EXCLUDING MINORS IN NONPLANAR GRAPHS OF GIRTH AT LEAST FIVE

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ABSTRACT

A graph is *quasi* 4-*connected* if it is simple, 3-connected, has at least five vertices, and for every partition (A, B, C) of $V(G)$ either $|C| \geq 4$, or G has an edge with one end in A and the other end in B, or one of A,B has at most one vertex. We show that any quasi 4-connected nonplanar graph with minimum degree at least three and no cycle of length less than five has a minor isomorphic to P_{10}^{-}, the Petersen graph with one edge deleted. We deduce the following weakening of Tutte's Four Flow Conjecture: every 2-edge connected graph with no minor isomorphic to P_{10}^{-} has a nowhere-zero 4-flow. This extends a result of Kilakos and Shepherd who proved the same for 3-regular graphs.

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1. Introduction

By a well-known result of Tait [17], the Four Color Theorem (4CT) [2, 3, 4, 9] is equivalent to the following.

Theorem 1.1. *Every 2-edge-connected 3-regular planar graph is edge 3-colorable.*

In this paper *graphs* are finite, and may have loops and multiple edges. A graph is a *minor* of another if the first can be obtained from a subgraph of the second by contracting edges. We say that a graph G has an H minor if G has a minor isomorphic to H. The *Petersen graph* (or *Petersen*) is the unique 3-regular graph on ten vertices with no cycle of length less than five. For a drawing of the Petersen graph, see [5] p. 99. Since the Petersen graph is nonplanar and taking minors preserves planarity, the following conjecture of Tutte [18] implies Theorem 1.1.

Conjecture 1.2. *Every 2-edge-connected 3-regular graph with no Petersen minor is edge 3-colorable.*

Kilakos and Shepherd [6] proved that Conjecture 1.2 holds if the Petersen graph is replaced by P_{10}^-, the graph obtained from the Petersen graph by deleting one edge. Note that the Petersen graph is edge transitive and therefore P_{10}^- is well defined. A proof of Conjecture 1.2 itself was announced by Robertson, Sanders, Seymour and Thomas [10, 11, 12, 14, 15], but the proof is long and has not yet been fully written.

In this paper we are concerned with an even stronger conjecture, also due to Tutte, which we now introduce. Let Γ denote the group $\mathbb{Z}_2 \times \mathbb{Z}_2$, and let G be a graph. A Γ-*flow* in G is a function $\phi : E(G) \to \Gamma - \{0\}$ such that for every vertex v of G, $\sum \phi(e) = 0$, where the sum is taken over all edges e incident with v. Let us remark that by a classical result of Tutte a graph has a Γ-flow if and only if it has a "nowhere-zero 4-flow". We omit the definition of nowhere-zero 4-flows, because we do not need them, and instead refer the reader to [16, 19] for more information on nowhere-zero flows in graphs. It follows immediately that if G is a 3-regular graph, then every Γ-flow in G is an edge 3-coloring, and conversely every edge 3-coloring gives rise to a Γ-flow in G by changing the colors to be the nonzero elements of Γ. Thus the following conjecture, due to Tutte [18], is stronger than Conjecture 1.2.

Conjecture 1.3. *Every 2-edge-connected graph with no Petersen minor has a Γ-flow.*

Our main theorem is a weaker form of Conjecture 1.3, an analogue of the result of Kilakos and Shepherd mentioned above, namely,

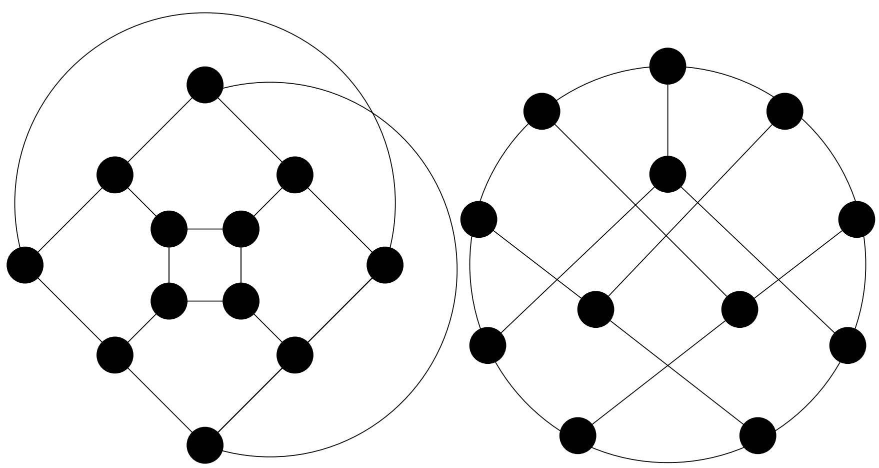
Theorem 1.4. *Every* 2*-edge-connected graph with no* P_{10}^{-} *minor has a* Γ*-flow.*

We prove Theorem 1.4 in Section 4. In fact, Theorem 1.4 follows by standard arguments from the following, which we prove in Section 3.

A *separation* of a graph G is a pair (A, B) of subsets of $V(G)$ such that $A \cup B = V(G)$, and there is no edge between $A - B$ and $B - A$. The *order* of (A, B) is $|A \cap B|$. The separation (A, B) is *nontrivial* if both $A - B$ and $B - A$ are nonempty. A graph G is *quasi* 4*-connected* if it is simple, 3-connected, has at least five vertices, and for every separation (A, B) of G of order three, either $|A| \leq 4$ or $|B| \leq 4$.

Theorem 1.5. *Any quasi* 4*-connected nonplanar graph with minimum degree at least three and no cycle of length less than five has a minor isomorphic to* P_{10}^{-}.

We deduce Theorem 1.5 from the following result, which we prove in Section 2. By *Dodecahedron* we mean the unique 3-regular planar graph with all faces of size five; *Triplex* and *Basket* are defined in Figure 1.1. For a drawing of the Dodecahedron see [5] p. 12.



Basket **Triplex**

FIGURE 1.1

Theorem 1.6. *Every graph with minimum degree at least three and no cycle of length less than five has a minor isomorphic to Triplex, Petersen, Dodecahedron or Basket.*

2. Graphs of girth at least five

We define a cycle to be *short* if it has length less than five. (Paths and cycles have no "repeated" vertices or edges.) Further, we define an edge of a graph G to be *special* if it is in all short cycles of G. Let $\mathcal{F}$ be the set of graphs with minimum degree at least three that have at least one special edge. Note that $\mathcal{F}$ includes all graphs of girth at least five. We say that a graph G is a *minor minimal* graph of $\mathcal{F}$ if G is in $\mathcal{F}$, but every proper minor of G is not in $\mathcal{F}$. Theorem 1.6 follows from the following, which is the main result of this section.

Theorem 2.1. *The minor minimal graphs of $\mathcal{F}$ are Triplex, Petersen, Dodecahedron and Basket.*

Before proving the theorem, we need several lemmas. The *girth* of a graph is the length of the shortest cycle, or infinity if the graph has no cycles. Let G be a graph, and let v be a vertex of G of degree two not incident with a loop. By *suppressing* v we mean contracting one of the edges incident with v; the other edge incident with v will be called the *surviving edge.*

Lemma 2.2. *Let G be a minor minimal graph of $\mathcal{F}$. Then G is connected and the girth of G is at least three.*

Proof. The graph G is clearly connected. To prove the other part suppose for a contradiction that G has a cycle C of length less than three, and let $e \in E(C)$. Let G' be the graph obtained from $G \backslash e$ by deleting the vertex of degree one if there is one, and then suppressing the resulting vertices of degree two. This is well-defined, because G is not the loopless two-vertex graph with three edges. Then $G' \in \mathcal{F}$, because either G' has girth at least five, or the surving edge is special, contrary to the minimality of G. □

Lemma 2.3. *Let G be a minor minimal graph of $\mathcal{F}$, and let u be an end of a special edge. Then u and all its neighbors have degree three in G.*

Proof. Let G and u be as stated, and suppose for a contradiction that the lemma is false. Let e be an edge incident with u such that e is special if u has degree at least four, and otherwise the other end of e has degree at least four. Let G' be obtained from $G \backslash e$ by suppressing the vertex of degree two if it exists; otherwise let $G' = G \backslash e$. Then $G' \in \mathcal{F}$, contrary to the minimality of G. □

The notation $G|A$ is used to mean the restriction of the graph G to the vertex set A.

Lemma 2.4. *Let G be a minor minimal graph of $\mathcal{F}$. Then G is 3-connected.*

Proof. Let G be a minor minimal graph of $\mathcal{F}$, and let e be a special edge in G. Then clearly $|V(G)| \geq 3$. For a contradiction, suppose that there is a nontrivial separation of order at most 2 of G. Let (A, B) be one such separation with $e \in E(G|B)$ and minimal order, and subject to that with $|A|$ minimum. By Lemma 2.2, G is connected and hence $|A \cap B| \geq 1$. If $|A \cap B| = 1$, then let G' be obtained from $G|A$ by adding a loop at the vertex in $|A \cap B|$; otherwise let G' be obtained from $G|A$ by adding an edge joining the two elements of $A \cap B$. Then $G' \in \mathcal{F}$, contrary to the minimality of G. □

Lemma 2.5. *Let $G \in \mathcal{F}$, let e_1, e_2, e_3 be edges of G such that $G \backslash \{e_1, e_2, e_3\}$ has two components G_1 and G_2 such that both G_1, G_2 have at least two vertices and neither is a cycle. Then G is not a minor minimal graph of $\mathcal{F}$.*

Proof. Let G, e_1, e_2, e_3, G_1,and G_2 be as in the lemma. By Lemma 2.4 the edges e_1, e_2, e_3 form a matching. From the symmetry we may assume that G_2 has no short cycles. For $i = 1, 2, 3$ let e_i have ends u_i and v_i with $u_i \in V(G_1)$. If the degree of v_1 in G is greater than three, then the graph obtained from G_2 by joining v_2 and v_3 with an edge is a minor of G and is in $\mathcal{F}$, therefore G is not a minor minimal graph of $\mathcal{F}$. We may therefore assume that each v_i has degree three. Let G' be the graph obtained from $G \backslash (V(G_1) - \{u_1, u_2, u_3\})$ by adding edges e_{12} from u_1 to u_2, e_{23} from u_2 to u_3, and e_{13} from u_1 to u_3. We know that G' is a minor of G since by Menger's Theorem and Lemma 2.4 there are three disjoint paths between $\{u_1, u_2, u_3\}$ and the vertices of a cycle in G_1. Moreover, G' is a proper minor of G, because G_1 is not a cycle. We claim that $G' \in \mathcal{F}$. To prove this claim suppose for a contradiction that no edge of G' is special. Thus G' must have a short cycle not using e_{12}, a short cycle not using e_{23} and a short cycle not using e_{13}. On the other hand, any short cycle in G' must use at least one of the edges e_{12}, e_{23}, or e_{13} since G_2 has no short cycles. We conclude that at least two edges of G have ends in $\{v_1, v_2, v_3\}$, say v_1v_2 and v_1v_3. Now $\{v_2, v_3\}$ is a vertex cut in G, contrary to Lemma 2.4. This proves our claim that $G' \in \mathcal{F}$, and hence G is not a minor minimal graph of $\mathcal{F}$. □

Lemma 2.6. *Let G be a minor minimal graph of $\mathcal{F}$. Then the girth of G is at least four.*

Proof. Let G be a minor minimal graph of $\mathcal{F}$. By Lemma 2.2 we know that the girth of G is at least three. Suppose, for a contradiction, that

G has a triangle with vertex set $\{u_1, u_2, u_3\}$. One of the edges of the triangle must be special. From Lemma 2.3 it follows that u_1, u_2 and u_3 have degree three. For $i = 1, 2, 3$ let v_i be the third neighbor of u_i. Now v_1, v_2 and v_3 are pairwise distinct, because otherwise no edge is special. If v_1 is adjacent to v_2 then u_1u_2 must a special edge in G. By Lemma 2.3 we know that v_1 and v_2 have degree three. Let v_1' and v_2' be the remaining neighbors of v_1 and v_2 respectively. Now the set $\{v_1v_1', v_2v_2', u_3v_3\}$ is a 3-edge cut contradicting Lemma 2.5 (because either $v_1' \neq v_3$ or $v_2' \neq v_3$, for otherwise no edge of G is special). Thus v_1 and v_2 are not adjacent, and similarly v_3 is not adjacent to v_1 or v_2. Thus G has a unique short cycle. Consequently, the edges u_1u_2, u_1u_3, u_2u_3 are special, and hence it follows from Lemma 2.3 that v_1, v_2, and v_3 all have degree three.

Let G' be obtained from $G \backslash u_1u_2$ by first suppressing u_1 and then suppressing u_2. Let f_1 be the surviving edge of u_1 and f_2 be the surviving edge of u_2. Since $G' \notin \mathcal{F}$, for any edge there is a short cycle that does not use it. Any short cycles in G' must use either f_1 or f_2, otherwise they would have been short in G. Since u_3v_3 is not special in G', there is a short cycle in G' not using u_3v_3. Therefore v_1 and v_2 have a common neighbor in G', say v_{12}. Using symmetry we see that v_2 and v_3 have a common neighbor, say v_{23}, and that v_1 and v_3 have a common neighbor, v_{13}, both in G'.

Suppose that $v_{12} = v_{13}$. If v_{12} has degree greater than three then let G' be the graph obtained from $G \backslash v_{12}v_1$ by suppressing v_1 and let f_1 be the surviving edge of v_1. Since $G' \notin \mathcal{F}$, for any edge there is a short cycle that does not use it. Any new short cycles in G' must use f_1, otherwise they would have been short in G. Since there is a short cycle using f_1 but not u_1u_2 we know that v_1 and v_3 have a common neighbor, say v_{13}', in G'. Now $\{v_1, v_{13}', v_3, v_{13}\}$ is the vertex set of a short cycle in G distinct from our unique short cycle on $\{u_1, u_2, u_3\}$, a contradiction. Therefore v_{12} has degree three in G. The remaining edges incident with v_1, v_2, and v_3 form a 3-edge cut contradicting Lemma 2.5. Therefore $v_{12} \neq v_{13}$ and by symmetry, v_{12}, v_{23}, and v_{13} are all pairwise distinct.

If v_{12} has degree greater than three then the graph obtained from $G \backslash v_1v_{12}$ by suppressing the vertex v_1 is in $\mathcal{F}$ because the edge u_1u_3 is special. Therefore v_{12} has degree three, and by symmetry v_{13} and v_{23} also have degree three. Let e_1, e_2, and e_3 be the remaining edges incident to v_{12}, v_{23}, and v_{13} respectively. Now $\{e_1, e_2, e_3\}$ is a three edge cut contradicting Lemma 2.5. Therefore G has no triangle and the girth of G is at least four. $\square$

Lemma 2.7. *Let G be a minor minimal graph of $\mathcal{F}$. If the girth of G is four, then there is exactly one short cycle.*

Proof. We know that G has no loops or parallel edges since the girth of G is four. Let e be a special edge in G, with ends u and v. Then u and v have degree three by Lemma 2.3. Let u_1 and u_2 be the remaining neighbors of u. Similarly, let v_1 and v_2 be the remaining neighbors of v. Then u_1, u_2, v_1, v_2 all have degree three by Lemma 2.3. Again, since the girth of G is four, all of u_1, u_2, v_1, and v_2 are distinct from each other and without loss of generality u_1 is adjacent to v_1. If there is more than one short cycle in G, then u_2 is adjacent to v_2. Let u_1', u_2', v_1', and v_2' be the remaining neighbors of u_1, u_2, v_1, and v_2 respectively. We know that all of u_1', u_2', v_1', v_2' are distinct from u_1, u_2, v_1, v_2, u, v, and that $u_1' \neq v_1'$, $u_2' \neq v_2'$, u_1' is not adjacent to v_1', and u_2' is not adjacent to v_2', since any of these cases would give a short cycle in G not containing e. Notice that if the degree of u_1' is greater than three then the graph obtained from $G\backslash u_1u_1'$ by suppressing u_1 is in $\mathcal{F}$ because the edge e is special. Therefore the degree of u_1' is three. Similarly, all of u_2', v_1', and v_2' have degree three. Suppose $u_1' = v_2'$, then since G is 3-connected by Lemma 2.4 and every short cycle must use the edge e, we know $u_2' \neq v_1'$. Let e_1 be the remaining edge incident with u_1'. Then $\{e_1, u_2u_2', v_1v_1'\}$ form a 3-edge cut contradicting Lemma 2.5. Therefore $u_1' \neq v_2'$ and by symmetry $u_2' \neq v_1'$, so u_1', u_2', v_1', v_2' are pairwise distinct.

Let G' be the graph obtained from $G\backslash e$ by suppressing both vertices of degree two. Let f_1 be the surviving edge of u, and f_2 be the surviving edge of v. Any short cycle in G' must use either f_1 or f_2. Since G' is not in $\mathcal{F}$, neither f_1 nor f_2 can be special in G'. Since f_1 is not special in G', $u_2' \neq v_1'$ and $u_1' \neq v_2'$, v_1' must be adjacent to v_2' in G'. Similarly, since f_2 is not special in G', u_1' must be adjacent to u_2' in G'.

Now let G' be the graph obtained from $G\backslash u_1u$ by suppressing both vertices of degree two. Let e be the surviving edge of u, and e_1 be the surviving edge of u_1. Any short cycle in G' must use either e or e_1. Since G' is not in $\mathcal{F}$, e cannot be special in G'. Thus u_1' and v_1' must have a common neighbor, say w_1. By symmetry we see that u_2' and v_2' must have a common neighbor, say w_2. If $w_1 = u_2'$ and u_1' is adjacent to v_2' then $\{u_1', u_2', v_1', v_2'\}$ is the vertex set of a short cycle in G not containing e. If on the other hand, $w_1 = u_2'$ and u_1' is not adjacent to v_2' then $\{u_1', v_2'\}$ is a vertex cut in G which is a contradition to Lemma 2.4. Therefore $w_1 \neq u_2'$ and by symmetry $w_1 \neq v_2'$. Similarly, we see that w_2 is also distinct from all of $u, u_1, u_2, u_1', u_2', v, v_1, v_2, v_1'$, and v_2'. Further, $w_1 \neq w_2$, for otherwise $\{w_1, u_1', u_2'\}$ is the vertex set of a short cycle in G that does not use e, and w_1 is not adjacent to w_2, because otherwise $\{w_1, w_2, u'_2, u'_1\}$ is the vertex set of a short cycle in G that does not use e. Now $\{w_1, w_2\}$ is a vertex cut in G, a contradiction to Lemma 2.4. Therefore G has at most one short cycle. □

Lemma 2.8. *Let G be a minor minimal graph of $\mathcal{F}$. If G has a short cycle, then G is isomorphic to Basket.*

Proof. Let G be a minor minimal graph of $\mathcal{F}$ with a short cycle. By Lemma 2.6 the girth of G is four. By Lemma 2.7 there is a unique cycle of length four. Let $\{u_1, u_2, u_3, u_4\}$ be the vertex set of the unique short cycle in G. Let e_{ij} be the edge between u_i and u_j. Since each e_{ij} is special, each of u_1, u_2, u_3, and u_4 have degree three by Lemma 2.3. Let u'_1, u'_2, u'_3, and u'_4 be the remaining neighbors of u_1, u_2, u_3, and u_4 respectively. By Lemma 2.3, each of u'_1, u'_2, u'_3, and u'_4 have degree three. Since there is only one short cycle, all of u'_1, u'_2, u'_3, and u'_4 are distinct from each other and from u_1, u_2, u_3, and u_4, and u'_i is not adjacent to u'_{i+1}, where u'_5 means u'_1. Let e_i be the edge between u_i and u'_i. Let G' be the graph obtained from $G\backslash e_{14}$ by suppressing both vertices of degree two. Let the surviving edge of u_1 be e_1 and the surviving edge of u_4 be e_4. Any short cycle in G' must use either e_1 or e_4. Since G' is not in $\mathcal{F}$ none of the edges can be special. Since e_{23} is not special, either u'_1 and u'_2 have a common neighbor or u'_3 and u'_4 have a common neighbor. Without loss of generality u'_1 and u'_2 have a common neighbor, say u_{12}. It follows that u_{12} is distinct from all vertices named. Since e_1 is not special in G', either u'_2 is adjacent to u'_4 in G' or u'_3 and u'_4 have a common neighbor in G'. Suppose u'_3 and u'_4 do not have a common neighbor in G', so u'_2 must be adjacent to u'_4 in G'. Since e_2 is not special in G', u'_1 must be adjacent to u'_3 in G'. Let G'' be the graph obtained from $G\backslash e_{12}$ by suppressing both vertices of degree two. Since e_{34} is not special in G'' and u'_3 is not adjacent to u'_4, either u_{12} is adjacent to u'_3 in G'' or u_{12} is adjacent to u'_4 in G''. Each case gives a two vertex cut $\{u_{12}, u'_4\}$ or $\{u_{12}, u'_3\}$, respectively, contradicting Lemma 2.4. Therefore, u'_3 and u'_4 must have a common neighbor in G'. Let u_{34} be the common neighbor of u'_3 and u'_4 in G'. If $u_{12} = u_{34}$ then let G'' be the graph obtained from $G\backslash u'_1 u_{12}$ by suppressing u'_1. Since e_{12} is not special in G'', either u'_1 is adjacent to u'_3 in G'' or u'_1 and u'_4 have a common neighbor in G''. Either of these cases give a short cycle in G distinct from the unique short cycle already present. Therefore, $u_{12} \neq u_{34}$.

By symmetry we know that u_2 and u_3 have a common neighbor in G, say u_{23}, that u_1 and u_4 have a common neighbor in G, say u_{14}, and that $u_{23} \neq u_{14}$. Suppose for a contradiction that $u_{23} = u_{12}$. Then $u_{14} \neq u_{34}$ since otherwise $\{u_{12}, u'_1, u_{14}, u'_3\}$ is the vertex-set of a short cycle in

G. Therefore u_{14} is distinct from all vertices previously named. The degree of u_{34} is three since otherwise the graph obtained from $G\backslash u'_4u_{34}$ by suppressing u'_4 is in $\mathcal{F}$ because the edge e_{14} would be special. By symmetry u_{14} has degree three. The vertex u'_2 is not adjacent to u_{14} or u_{34}, for otherwise $\{u_{12}, u_{34}\}$ or $\{u_{12}, u_{14}\}$ is a vertex-cut, contrary to Lemma 2.4. It follows that the degree of u_{12} is three since otherwise the graph obtained from $G\backslash u'_2u_{12}$ by suppressing u'_2 has a unique short cycle, and hence is in $\mathcal{F}$. Now the remaining edges leaving u'_2, u_{34}, and u_{14} are a three edge cut meeting the criteria of Lemma 2.5, and hence G is not minimal, a contradiction. Therefore $u_{23} \neq u_{12}$ and by symmetry $u_{12}, u_{23}, u_{34}, u_{14}$ are pairwise distinct.

If the degree of u_{12} is greater than three then the graph obtained from $G\backslash u'_1u_{12}$ by suppressing u'_1 is in $\mathcal{F}$, because the edge u_1, u_4 is special. Therefore the degree of u_{12} is three. Similarly the degrees of each of u_{23}, u_{34}, and u_{14} are all three. Let u'_{12} be the neighbor of u_{12} other than u'_1 and u'_2, and let $u'_{23}, u'_{34}, u'_{14}$ be defined analogously. If $u'_{12} = u_{34}$, then $u'_{23} = u_{14}$ (otherwise $\{u_{14}, u_{23}\}$ is a vertex-cut violating Lemma 2.5), and hence G is isomorphic to Basket, as desired. We may therefore assume that $u'_{12} \neq u_{34}$, and likewise $u'_{23} \neq u_{14}$. It follows that the vertices $u'_{12}, u'_{23}, u'_{34}, u'_{14}$ are distinct from all the vertices considerd so far, and it also follows that consecutive vertices in the sequence $u'_{12}, u'_{23}, u'_{34}, u'_{14}, u'_{12}$ are distinct.

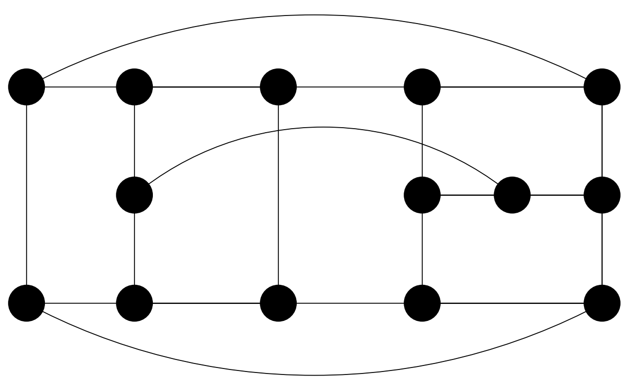
Let G' be the graph obtained from $G\backslash u'_1u_{12}$ by suppressing both vertices of degree two. Since G' is not in $\mathcal{F}$, the edge u_1u_4 is not special, and hence u'_{12} and u'_{23} must be adjacent. Similarly, u'_{23} is adjacent to u'_{34}, u'_{34} is adjacent to u'_{14} and u'_{14} is adjacent to u'_{12}. Thus if $u'_{12}, u'_{23}, u'_{34}, u'_{14}$ are pairwise distinct then they form the vertex set of a short cycle sharing no edge with the short cycle on $\{u_1, u_2, u_3, u_4\}$, a contradiction. Thus, we may assume that $u'_{12} = u'_{34}$. Since G is 3-connected, there exists a path in G with ends u'_{14} and u'_{23} and otherwise disjoint from all the vertices which have been named. Thus G has a proper minor isomorphic to Basket, a contradiction. □

Let G be a graph, and let $X \subseteq V(G)$. We define $\delta(X)$ to be the set of all edges of G with one end in X and the other end in $V(G) - X$. A 3-regular graph G is *cyclically 5-connected* if it has at least six vertices, and for every set $X \subseteq V(G)$ with $|\delta(X)| \leq 4$, one of $G \mid X$, $G\backslash X$ has no cycles.

Lemma 2.9. *Let G be a minor minimal graph of $\mathcal{F}$ with no short cycles. Then G is cyclically 5-connected.*

Proof. Let G be as stated. Then every edge of G is special, and hence G is 3-regular by Lemma 2.3. It follows that G has at least six vertices. Suppose for a contradiction that G has a set X of vertices such that $|\delta(X)| \leq 4$ and both $G \mid X$ and $G\backslash X$ have cycles. Let us choose such a set X with $|\delta(X)|$ minimum, and, subject to that, with $|X|$ minimum. Since G is 3-regular, we see that $\delta(X)$ is a matching, and from Lemmas 2.4 and 2.5 we deduce that $|\delta(X)| = 4$. Let $u, v \in X$ be distinct vertices incident with edges of $\delta(X)$. We claim that u and v are not adjacent in G. To prove this claim suppose for a contradiction that u and v are adjacent. Since G is 3-regular and has no short cycles it is easy to see that $G \mid (X - \{u, v\})$ has a cycle. Thus $X - \{u, v\}$ contradicts the minimality of $|X|$.

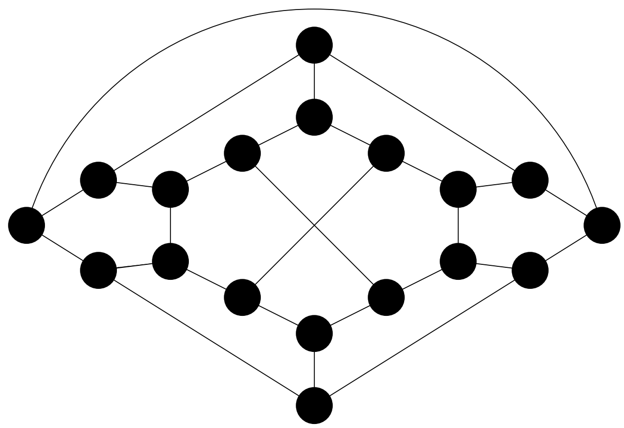
Let u_1, u_2, u_3, u_4 be the ends of edges in $\delta(X)$ that belong to $V(G) - X$, and let C be a cycle in $G\backslash X$. By Menger's theorem and the minimality of $|\delta(X)|$ there exist four disjoint paths P_1, P_2, P_3, P_4 between $\{u_1, u_2, u_3, u_4\}$ and $V(C)$. Let the ends of P_i be u_i and v_i; we may assume that v_1, v_2, v_3, v_4 occur on C in the order listed. Let G' be obtained from G by deleting $V(G) - (X - \{u_1, u_2, u_3, u_4\})$ and all edges incident with $\{u_1, u_2, u_3, u_4\}$ except those in $\delta(X)$, and adding the edges $u_1u_2, u_2u_3, u_3u_4, u_4u_1$. By contracting the paths P_1, P_2, P_3, P_4 and certain edges of C we see that G' is isomorphic to a minor of G. By the claim of the previous paragraph the graph G' has a unique short cycle, and hence $G' \in$, contrary to the minimality of G. □



Box

FIGURE 2.1

The following is a result of McCuaig [7, 8], independently obtained by Aldred, Holton and Jackson [1]. *Box* and *Ruby* are defined in Figures 2.1 and 2.2.



Ruby

FIGURE 2.2

Lemma 2.10. *Every cyclically* 5*-connected* 3*-regular graph of girth at least five has a minor isomorphic to Ruby, Triplex, Dodecahedron, Petersen, or Box.*

Proof of Theorem 2.1. Let G be a minor minimal graph of $\mathcal{F}$. If G has any short cycles, then it has exactly one short cycle by Lemma 2.7 and G is isomorphic to Basket by Lemma 2.8. On the other hand, if G has no short cycles, then every edge of G is special, and hence G is 3-regular by Lemma 2.3. By Lemma 2.9 we know that G is cyclically 5-connected. It now follows from Lemma 2.10 that G has a minor isomorphic to one of Ruby, Triplex, Dodecahedron, Petersen, or Box. Both Ruby and Box have a minor isomorphic to Basket, and therefore G must be one of Triplex, Petersen, Dodecahedron or Basket. □

3. NONPLANAR GRAPHS OF GIRTH AT LEAST FIVE

Recall that a graph G is *quasi* 4*-connected* if it is simple, 3-connected, has at least five vertices, and for every separation (A, B) of G of order three, either $|A| \leq 4$ or $|B| \leq 4$. A graph K is a *subdivision* of a graph G if K is obtained from G by replacing its edges by internally disjoint nonzero length paths with the same ends.

Let H be a simple 3-connected planar graph. Then H has a unique planar embedding. In particular, a cycle in H bounds a region in some planar embedding of H if and only if it bounds a region in every planar embedding of H. Such cycles will be called *peripheral.* Let u, v be two vertices of H such that no peripherial cycle includes both of them, and let H_1 be obtained from H by adding an edge with ends u and v. We say that H_1 is a *jump extension* of H. Let C be a peripheral cycle in G on at least four vertices, and let u, v, x, y be distinct vertices of C appearing on C in the order listed. Let H_2 be obtained from H by adding two edges, one with ends u and x, and the other with ends v and y. We say that H_2 is a *cross extension* of H. The following is shown in [13].

Lemma 3.1. *Let H be a quasi 4-connected planar graph with no cycles of length three, and let G be a quasi 4-connected nonplanar graph such that G has a subgraph isomorphic to a subdivision of H. Then a jump extension or a cross extension of H is isomorphic to a minor of G.*

Now we are ready to prove Theorem 1.5, which we restate.

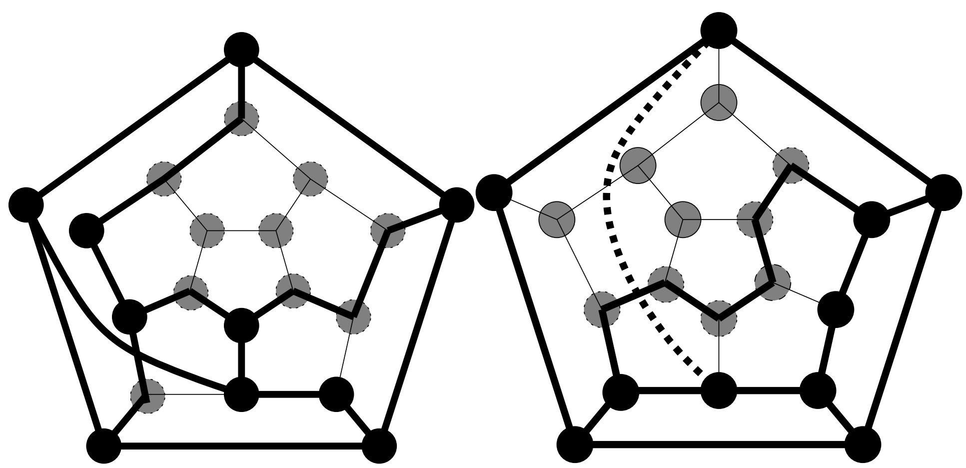


FIGURE 3.1. Jump extensions of the Dodecahedron

Theorem 3.2. *Any quasi 4-connected nonplanar graph with minimum degree at least three and no cycle of length less than five has a minor isomorphic to P_{10}^-.*

Proof. Let G be a quasi 4-connected nonplanar graph with minimum degree at least three and no cycle of length less than five. By Theorem 1.6, G has a minor isomorphic to Triplex, Petersen, Basket, or Dodecahedron. Each of Triplex, Petersen and Basket in turn have a minor isomorphic to P_{10}^-. Let D denote the Dodecahedron; we may therefore assume that G has a minor isomorphic to D. Since D is 3-regular, it follows that G has a subgraph isomorphic to a subdivision of D. The graph D is quasi 4-connected, as can be seen by inspection. By Lemma 3.1 applied to $G = G$ and $H = D$ we deduce that a jump extension or a cross extension of D is isomorphic to a minor of G. But Figure 3.1 shows that every jump extension of D has a minor isomorphic to P_{10}^{-}, and Figure 3.2 shows that the same holds for the unique (up to isomorphism) cross extension of D, as desired. □

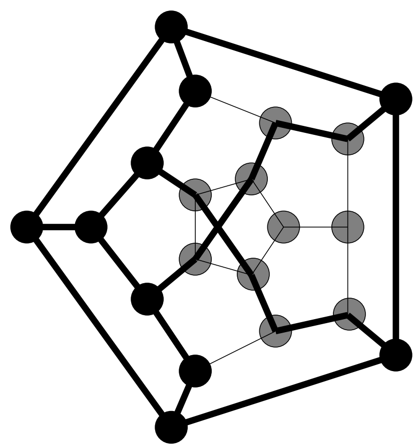


FIGURE 3.2. Cross extension of the Dodecahedron

4. Γ-FLOWS

We will now proceed with the proof of Theorem 1.4. An *isthmus* in a graph is an edge whose deletion increases the number of connected components. We need four lemmas. The first [19, Lemma 2.8.4] is well-known and straightforward to prove.

Lemma 4.1. *Let G be a minor minimal graph with no isthmus and no Γ-flow. Then G is 2-connected and has minimum degree at least three.*

Lemma 4.2. *Let G be a minor minimal graph with no isthmus and no Γ-flow. Then G is 3-connected.*

Proof. Let G be a minor minimal graph with no isthmus and no Γ-flow. Then G is 2-connected by Lemma 4.1. For a contradiction, suppose there is a nontrivial separation (A_1, A_2) of order two. Let $A_1 \cap A_2 = \{u, v\}$. Let G_1 be obtained from $G|A_1$ by adding an edge e_1 with ends u and v and let G_2 be obtained from $G|A_2 - E(G|A_1)$ by adding an edge e_2 with ends u and v. Since G is 2-connected and has no isthmus, neither G_1 nor G_2 can have an isthmus. Therefore, by the minimality of G, there exist Γ-flows ϕ_1 and ϕ_2 in G_1 and G_2 respectively. By the transitivity of Γ we can assume $\phi_1(e_1) = \phi_2(e_2)$. Now, the flow in G given by $\phi(e) = \phi_1(e)$ if $e \in E(G_1) - \{e_1\}$ and $\phi(e) = \phi_2(e)$ if $e \in E(G_2) - \{e_2\}$ is a Γ-flow, a contradiction. □

Our third lemma follows from [19, Theorem 3.8.10], but for the convenience of the readers we give a proof from first principles.

Lemma 4.3. *Let G be a minor minimal graph with no isthmus and no Γ-flow. Then the girth of G is at least five.*

Proof. Let G be a minor minimal graph with no isthmus and no Γ-flow. For a contradiction, suppose that C is a short cycle in G. Let $F \subseteq E(C)$ be such that $|F| = 1$ if C has at most three edges, and let F consist of two diagonally opposite edges of C otherwise. We claim that we may assume that $G \backslash F$ has an isthmus. Indeed, otherwise $G \backslash F$ has a Γ-flow by the minimality of G, and hence there exists a function $\phi : E(G) \to \Gamma$ such that $\phi(e) = 0$ if and only if $e \in F$, and $\sum \phi(e) = 0$ for every vertex $v \in V(G)$, where the summation is over all edges of G incident with v. Let γ be a nonzero element of Γ such that $\phi(e) \neq \gamma$ for every $e \in E(C) - F$, and let $\phi' : E(G) \to \Gamma$ be defined by $\phi'(e) = \phi(e)$ if $e \in E(G) - E(C)$ and $\phi'(e) = \phi(e) + \gamma$ if $e \in E(C)$. Then ϕ' is a Γ-flow in G, as desired. This proves our claim that we may assume that $G \backslash F$ has an isthmus.

If $|E(C)| \leq 3$, then the fact that $G \backslash F$ has an isthmus implies that either G is not 2-connected, or it has a vertex of degree two, contrary to Lemma 4.1. Thus we may assume that $|E(C)| = 4$. Let the vertices of C be v_1, v_2, v_3, v_4 (in order), and let the edges of C be e_1, e_2, e_3, e_4 in such a way that e_i has ends v_i and v_{i+1}, where v_5 means v_1. Let f_1 be an isthmus of $G \backslash \{e_1, e_3\}$, and let A, B be the vertex-sets of the two components of $G \backslash \{e_1, e_3, f_1\}$. Let f_2 be an isthmus of $G \backslash \{e_2, e_4\}$, and let C, D be the vertex-sets of the two components of $G \backslash \{e_2, e_4, f_2\}$. By symmetry we may assume that $v_1 \in A$; since G is 2-connected and has minimum degree at least three we deduce that $f_1 \notin \{e_1, e_3\}$, and hence $u_2 \in A$ and $u_3, u_4 \in B$. Similarly, we may assume that $u_2, u_3 \in C$ and $u_1, u_4 \in D$. If f_1 has one end in $A \cap C$ and the other end in $B \cap D$, then $f_1 = f_2$. Thus at least one of the sets $A \cap C, A \cap D, B \cap C, B \cap D$ includes no end of f_1 or f_2, say $A \cap C$ does. Then $G \backslash \{e_1, e_4\}$ has no path between u_1 and u_2, contrary to the fact that G is 2-connected and has minimum degree at least three. □

Lemma 4.4. *Let G be a minor minimal graph with no isthmus and no Γ-flow. Then G is quasi 4-connected.*

Proof. Let G be a minor minimal graph with no isthmus and no $\Gamma-$flow. For a contradiction, suppose G is not quasi 4-connected. By Lemma 4.3 and Lemma 4.2, G is simple and 3-connected. Thus, there is a separation (A_1, A_2) of order three such that $|A_1|, |A_2| > 5$. Let $A_1 \cap A_2 = \{u_1, u_2, u_3\}$. Let G_1 be obtained from $G|A_1$ by adding a new vertex

v_1 and edges e_i with ends u_i and v_1 for $i = 1, 2, 3$. Let G_2 be obtained from $G|A_2 - E(G|A_1)$ by adding a new vertex v_2 and edges f_i with ends u_i and v_2 for $i = 1, 2, 3$. Since G is 3-connected, neither $G|A_1$ nor $G|A_2$ can have an isthmus. Therefore, by the minimality of G, there exist Γ-flows ϕ_1 and ϕ_2 in G_1 and G_2 respectively. It follows that the values $\phi_1(e_1), \phi_1(e_2), \phi_1(e_3)$ are distinct, and similarly for $\phi_2(f_1), \phi_2(f_2), \phi_2(f_3)$. By the transitivity of Γ we can assume $\phi_1(e_i) = \phi_2(f_i)$ for $i = 1, 2, 3$. Now, the flow in G given by $\phi(e) = \phi_1(e)$ if $e \in E(G_1) - \{e_1, e_2, e_3\}$ and $\phi(e) = \phi_2(e)$ if $e \in E(G_2) - \{f_1, f_2, f_3\}$ is a Γ-flow, a contradiction.

□

The following implies Theorem 1.4.

Theorem 4.5. *Every graph with no isthmus and no P_{10}^- minor has a Γ-flow.*

Proof. Let G be a minor minimal graph with no isthmus and no Γ-flow. We will show that G has a P_{10}^- minor. By Lemma 4.3 we know that the girth of G is at least five. By Lemma 4.1 we know that the minimum degree of G is at least three. If G is planar then it has a Γ-flow by the four color theorem and Theorem 1.1 of [16]. By Lemma 4.4, G is quasi 4-connected. Thus G is quasi 4-connected, nonplanar, has girth at least five, and hence it has a P_{10}^- minor by Theorem 1.5, as desired. □

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