, we can say that:*

$$N \leq \log(a([t]^n)) \leq N\left(1 + \frac{11t^2 \log t \ (\log n)^{\frac{3}{2}}}{n^{\frac{1}{4}}}\right).$$

The $t = o(n^\epsilon)$, $0 < \epsilon < \frac{1}{8}$ hypothesis is necessary to ensure that this theorem gives matching first order terms at the logarithmic level. In the proof itself we use a weaker hypothesis on the relationship of t and n– the theorem remains true when $t = \omega(n^{\frac{1}{8}})$, however in this case the argument gives a weaker result.

We follow closely a method used by Pippenger [17], which he used to show a similar result for the number of antichains contained in the Boolean lattice:

Theorem 2.2 *Let $a(\mathfrak{B}_n)$ be the number of antichains in the Boolean lattice. Then:*

$$\binom{n}{\lfloor \frac{n}{2} \rfloor} \le \log(a(\mathfrak{B}_n)) \le \binom{n}{\lfloor \frac{n}{2} \rfloor} \left(1 + \frac{(\log n)^{\frac{3}{4}}}{n^{\frac{1}{4}}}\right). \tag{3}$$

We know that Pippenger's result does not give the best known asymptotics for the number of antichains in the Boolean Lattice. In particular, the second order term is far from what we know to be the truth in this case, which was discovered by Korshunov [15], who developed asymptotics for the function $a(\mathfrak{B}_n)$ directly. The detailed case analysis involved in his argument is very precise, and difficult to reproduce in a ranked poset which is not bi-regular. A result of this strength is seemingly out of reach with current entropy techniques. We intend Theorem 2.1 as a preliminary result, to establish that N gives the correct first order term at the logarithmic level for the number of antichains in $[t]^n$.

For our information theoretic approach, we use two functions extensively, $H(X)$, the entropy of a random variable (see [3] for information on entropy), and $h_1(p)$, the truncated binary entropy function. We define

$$h_1(q) = \begin{cases} -q \log q - (1-q)\log(1-q) & \text{if } 0 \le q \le \frac{1}{2} \\ 1 & \text{if } \frac{1}{2} \le q \le 1. \end{cases} \tag{4}$$

A function f defined on a poset $(P, \preceq)$ is monotone if $x \preceq y$ implies that $f(x) \le f(y)$. We note that there is a one-to-one correspondence between antichains and monotone Boolean functions in a ranked poset. We can represent an antichain, $\mathcal{A}$, with a Boolean function g, by taking $g(a) = 1$ for all $a \in \mathcal{A}$, and $g(b) = 0$ otherwise. From each such function g, we can form the associated monotone Boolean function f, by "closing g upwards," i.e. by setting $f(z) = 1$ if $x \preceq z$ for some x for which $g(x) = 1$. Similarly, given a monotone Boolean function f, note that the set of minimal elements x, for which $f(x) = 1$, defines an antichain. This correspondence allows us to count antichains by counting monotone (Boolean) functions in $[t]^n$.

We use the following lemma from [17] in the course of our argument.

Lemma 2.3 *Suppose the random variable K takes values in $\{0, 1, \ldots, n\}$, and for some $k \geq 1$ and $0 \leq q \leq 1$,*

$$\mathbb{P}(K \geq k) \leq q,$$

Then $H(K) \leq h_1(q) + \log k + q \log n$.

Note that $[t]^n$ is a Sperner poset, so the largest antichain in $[t]^n$ is a level set. This follows from the fact that it is the product of chains, which are Sperner. Together with Dilworth's Theorem, which states that there is a chain partition whose size is equal to the size of the largest antichain, we are guaranteed that there is a chain partition of $[t]^n$ whose size is exactly N. (See [7] for additional details on Sperner posets and chain partitions). We fix one such chain partition for the remainder of our argument. We enumerate the elements of this chain partition with $\{C_1, C_2, \ldots, C_N\}$. For a fixed monotone function, g, there is a unique point in each chain where the values of the function change from zero to one. For each chain C_j in the partition, we can define a parameter to capture this information,

$$\gamma_j(g) = \#\{x \in C_j : g(x) = 1\}.$$

Our strategy for counting the number of monotone functions hinges on this property. Let f be a monotone Boolean function, chosen uniformly from the set of all monotone Boolean functions. We know from a basic property of entropy that

$$H(f) = \log(a([t]^n)).$$

We define a pair of variables $(\hat{\delta}, \tilde{\delta})$, which in turn determines f. This allows us to use the subadditivity of entropy again to give an upper bound on $H(f)$:

$$H(f) \leq H(\hat{\delta}, \tilde{\delta}) \leq H(\hat{\delta}) + H(\tilde{\delta}). \tag{5}$$

To define $\tilde{\delta}$ we first need another description. For each point $x \in [t]^n$, and $\ell = 0, \ldots, t-1$, define $d_\ell(x)$ to be the number of coordinates of x which take value ℓ. Let $d_{\text{down}}(x)$ be the down-degree of x, the number of neighbors of x on the immediately preceding level. We can think of $d_\ell(x)$ as the ℓ^{th} down-degree of x, as $\sum_{\ell=1}^{t-1} d_\ell(x) = d_{\text{down}}(x)$. We call a point in $[t]^n$ *low* if $d_j(x) < \frac{n}{2t}$ for some $1 \leq j \leq t-1$. The term "low" is a vestigial artifact of Pippenger's proof, where he called points low if $d_1(x) \leq \frac{n}{4}$, i.e., if they occur on the lowest $\frac{n}{4}$ levels. In the context that we use the word low, it is important to note that it is possible to have two points in the same level set where one is low and the other is not, so low is not purely rank dependent. *This is a new ingredient of our proof.* We call a chain low if it contains a low point.

We take f and selectively "forget" information from some of the chains in the chain partition. To be more precise, let $v_1, v_2, \ldots, v_N$ be independent random variables assigned to each chain in the chain partition. Let $p = \frac{(\log n)^{\frac{1}{2}}}{n^{\frac{1}{4}}}$. With each chain C_j, we associate a variable as follows:

$$v_j = \begin{cases} 1 \text{ if } C_j \text{ is low} \\ 1 \text{ with probability } p \text{ for } C_j \text{ not low} \\ 0 \text{ with probability } 1-p \text{ for } C_j \text{ not low} \end{cases} \tag{6}$$

From f we form the function $\tilde{\delta} = (\tilde{\delta}_1, \tilde{\delta}_2, \ldots, \tilde{\delta}_N)$, by taking $\delta_j = \gamma_j(f)v_j$. Now $\tilde{\delta}$ gives us enough information to reconstruct f on all low chains and on any chain C_j, with $v_j = 1$. Let $\tilde{f}$ be the smallest monotone function which is consistent with $\tilde{\delta}$, i.e., the pointwise-least function so that $\gamma_j(\tilde{f}) \geq \tilde{\delta}_j$, for all $1 \leq j \leq N$. We record the missing information about f in a variable $\hat{\delta} = (\hat{\delta}_1, \hat{\delta}_2, \ldots, \hat{\delta}_N)$ where $\hat{\delta}_j = \gamma_j(f) - \gamma_j(\tilde{f})$. From these definitions, it is easy to see that f is determined by $\delta = (\tilde{\delta}, \hat{\delta})$; indeed, $\tilde{\delta}$ contains the information about g on all chains which are not forgotten, and we can reclaim information about the rest of the chains using the information from $\hat{\delta}$.

We now want to bound $H(\tilde{\delta})$. First we use the subadditivity of entropy to say that:

$$H(\tilde{\delta}) \leq \sum_{j=1}^{N} H(\tilde{\delta}_j). \tag{7}$$

For each fixed j, we now bound $H(\tilde{\delta}_j)$ using Lemma 2.3. Observe that $\tilde{\delta}_j \geq 1$ only if $v_j = 1$. We can use $\mathbb{P}(v_j = 1)$ in place of q in the lemma, so that $\mathbb{P}(\tilde{\delta}_j \geq 1) \leq \mathbb{P}(v_j = 1)$ implies that $H(\tilde{\delta}_j) \leq h_1(q) + q \log n$. If C_j is low then $v_j = 1$ with probability 1. Then $\tilde{\delta}_j = \gamma_j$, a random variable taking either value 0 or a value in $\{1, \ldots, n\}$, since this records how many ones are in our low chain. Therefore if C_j is low, $H(\tilde{\delta}_j) \leq 1 + \log(n)$. If C_j is not low, we can use Lemma 2.3 with $q = p$. Letting M be the number of low chains, we can separate the terms of the sum as follows:

$$H(\tilde{\delta}) \leq M(1 + \log n) + (N - M)(h_1(p) + p \log n). \tag{8}$$

To proceed, we need to give a bound on M, the number of low chains. We can bound this from above by the number of low points. In order for a point x to be low, we need there to be some j for which $d_j(x) \leq \frac{n}{2t}$. We can think of each coordinate x_i, $i = 1, \ldots, n$, as a uniform random variable chosen from $\{0, 1, \ldots, t-1\}$. Then $\mathbb{P}(x_i = j) = \frac{1}{t}$, so the expected value for $d_j(x) = \frac{n}{t}$. Using a version of Chernoff's inequality from [10], we see that

$$\mathbb{P}\left(d_j(x) \leq \frac{n}{2t}\right) \leq \exp\left(-\frac{(\frac{n}{2t})^2}{2\frac{n}{t}}\right).$$

Since this is for a single value of j, we multiply by both t and the total number of points, to see that there are at most $t^{n+1}\exp(-n/8t)$ low points.

Plugging this estimate and the value for p back into (8) and recalling the value of N from (1), we see that:

$$\begin{aligned} H(\tilde{\delta}) &\leq t^{n+1}e^{-\frac{n}{8t}}(1+\log n) + N\left(\frac{3(\log n)^{\frac{3}{2}}}{n^{\frac{1}{4}}}\right) \\ &\leq N\left(\frac{4(\log n)^{\frac{3}{2}}}{n^{\frac{1}{4}}}\right). \end{aligned} \tag{9}$$

In the first inequality above, we use that for p small, $h_1(p) \leq 2p\log(1/p)$. In the second inequality we use the fact that $t = o(n^{\epsilon})$ (though, for this calculation, it suffices that $t \leq n/(16\log n)$).

Now we need to bound $H(\hat{\delta})$. Working with $\hat{\delta}$, we want to explore the possible discrepancy between f and $\tilde{f}$. We call a chain C_j bad if $\hat{\delta}_j \geq 2$. Again, the first property we use is the subadditivity of entropy to say that:

$$H(\hat{\delta}) \leq \sum_{i=1}^{N} H(\hat{\delta}_i).$$

We can again apply Lemma 2.3, now allowing $k = 2$ and setting $q_j = \mathbb{P}(\hat{\delta}_j \geq 2)$. Let $Q = \sum_{j=1}^{N} q_j$, noting that this is the expected number of bad chains. Then we can continue:

$$\begin{aligned} H(\hat{\delta}) &\leq \sum_{j=1}^{N}(h_1(q_j) + 1 + q_j\log n) \\ &\leq Nh_1\left(\frac{Q}{N}\right) + N + Q\log n \\ &= N\left(1 + h_1\left(\frac{Q}{N}\right) + \frac{Q}{N}\log n\right). \end{aligned} \tag{10}$$

where we use the concavity of entropy and Jensen's Inequality in the second inequality.

We proceed by looking at points which cause chains to be bad, and then observing again that the total number of bad chains is less than the number of these troublesome points. We break the points which contribute to γ_j into two cases and then count those points to find a bound on Q. A point x in $[t]^n$ is called bad if (i) x is not low, (ii) the chain C_j which contains x also contains some y so that y is a neighbor of x on the immediately preceding level with $f(y) = 1$, and (iii) $\tilde{f}(x) = 0$.

Each one of x's immediately preceding neighbors arises by decreasing one of the nonzero coordinates of x by one. If y differs from x in a coordinate where the value of x is k, we refer to y as a k-neighbor of x. We classify bad points into two groups, points which are bad because they have many k-neighbors for some $k \geq 1$ which violate condition (ii), and points which have relatively few k-neighbors violating condition (i i) for all $1 \leq k \leq t-1$. To formalize this, let $s = n^{\frac{1}{4}}(\log n)^{\frac{1}{2}}$. We call a point x heavy if for any k, x has more than s k-neighbors. In order for a heavy x to be bad, we need each of the (at least s) chains containing k-neighbors y w ith $f(y) = 1$ to be assigned a value of 0 in γ. This happens with probability $(1-p)$ for each chain. Using $p = \frac{(\log n)^{\frac{1}{2}}}{n^{\frac{1}{4}}}$ allows the following calculation:

$$\mathbb{P}(x \text{ is heavy and bad}) \leq (t-1)(1-p)^s \leq (t-1)e^{-ps} \leq \frac{t}{n}.$$

The factor of $(t-1)$ appears because there are $t-1$ different k values for which x can be heavy. If x is not heavy, it means that for every k, the number of k-neighbors of x is less than s. We apply the group Sym(n), the group of all permutations on $[n]$. The subgroup Stab(x) of permutations which fix x acts transitively on each collection of k-neighbors. Let y be a k-neighbor of x so that $f(y) = 1$. We can average over the whole orbit of y, since x is only bad if the chain containing it also contains y. Then the probability that x is bad but not heavy is bounded by

$$\frac{s}{\frac{n}{2t}} = \frac{2ts}{n} = \frac{2t(\log n)^{\frac{1}{2}}}{n^{\frac{3}{4}}}.$$

Combining these two estimates, we see that for $n \geq 4$,

$$\mathbb{P}(x \text{ is bad}) \leq \max\left(\frac{t}{n}, \frac{2t(\log n)^{\frac{1}{2}}}{n^{\frac{3}{4}}}\right) = \frac{2t(\log n)^{\frac{1}{2}}}{n^{\frac{3}{4}}}.$$

Therefore $Q = \mathbb{E}(\text{bad points}) \leq t^n \frac{2t(\log n)^{\frac{1}{2}}}{n^{\frac{3}{4}}}$. We use bounds for Q and N in (10) to see that:

$$\begin{aligned}
H(\hat{\delta}) &\leq N\left(1 + h_1\left(\frac{Q}{N}\right) + \frac{Q}{N}\log n\right) \\
&\leq N + \left(\sqrt{\frac{\pi}{24}}\right)\frac{t^2(\log n)^{\frac{3}{2}}}{n^{\frac{1}{4}}} + \left(2\sqrt{\frac{\pi}{6}}\right)\frac{t^2(\log n)^{\frac{3}{2}}}{n^{\frac{1}{4}}} \qquad (11) \\
&\leq N + \frac{7t^2(\log n)^{\frac{3}{2}}\log t}{n^{\frac{1}{4}}}.
\end{aligned}$$

Now we are ready to give estimates for both parts of (5), to finally conclude that:

$$\begin{aligned} H(f) &\leq H(\tilde{\delta}) + H(\hat{\delta}) \\ &\leq N + \frac{7t^2(\log n)^{\frac{3}{2}}\log t}{n^{\frac{1}{4}}} + N\left(\frac{4(\log n)^{\frac{3}{2}}}{n^{\frac{1}{4}}}\right) \\ &\leq N\left(1 + \frac{11t^2(\log n)^{\frac{3}{2}}\log t}{n^{\frac{1}{4}}}\right), \end{aligned} \tag{12}$$

as claimed.

2.2 Number of Antichains in $F_{n,k}$

Sharp estimates for the number of antichains in $F_{n,k}$ for $5 \leq k \leq 11$ were given by Andreeva [4]. In this section, our goal is to give an estimate for $\log(a(F_{n,k}))$ which has no dependence on k. The following theorem of Kahn ([11]) plays a key role:

Theorem 2.4 (Kahn) *Let P be a graded poset with levels $P_1, P_2, \ldots, P_m$, with $|P_m| \leq M$. Assume that there exists an $s \in \mathbb{N}$ with $s \geq d_{up}(v)$ for all $v \in P_1, P_2, \ldots, P_{m-1}$ and $s \leq d_{down}(v)$ for all $v \in P_2, \ldots, P_m$, then*

$$a(P) \leq (m2^s - (m-1))^{\frac{M}{s}}. \tag{13}$$

We note that Kahn proved this theorem so he could bound the number of antichains in the Boolean Lattice $\mathcal{B}_n \cong F_{n,1}$. However, the theorem cannot be applied directly, as neither $F_{n,k}$ nor $\mathcal{B}_n$ satisfy all of the hypotheses of the theorem. Kahn proves a technical lemma allowing him to apply his theorem indirectly to $\mathcal{B}_n$; we provide a construction and similar technical lemma which allows us to extend his result to $F_{n,k}$.

The i-th level set in $F_{n,k}$ has size $|P_i| = \binom{n}{i}k^i$, for $i = 0, \ldots, n$. Recall $i_{\max}$ is the index of the largest level set in $F_{n,k}$.

This leads directly to the lower bound in the following theorem, as any subset of the largest level set is itself an antichain.

Theorem 2.5 $2^{|P_{i_{\max}}|} \leq a(F_{n,k}) \leq \left(n2^{i_{\max}} - (i_{\max} - 1)\right)^{\frac{|P_{i_{\max}}|}{i_{\max}}}$.

Calculating the quantity $\dfrac{\log(a(F_{n,k}))}{|P_{i_{\max}}|}$ gives us a way of seeing how close the bounds are. As we can see, this result gives matching first order terms at the logarithmic level.

Corollary 2.6
$$1 \leq \frac{\log(a(F_{n,k}))}{|P_{i_{\max}}|} \leq 1 + O\left(\frac{\log(n)}{i_{\max}}\right).$$

$F_{n,k}$ does not satisfy the hypotheses for Kahn's theorem, as its rank-sequence is unimodal, and there is no uniform bound on degrees which applies to all levels. However, if we truncate $F_{n,k}$ at level $i_{\max}$, the mode of the rank sequence, the truncated poset has a strictly increasing rank sequence and it satisfies that $d_{\text{up}}(x) \geq i_{\max}$ for all $x \in P_1 \cup P_2 \ldots \cup P_{i_{\max}-1}$ and $d_{\text{down}}(x) \leq i_{\max}$ for all $x \in P_2 \cup P_3 \ldots \cup P_{i_{\max}}$. Note that for $x \in P_i$, $d_{\text{up}}(x) = k(n-i)$ and $d_{\text{down}}(x) = i$; so $i_{\max}$ appropriately plays the role of s in Theorem 2.4. Since we can use Kahn's theorem to count the number of antichains in the truncated poset, we seek a way to relate that number to the number of antichains in all of $F_{n,k}$.

Definition 2.7 *A poset Q is a relaxation of a poset P, if P and Q have the same groundset and $y \lessdot x \in Q \Rightarrow y \lessdot x \in P$.*

It is a simple consequence of the definition of relaxation that if Q is a relaxation of P, then $a(Q) \geq a(P)$. The Hasse diagram of P may contain more edges than that of Q, but adding edges only decreases the number of possible antichains. In order to establish the upper bound in Theorem 2.5, we seek a poset which satisfies the hypotheses of Kahn's theorem and contains a relaxation of $F_{n,k}$ as a subposet. Kahn has already constructed such a poset for $F_{n,1} \cong \mathcal{B}_n$, and here we prove the following technical lemma which provides an construction when $k \geq 2$:

Lemma 2.8 *Fix n and $k \in \mathbb{N}$ with $k \geq 2$; there exists a graded poset $A_{n,k}$ ranked by $\{0, 1, \ldots, n\}$ which:*

- *Contains a relaxation of $F_{n,k}$;*
- *Satisfies $d_{up}(x) \geq i_{\max}$ for all $x \in A_1 \cup A_2, \ldots \cup A_{n-1}$ and $d_{down}(x) \leq i_{\max}$ for all $x \in A_2 \cup A_3, \ldots \cup A_n$;*
- *Satisfies $|A_i| \leq |P_{i_{\max}}| = \binom{n}{i_{\max}} (k^{i_{\max}})$ for all level sets $\{A_i\}_{i=0}^{n}$.*

Proof. Take the Hasse diagram for $F_{n,k}$, with $k \geq 2$, and consider it for the moment as a graph. Our objective is to manipulate it to form the Hasse diagram of an appropriate poset $A_{n,k}$, and then take the transitive closure of the covering relations represented as edges of this Hasse diagram to form the desired poset. We do not need to make any modifications below level $i_{\max}$, as these vertices already satisfy the degree bounds, and the ranks are increasing up to level $i_{\max}$. We follow three steps to transform the Hasse diagram, and then justify that these steps can indeed be carried out.

1. For $i > i_{\max}$, remove $i - i_{\max}$ down edges from each vertex on level i. These edges can be chosen arbitrarily. This gives us $d_{\text{down}}(y) = i_{\max}$ for every vertex $y \in P_{i_{\max}} \cup \ldots \cup P_n$.

2. For $i > i_{\max}$, add vertices so there are $k^{i_{\max}}\binom{n}{i_{\max}}$ vertices on level i. Let us call the number of vertices needed on level i, j_i, and note that $j_i = k^{i_{\max}}\binom{n}{i_{\max}} - k^i\binom{k}{i}$. This insures that the rank sequence of $A_{n,k}$ is increasing.

3. For $i \geq i_{\max}$, add edges between level i and $i+1$ to ensure that both the up- and down-degrees of every vertex above level $i_{\max}$ are exactly equal to $i_{\max}$, while maintaining that $F_{n,k}$ is contained as a relaxation.

The feasibility of the first two steps is clear. However in the third step, we need to verify that we can add enough edges to satisfy the degree requirements without adding additional edges between vertices in the original $F_{n,k}$ ground set to guarantee that our new graph contain a relaxation of $F_{n,k}$.

After the first step, counting edges by their top endpoint, we see that there are $i_{\max}\left(k^{i+1}\binom{n}{i+1}\right)$ edges remaining between levels i and $i+1$. Since we know vertices in level i require up degree $i_{\max}$, we can see that the number of edges which we need to add between level i and $i+1$ is $i_{\max}\left(k^i\binom{n}{i}\right) - i_{\max}\left(k^{i+1}\binom{n}{i+1}\right)$. We need to be able to add all of these edges between vertices in $F_{n,k}$ on level i and new vertices on level $i+1$ in order to increase the up degree for vertices in level i to at least $i_{\max}$.

Since the down-degree of new vertices on level $i+1$ needs to be $i_{\max}$, we can use all of the available edges from $L = k^i\binom{n}{i} - k^{i+1}\binom{n}{i+1}$ new level $i+1$ vertices to supplement the up degree of original vertices from level i. This leaves $k^{i_{\max}}\binom{n}{i_{\max}} - k^i\binom{n}{i}$ vertices on level $i+1$ which currently have degree 0 (Notice that this is exactly the number of added vertices on level i). At this point, all vertices have the correct degree except these new vertices which remain isolated on level $i+1$ and all of the added vertices on level i which have not been modified in the above procedure (all of them still have up degree 0 at this point).

We can label these isolated vertices in levels i and $i+1$ with labels $1 \ldots j_i$ and $1 \ldots j_{i+1} - L$ respectively. Now we add edges so that every vertex x_l on the bottom gets connected with each vertex on the top y_k for $k \in \{l, l+1, \ldots, l + i_{\max}\}$ where sums are evaluated modulo j_i. Since $i_{\max} \leq j_i$ this process creates a simple graph also known as the circulant bipartite graph with degree $i_{\max}$.

This now determines the complete Hasse diagram for $A_{n,k}$ by giving all of its covering relations. Taking the transitive closure of these relations gives us a poset satisfying the hypotheses of Theorem 2.4. $\square$

The proof of Theorem 2.5 follows by applying Theorem 2.4 on $A_{n,k}$ to give an upper bound for $a(A_{n,k})$, noting that $a(F_{n,k}) \leq a(A_{n,k})$, since we ensured that it contained a relaxation of $F_{n,k}$. We use the values $m = n$, $s = i_{\max}$, $M = k^{i_{\max}}\binom{n}{i_{\max}}$.

3 Counting Linear Extensions

In this section we use general known bounds on (the logarithm of) the number of linear extensions of ranked posets and LYM posets (see the definition below) in estimating the same for the posets $[t]^n$ and $F_{n,k}$. As a result, the propositions below might be viewed as observations with some computations. Tighter estimates, accurate up to the second order terms of the logarithm of the number of linear extension would indeed be interesting to establish.

For the lower bounds, the following is standard.

Proposition 3.1 *Let P be a ranked poset with height K. If we enumerate the levels of P as P_i for $i = 1, \ldots, K$, with associated rank sequence $\{r_i\}_{i=1}^{K}$, a lower bound for $L(P)$ is given by the expression*

$$\prod_{i=1}^{K} r_i! \leq L(P)\,. \tag{14}$$

Proof. The left hand side counts the number of linear extensions which can be formed by ordering each level set individually and then concatenating the orderings of each level, putting all elements of a given level before all of the elements in a higher level.

A non-trivial upper bound for $L(\mathfrak{B}_n)$ was given by Sha and Kleitman [20]:

Theorem 3.2 *(Sha and Kleitman) If r_i is the rank function for $\mathfrak{B}_n$,*

$$L(\mathfrak{B}_n) \leq \prod_{i=1}^{K} (r_i)^{r_i}\,.$$

This result was extended in various ways by Shastri [21], Kahn-Kim [12] (see the discussion in the last section of [5]), and the version we need is as follows. First we recall a definition.

Definition 3.3 *For an element x in a ranked poset P as above, the weight of $x \in P_i$ is $1/r_i$. An LYM poset is a ranked poset which satisfies that for every antichain A, the sum of all of the weights of elements in A is at most 1.*

Theorem 3.4 *(Brightwell and Tetali) For P be an LYM poset, with the rank function $\{r_i\}_{i=1}^{K}$, we have*

$$L(P) \leq \prod_{i=1}^{K} (r_i)^{r_i}\,.$$

It follows from results about products of posets in [9] and [8] that $[t]^n$ is an LYM poset. In the next section, we use this result to get an upper bound for the number of its linear extensions.

Although $F_{n,k}$ is also an LYM poset, to derive an upper bound on the number of linear extensions of $F_{n,k}$, we instead use a different theorem of Brightwell-Tetali also proven in [5]. In [5], the main focus was on deriving a tight bound (correct up to a second order term) on the logarithm of the number of linear extensions of the Boolean lattice. In addition to this, the authors gave a general upper bound for the number of linear extensions of posets with degree-regularity: Let u_j be the up degree for all vertices on level j, and similarly let d_j be the down-degree of all vertices on level j. Letting r be the following harmonic average of the down-degrees:

$$r = \sum_{j=2}^{K} \frac{r_{j-1}}{d_j}\,, \tag{15}$$

they showed:

Theorem 3.5 *(Brightwell and Tetali) For a regular poset, P with $|P| = N$ and height $K > 0$, as above:*

$$L(P) \leq \prod_{i=1}^{K} r_i! \left(\frac{2e(K-1)N}{r}\right)^r . \tag{16}$$

3.1 Linear Extensions of $[t]^n$

The purpose of the following proposition is to motivate a natural question, that of precisely identifying the second term in the rate function of the number of linear extensions of $[t]^n$.

Proposition 3.6 *There exists a constant $C > 0$ such that*

$$\log(N(t,n)) - C + o(1) \leq \frac{\log\Big(L([t]^n)\Big)}{t^n} \leq \log(N(t,n))\,,$$

where $N(t,n)$ denotes the largest rank (the size of the middle level of $[t]^n$), and is given in (1) *.*

Proof. The upper bound follows simply by observing that,

$$\log\Big(\prod_{j=0}^{n(t-1)} r_j^{r_j}\Big) = \sum_j r_j \log r_j \leq [\log N(t,n)](\sum_j r_j) = t^n[\log N(t,n)]\,.$$

To prove the lower bound, we use convexity and standard estimates. By Proposition 3.1, a lower bound for the number of extensions of any ranked poset is the product of the factorials of its rank sequence. Let $r_0, \ldots, r_{n(t-1)}$ denote the rank sequence of the poset $[t]^n$, i.e., $r_j = |P_j|$. We claim that, for any sequence of positive integers $b_1, \ldots, b_s$,

$$\prod_{j=1}^{s} b_j! \geq \left(\left(s^{-1}\sum_{j=1}^{s} b_j\right)!\right)^{s}, \tag{17}$$

where $x! = \Gamma(x+1)$ for $x \in \mathbb{R}$. To see this, take the natural log of both sides:

$$\sum_{j=1}^{s} \ln(b_j!) \geq s \ln\left(s^{-1}\sum_{j=1}^{s} b_j\right)!$$

This equation clearly holds, by Jensen's inequality, once we show that $\ln x!$ is a convex function of x. The second derivative of $\ln \Gamma(x)$ is given by the *trigamma function*, with well-known representation:

$$\frac{d^2}{dx^2} \ln \Gamma(x) = \sum_{j=0}^{\infty} \frac{1}{(j+x)^2},$$

a quantity which is clearly positive for $x > 0$. Therefore, we may conclude that (17) holds.

To simplify our calculations, we note that $r_0 = 1$ and that we can extend our rank sequence by defining $r_{n(t-1)+1}, \ldots, r_{tn} = 0$, allowing us to apply our bound for $r_1, \ldots, r_{tn}$, yielding

$$L([t]^n) \geq \prod_{j=1}^{n(t-1)} r_j! \geq \left(\left(\frac{1}{tn}\sum_{j=1}^{nt} r_j\right)!\right)^{tn} = \left(\left(\frac{t^{n-1}}{n}\right)!\right)^{tn} \geq \left(\frac{t^{n-1}}{ne}\right)^{t^n},$$

with the final calculation following from Stirling's formula, and the penultimate calculation using a Stirling-type estimate of the form $\Gamma(x+1) \geq (x/e)^x$. This establishes the lower bound in Proposition 3.6, in view of (1).

One may also observe that the bounds provided by Proposition 3.1 and Theorem 3.4 are in general tight up to a constant at the level of the rate function: by Stirling's approximation,

$$\log\Big(\prod_{j=0}^{n(t-1)} r_j^{r_j}\Big)/t^n = \log\Big(\prod_{j=0}^{n(t-1)} \frac{(r_j!\ e^{r_j})}{\sqrt{2\pi r_j}}(1+o(1))\Big)/t^n = \log\Big(\prod_{j=0}^{n(t-1)} r_j!\Big) + \log e + o(1)\,.$$

3.2 Linear Extensions of $F_{n,k}$

Recall that for $F_{n,k}$, the number of levels in the poset is $K = n + 1$, the rank function is $r_i = \binom{n}{i}k^i$, and $|F_{n,k}| = (k+1)^n$. For ease of notation let

$$\mathcal{L}_{n,k} = \frac{1}{(k+1)^n} \log \left(\prod_{i=0}^{n} \left(\binom{n}{i} k^i \right)! \right).$$

Theorem 3.7

$$\mathcal{L}_{n,k} \leq \frac{\log L(F_{n,k})}{|F_{n,k}|} \leq \mathcal{L}_{n,k} + O\left(\frac{\log n}{n}\right).$$

More specifically,

$$n\log(k+1) - \log(n+1) - \log e \leq \frac{\log(L(F_{n,k}))}{|F_{n,k}|} \leq n\log(k+1) - \log e + o(1).$$

Proof. Using the results from (14) and Theorem 3.5, for a ranked poset P with levels indexed by $1, \ldots, K$ and rank sequence $\{r_i\}_{i=1}^{K}$ we know that:

$$\frac{1}{|P|} \log \prod_{i=1}^{K} r_i! \leq \frac{1}{|P|} \log L(P) \leq \frac{1}{|P|} \log \left(\prod_{i=1}^{K} r_i! \left(\frac{2e(K-1)N}{r} \right)^r \right). \tag{18}$$

The bound in the second half of Theorem 3.7 follows by giving an expansion for $\mathcal{L}_{n,k}$, since this term appears in both the upper and lower bounds.

$$\begin{aligned} \mathcal{L}_{n,k} &= \frac{1}{(k+1)^n} \log \prod_{i=0}^{n} \left(\binom{n}{i} k^i \right)! \\ &= \sum_{i=0}^{n} \frac{\log \left(\sqrt{2\pi \binom{n}{i} k^i} \left(\frac{\binom{n}{i}k^i}{e} \right)^{\binom{n}{i}k^i} (1 + o(1)) \right)}{(k+1)^n} \\ &= \sum_{i=0}^{n} \frac{\binom{n}{i}k^i \log \left(\binom{n}{i} k^i \right)}{(k+1)^n} - \log e + o(1). \end{aligned} \tag{19}$$

To bound the sum in this term, we use standard techniques: Stirling's formula, the derivative of the Binomial Theorem, bounds for binomial coefficients, and Jensen's inequality. We see that

$$-\log\left(\frac{1}{(k+1)^n}(n+1) \right) \leq \sum_{i=0}^{n} \frac{\binom{n}{i}k^i \log\left(\binom{n}{i}k^i\right)}{(k+1)^n} \leq \left[\frac{k \log k}{k+1} + h\left(\frac{k}{k+1} \right) \right] n \tag{20}$$

Here the function $h(x)$ is the binary entropy function.

The result follows once it is established that the error term in the upper bound from the additional term in the Brightwell-Tetali bound (16), namely

$$\frac{\log\left(\left(\frac{2e(K-1)N}{r}\right)^r\right)}{|F_{n,k}|} = \frac{r\log\left(\frac{(k+1)^n}{r}\right) + r\log(2en)}{(k+1)^n}\,, \tag{21}$$

has a smaller order than the linear order of the leading terms; in particular we can see that it has order $O\left(\frac{\log n}{n}\right)$. First, we find the value for the degree parameter r described above.

$$r = \sum_{i=1}^{n} \frac{r_{i-1}}{d_i} = \sum_{j=0}^{n-1} \frac{k^j\binom{n}{j}}{j+1} = \frac{(k+1)^{n+1}-1}{k(n+1)} - \frac{k^n}{n+1} = \frac{(k+1)^{n+1}-k^{n+1}-1}{k(n+1)}. \tag{22}$$

In the simplification of this term, we use a standard equation with closed form given in Page 34 of [18].

Using the value for r from (22) in expression (21), and combining the results from (19), and (20), we establish the bounds:

$$n\log(k+1) - \log(n+1) - \log e \le \frac{\log(L(F_{n,k}))}{|F_{n,k}|} \le n\log(k+1) - \log e + o(1)\,,$$

as desired.

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