MINOR-MINIMAL PLANAR GRAPHS OF EVEN BRANCH-WIDTH

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Abstract. Let $k \geq 1$ be an integer, let H be a minor-minimal graph in the projective plane such that every homotopically non-trivial closed curve intersects H at least k times, and let G be the planar double cover of H obtained by lifting G into the universal covering space of the projective plane, the sphere. We prove that G is minor-minimal of branch-width $2k$. We also exhibit examples of minor-minimal planar graphs of branch-width 6 that do not arise this way.

1. Introduction

The projective plane $\mathbb{P}$ is obtained from a closed disk by identifying diagonally opposite pairs of points on the boundary of the disk. Given a graph H embedded in $\mathbb{P}$ its *planar double cover* G is the lift of H into the universal covering space of $\mathbb{P}$, the sphere. Thus to every vertex v of H there correspond two vertices v_1, v_2 of G; we say that v is the *projection* of v_1 and v_2, and that v_1 and v_2 are the *lifts* of v. Similarly, we speak of projections and lifts of paths, cycles, walks, and faces. This construction is illustrated in Figure 1, where the Dodecahedron is shown to be the planar double cover of the Petersen graph. In particular, if W is a walk in G with ends the two lifts of a vertex $v \in V(H)$, then the projection of W is a homotopically non-trivial closed walk in H with both ends v.

Let H and G be as in the above paragraph. Our objective is to relate the representativity of H (also known as face-width) and the branch-width of G, two important parameters that we now review. Let $k \geq 0$ be an integer. We say that H is *k-representative* if every homotopically non-trivial closed curve intersects H at least k times. This concept has received a lot of attention in the literature; we refer to [7] for more information. A graph is a *minor* of another if the first can be obtained from a subgraph of the second by contracting edges. Randby [8] proved that given two minor-minimal k-representative graphs in $\mathbb{P}$, each can be obtained from the other by means of repeated application of ΔY- and $Y\Delta$-exchanges. Since for each k there is a natural example of a minor-minimal k-representative projective planar graph, namely the $k \times k$ projective grid, Randby's result gives a convenient way to generate all minor-minimal k-representative projective planar graphs. Incidentally, Schrijver [12] proved an analogue of Randby's result for the torus, as well as a related result for arbitrary orientable surfaces [11].

A *branch-decomposition* of a graph G is pair (T, η), where T is a tree with all vertices of degree one or three, and η is a bijection between the leaves (vertices of degree one) of T and $E(G)$. For $f \in E(T)$ let T_1, T_2 be the two components of $T \backslash f$, and let X_i be the set of leaves of T that belong to T_i. We define the *order* of f to be the number of vertices of G incident both with an edge in $\eta(X_1)$ and an edge in $\eta(X_2)$. The *width* of (T, η) is the maximum order of an edge of T, and the *branch-width* of G is the minimum width of a branch-decomposition of G, or 0 if $|E(G)| \leq 1$, in which case G has no branch-decomposition.

Computing branch-width is NP-hard [13], but there is a polynomial time algorithm when G is planar [13]. Thus one might expect that planar branch-width is better behaved in other respects as well, such as in terms of excluded minors. Since taking minors does not

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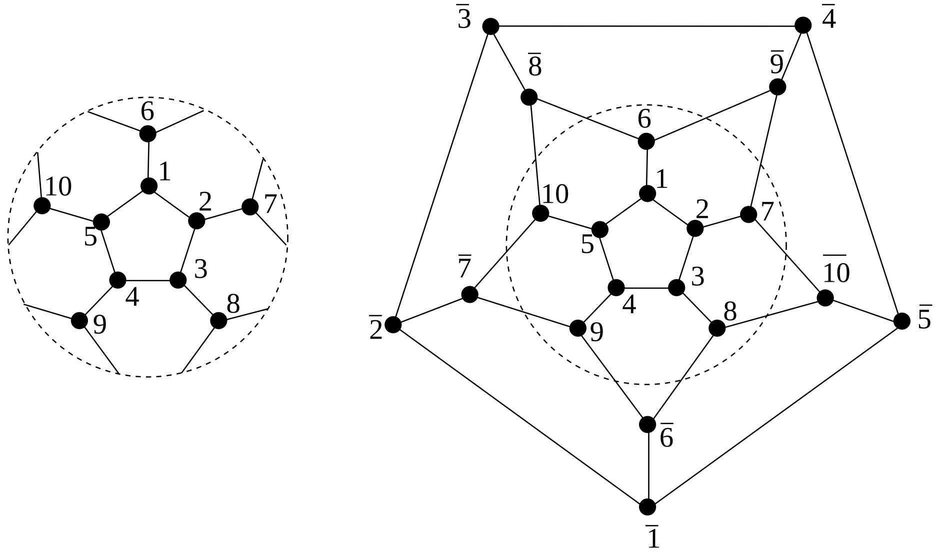

FIGURE 1. An embedding of the Petersen graph in $\mathbb{P}$ and its planar double cover, the Dodecahedron

increase branch-width, as is easily seen, graphs of branch-width at most t are characterized by *excluded minors*, the list of minor-minimal graphs of branch-width $t+1$. It follows from [9] that this list is finite.

Our first result is that planar double covers of minor-minimal k-representative projective planar graphs are excluded minors for odd branch-width:

Theorem 1.1. *Let $k \geq 1$ be an integer, let H be a minor-minimal k-representative projective planar graph, and let G be the double cover of H. Then G is a minor-minimal graph of branch-width $2k$.*

Thus, in view of Randby's theorem mentioned above, this gives a description of a class of minor-minimal graphs of branch-width $2k$. Since for $k = 1$ and $k = 2$ this class actually includes all minor-minimal graphs of branch-width $2k$ [2, 3, 5], one might wonder whether this holds in general. Unfortunately, it does not. In Section 4 we exhibit examples of minor-minimal planar graphs of branch-width 6 that are not double covers of any graph.

The paper is organized as follows. We prove the lower bound for Theorem 1.1 in Section 2 and the corresponding upper bound in Section 3. The examples are presented in Section 4.

The second author would like to acknowledge helpful conversations with P. D. Seymour from the summer of 1990; in particular, the question whether all minor-minimal graphs of even branch-width arise as in Theorem 1.1 was inspired by those conversations. The second author would also like to acknowledge that the results of this paper appeared (in a slightly different form) in the PhD dissertation [4] of the first author.

2. Lower Bound

We will make use of the result of [13] that the branch-width of a planar graph G is equal to one half the "carving-width" of the "medial graph" of G. Here are the definitions. Let G be a graph drawn in a surface in such that every face of G is homeomorphic to an open disk. Let us choose, for each vertex $v \in V(G)$, one of the two cyclic orderings of edges incident with v (with each loop occurring twice in the ordering) and designate it as *clockwise*. By an *angle* at v we mean an ordered pair (e, e') of edges incident with v such that e' immediately follows e in the clockwise order around v. To each angle there naturally corresponds a face

f of G incident with e and e'. The *medial graph* of G is the graph M defined as follows. For each edge $e \in E(G)$ choose a vertex x_e positioned in the interior of e, and for each angle (e, e') choose an edge joining x_e and $x_{e'}$ inside the face of G that corresponds to the angle (e, e') in a small neighborhood of e and e'. There is a certain ambiguity when the same face corresponds to different angles, but there is a natural interpretation under which the medial graph is unique up to homotopic shifts of edges. Each face of M corresponds to either a unique vertex of G or a unique face of G, and if G has no loops or cut edges, then every face of M is bounded by a cycle.

A *carving* in a graph G is a pair (T, η), where T is a tree with all vertices of degree one or three, and η is a bijection from the leaves of T to $V(G)$. Similarly as in a branch-decomposition, each edge f of T determines a cut in G. The *width* of (T, η) is the maximum order of those cuts, over all $f \in E(T)$. The *carving-width* of G is the minimum width of a carving in G. The following is [13, Theorem 7.2].

Theorem 2.1. *Let G be a connected plane graph with at least two edges, and let M be its medial graph. Then the branch-width of G is half the carving-width of M.*

Thus it suffices to bound the carving-width of the medial graph of G from Theorem 1.1. We will use the concept of an "antipodality," introduced in [13]. Let G be a non-null connected plane graph with planar dual G^*, let $F(G)$ be the faces of G, and let $k \geq 0$ be an integer. An *antipodality* in G of range $\geq k$ is a function α with domain $E(G) \cup F(G)$, such that for all $e \in E(G)$, $\alpha(e)$ is a non-null subgraph of G and for all $f \in F(G)$, $\alpha(f)$ is a non-empty subset of $V(G)$, satisfying:

(A1) If $e \in E(G)$, then no end of e belongs to $V(\alpha(e))$
(A2) If $e \in E(G)$, $f \in F(G)$, and e is incident with f, then $\alpha(f) \subseteq V(\alpha(e))$, and every component of $\alpha(e)$ has a vertex in $\alpha(f)$
(A3) If $e_1 \in E(G)$ and $e_2 \in E(\alpha(e_1))$ then every closed walk of G^* using e_1^* and e_2^* has length $\geq k$.

The following is a special case of [13, Theorem 4.1]; we will only need the easier "if" part.

Theorem 2.2. *Let M be a connected plane graph on at least two vertices, and let $k \geq 0$ be an integer. Then M has carving-width at least k if and only if either some vertex of M has degree at least k, or M has an antipodality of range $\geq k$.*

As we will see at the end of the next section, the following lemma and Theorems 2.1 and 2.2 imply that the planar double cover of a minor-minimal k-representative projective planar graph has branch-width at least $2k$.

Lemma 2.3. *Let $k \geq 2$ be an integer, let H be a k-representative projective planar graph, let G be a planar double cover of H, and let M be the medial graph of G. Then M has an antipodality of range $\geq 4k$.*

Proof. First we notice that since $k \geq 2$ we may assume (by considering a subgraph of H) that the graph G has no loops or cut edges. For $v \in V(G)$ and a face $f \in F(G)$ incident with v there is a unique edge $a_{v,f}$ of M incident with v^* and f^*, where v^* and f^* denote the faces of M corresponding to v and f, respectively. For $v \in V(G)$ let $v' \in V(G)$ be the other vertex of G with the same projection as v, and for $f \in F(G)$ let f' be defined analogously. Finally, for $x \in V(G) \cup F(G)$ let C_x denote the cycle bounding the face of M corresponding to x. We note that C_x is indeed a cycle, because G has no loops or cut edges. We define the function α as follows:

$$\alpha(a_{v,f}) = C_{v'} \cup C_{f'} \text{ for every } v \in V(G) \text{ and } f \in F(G)$$
$$\alpha(f) = V(C_{f'}) \text{ for every } f \in F(M)$$

We claim that α defines an antipodality of range $4k$ in M. Conditions (A1) and (A2) follow easily from the fact that G has no loops or cut edges and that each $\alpha(a_{v,f})$ is connected.

To prove (A3) suppose for a contradiction that there is a closed walk W in M^* of length at most $4k-1$ that uses a^* and b^* for some $a = a_{v,f} \in E(M)$ and $b \in E(\alpha(a_{v,f}))$. Thus W uses both v^* and f^*, and at least one of $(v')^*$, $(f')^*$. In either case, W includes a subwalk W' of length at most $2k-1$ from x^* to $(x')^*$ for some $x \in V(G) \cup F(G)$. Now the projection of W' is a homotopically non-trivial closed walk of length at most $2k-1$ in the dual of the medial graph of H. That, in turn, implies that H is not k-representative, a contradiction. Hence α is an antipodality, as desired. □

3. Upper bound

In this section we show that if G is as in Theorem 1.1, then every proper minor of G has branch-width at most $2k-1$. Again, we find it convenient to work with carving-width of the medial graph. Thus we need to translate minimal k-representativity into the language of medial graphs. Let G be a graph drawn in a surface, let v be a vertex of G of degree four, let e_1, e_2, e_3, e_4 be the four edges of G incident with v listed in the cyclic order of appearance around v, let v_i be the other end of e_i, and let f_i be the face of G incident with e_i and e_{i-1}, where e_0 means e_4. Let G' be obtained from $G\backslash v$ by inserting two edges, one with ends v_1 and v_4 in the face f_1 and the other with ends v_2 and v_3 in the face f_3. We say that G' was obtained from G by *opening at v through the faces f_2 and f_4*, or simply by *opening at v*. Let $k \geq 1$ be an integer. A 4-regular graph G drawn in $\mathbb{P}$ is *k-tight* if every homotopically non-trivial closed walk in G^* has length at least k, and for every graph J obtained from G by opening at some vertex there exists a homotopically non-trivial closed walk in J^* of length at most $k-1$. We say that G is *tight* if it is k-tight for some integer $k \geq 1$. The following is shown in [11] and is also easy to see.

Lemma 3.1. *Let $k \geq 1$ be an integer, let G be a connected graph in the projective plane, and let M be the medial graph of G. Then G is a minor-minimal k-representative graph in $\mathbb{P}$ if and only if M is $2k$-tight.*

We will need the following characterization of tight graphs in terms of straight ahead decompositions. Let G be a graph with all vertices of degree four or one drawn in a surface. Let v be a vertex of G of degree four, and let e_1, e_2, e_3, e_4 be the edges of G incident with v listed in the order in which they appear around v. We say that the edges e_1 and e_3 are *opposite*. Let $F_1, F_2, \ldots, F_r$ be the equivalence classes of the transitive closure of the opposite relation, and let G_i be the subgraph of G with edge-set F_i and vertex-set all vertices incident with edges of F_i. We say that $G_1, G_2, \ldots, G_r$ is the *straight ahead decomposition* of G. The next result follows from a theorem of Lins [6]. Schrijver [10] obtained an analogous result for orientable surfaces.

Theorem 3.2. *Let G be a 4-regular graph drawn in the projective plane. Then G is tight if and only if the straight ahead decomposition of G consists of homotopically non-trivial cycles such that every two intersect exactly once.*

As usual, a walk in a graph G is an alternating sequence of vertices and edges of G. It has an *origin* and *terminus*, called its *ends*. A walk is *closed* if its ends are equal. Even though any vertex of a closed walk may be regarded as its origin and terminus, for our purposes changing the ends results in a different walk. This subtle point will be important later.

Let G be a graph drawn in a surface, let f, f' be two faces of G, and let W_1, W_2 be two walks in G^* with origin f^* and terminus $(f')^*$. Let v be a vertex of G of degree four, let

e_1, e_2, e_3, e_4 be the four edges of G incident with v listed in the cyclic order of appearance around v, and let f_i be the face of G incident with e_i and e_{i-1}, where e_0 means e_4. If W_1 includes the subwalk $f_1^*, e_1^*, f_2^*, e_2^*, f_3^*$, and W_2 is obtained from W_1 by replacing (one occurence of) that subwalk by the walk $f_1^*, e_4^*, f_4^*, e_3^*, f_3^*$, then we say that W_2 was obtained from W_1 by a $\Delta\nabla$-*exchange*. We write $W_1 = W_2 * v$; then also $W_2 = W_1 * v$. Let us remark the obvious fact that W_1 and W_2 have the same origin and terminus.

Let Δ denote the closed unit disk in $\mathbb{R}^2$. Let G be a graph drawn in Δ such that the vertices $v_1, v_2, \dots, v_{2k}$ and only these vertices are drawn on the boundary of Δ in the order listed, where $v_1, v_2, \dots, v_{2k}$ have degree one and all other vertices of G have degree four. Assume further that the straight ahead decomposition of G is of the form $P_1, P_2, \dots, P_k$, where each P_i is a path with one end in $\{v_1, v_2, \dots, v_k\}$ and the other end in $\{v_{k+1}, v_{k+2}, \dots, v_{2k}\}$. Finally, assume that for $i \neq j$ the paths P_i and P_j intersect in at most one vertex. In those circumstances we say that G is a *graft*, and that $v_1, v_2, \dots, v_{2k}$ are its *attachments*. For $i = 1, 2, \dots, 2k$ let e_i denote the unique edge of G incident with v_i, and let f_i be the face of G incident with e_{i-1} and e_i, where e_0 means e_{2k}. Let us emphasize that G is embedded in Δ, and hence f_i and f_{i+1} are distinct faces, each incident with a segment of the boundary of Δ. Let G^* denote the geometric dual of G. By a *broom* in G we mean a walk in G^* from f_1^* to f_{k+1}^* of length exactly k. (The existence of $P_1, P_2, \dots, P_k$ implies that every walk in G^* from f_1^* to f_{k+1}^* has length at least k.) There are two natural examples of brooms, namely the walks with edge-sets $f_1^*, f_2^*, \dots, f_{k+1}^*$ and $f_1^*, f_{2k}^*, f_{2k-1}^*, \dots, f_{k+1}^*$. Those two brooms will be called the *extreme brooms* of G.

Let G be a graph drawn in a surface with every vertex of degree four or one, let f, f' be two faces of G, and let W_0 and W be two walks in G^* with origin f and terminus f'. We say that G is *sweepable* from W_0 to W if the vertices of G of degree four can be numbered $v_1, v_2, \dots, v_n$ so that $W_i := W_{i-1} * v_i$ is well-defined for all $i = 1, 2, \dots, n$ and $W_n = W$.

Lemma 3.3. *Let $k \geq 1$ be an integer, let G be a graft with attachments $v_1, v_2, \dots, v_{2k}$, and let W, W' be the extreme brooms of G. Then G is sweepable from W to W'.*

Proof. Let $P_1, P_2, \dots, P_k$ be the straight ahead decomposition of G, numbered so that v_i is an end of P_i for $i = 1, 2, \dots, k$. If the paths $P_1, P_2, \dots, P_k$ are pairwise disjoint, then G is a matching, and hence $W = W'$ and the theorem holds. Thus we may assume that some two of the paths P_i intersect, and we proceed by induction on $|V(G)|$. By a *wedge* we mean an ordered pair (i, j) of distinct integers from $\{1, 2, \dots, k\}$ such that the paths P_i and P_j intersect in a (unique) vertex v, and $v_i P_i v$, the subpath of P_i with ends v_i and v, is not intersected by any other $P_{j'}$. Since some two paths P_i intersect, there exists a wedge, and so we may select a wedge (i, j) with $|i - j|$ minimum.

We claim that $|i - j| = 1$. To prove this claim we may assume that $i + 1 < j$. By planarity the path P_{i+1} intersects $P_i \cup P_j$, and hence there exists a wedge $(i + 1, j')$. We deduce from the minimality of $|i - j|$ that $j' \notin \{i, i + 1, \dots, j\}$, and hence $P_{j'}$ intersects $v_i P_i v \cup v P_j v_j$ by planarity. In fact, it intersects $v_i P_i v \cup v P_j v_j$ an even number of times. But $P_{j'}$ does not intersect $v_i P_i v$, because (i, j) is a wedge, and hence $P_{j'}$ intersects P_j at least twice, contrary to the definiton of a graft. This contradiction proves our claim that $|i - j| = 1$. It follows that both $v_i P_i v$ and $v_j P_j v$ have length one.

Thus we can apply a $\Delta\nabla$-exchange at the vertex v to one of the brooms W, W', say to W', to obtain a broom $W'' := W' * v$. Let x_i be the neighbor of v on P_i other than v_i, and let x_j be defined analogously. Let G' be obtained from G by deleting v, v_i, v_j and adding two new vertices of degree one, joined to x_i and x_j, respectively. Then G' with its natural drawing in a disk forms a graft, and W and W'' can be regarded as the two extreme brooms in G'. By the induction hypothesis the graft G' is sweepable from W to W''; by considering the corresponding ordering of vertices of G' and appending v at the end we obtain a desired ordering of the vertices of G, showing that G is sweepable from W to W'. □

Theorem 3.4. *Let $k \geq 1$ be an integer, let H be a 4-regular k-tight graph in $\mathbb{P}$, and let W be a homotopically non-trivial closed walk in H^* of length k. Then H is sweepable from W to W.*

Proof. By Theorem 3.2 the straight ahead decomposition of H consists of k homotopically non-trivial cycles such that every two of them intersect exactly once. We cut H open along W and construct a graft G as follows. For every edge e^* of W we cut the corresponding edge e into two by inserting two new vertices of degree one "in the middle of" e. The theorem follows by applying Theorem 3.3 to the resulting graft G. □

Lemma 3.5. *Let $k \geq 1$ be an integer, let H be a 4-regular k-tight graph in $\mathbb{P}$, and let G be the planar double cover of H. Let f_1, f_2 be the two lifts of some face f of H, and let W_1 be a walk of length k in G^* with origin f_1 and terminus f_2. Then G is sweepable from W_1 to W_1.*

Proof. Let W be the projection of W_1, and let W_2 be the other lift of W. Then W is a homotopically non-trivial closed walk in H^*. By Theorem 3.4 H is sweepable from W to W; let $v_1, v_2, \dots, v_n$ be the corresponding ordering of the vertices of H. Then each v_i has a lift $v_i^1 \in V(G)$ such that $W_2 = W_1 * v_1^1 * v_2^1 * \cdots * v_n^1$. Now let v_i^2 be the other lift of v_i. It follows that $W_1 = W_2 * v_1^2 * v_2^2 * \cdots * v_n^2$, and hence the sequence $v_1^1, v_2^1, \dots, v_n^1, v_1^2, v_2^2, \dots, v_n^2$ shows that G is sweepable from W_1 to W_2. □

Lemma 3.6. *Let $k \geq 1$ be an integer, let H be a 4-regular k-tight graph in $\mathbb{P}$, and let G be the planar double cover of H. Let G_1 be obtained from G by opening at $\overline{u} \in V(G)$. Then G_1 has carving-width at most $2k-1$.*

Proof. Let $\overline{e}_1, \overline{e}_2, \overline{e}_3, \overline{e}_4$ be the edges of G incident with $\overline{u}$ listed in cyclic order around $\overline{u}$, and let $\overline{f_i}$ be the face of G incident with $\overline{e}_i$ and $\overline{e}_{i-1}$, where $\overline{e}_0$ means $\overline{e}_4$. Let the numbering be such that G_1 is obtained by opening at $\overline{u}$ through the faces $\overline{f_2}$ and $\overline{f_4}$. Let u, e_i, f_i be the projections of $\overline{u}, \overline{e}_i, \overline{f_i}$, respectively. Let H_1 be the graph obtained from H by opening at u through the faces f_2 and f_4, and let f_{24} be the resulting new face. Since H is k-tight there exists a homotopically non-trivial closed walk Z_1 in H_1^* of length $k-1$ with origin and terminus f_{24}^*. Since H is k-tight, we deduce that f_{24}^* is the only repeated vertex in Z_1. To Z_1 there corresponds a closed walk Z in H^* starting with $f_2^* e_2^* f_3^* e_3^* f_4^*$ and ending in f_2^*. Let $\overline{Z}$ be the lift of Z that starts with $(\overline{f_2})^*(\overline{e}_2)^*(\overline{f_3})^*(\overline{e}_3)^*(\overline{f_4})^*$ and ends in $(\tilde{f}_2)^*$, where $\tilde{f}_2$ is the other lift of f_2. There is a corresponding walk $\overline{Z}_1$ in G_1^* with origin $(\overline{f}_{24})^*$ and terminus $(\tilde{f}_2)^*$, where $\overline{f}_{24}$ is the face resulting from the opening of G at $\overline{u}$ that creates G_1.

By Lemma 3.5 the graph G is sweepable from $\overline{Z}$ to $\overline{Z}$; let $v_1, v_2, \dots, v_n$ be the corresponding ordering of the vertices of G, and let $W_0 := \overline{Z}$ and $W_i := W_{i-1} * v_i$ be the corresponding walks. Since each W_i starts at $(\overline{f_2})^*$ we deduce that the $\Delta\nabla$-exchange at $\overline{u}$ replaces $(\overline{f_2})^*(\overline{e}_1)^*(\overline{f_1})^*(\overline{e}_4)^*(\overline{f_4})^*$ by $(\overline{f_2})^*(\overline{e}_2)^*(\overline{f_3})^*(\overline{e}_3)^*(\overline{f_4})^*$ or vice versa. From the symmetry we may assume that it replaces the former by the latter; then we may further assume that $\overline{u} = v_n$. We deduce that for all $i = 1, 2, \dots, n-2$ the concatenation of $\overline{Z}_1$ and W_i separates $\{v_1, v_2, \dots, v_i\}$ from $\{v_{i+1}, v_{i+2}, \dots, v_{n-1}\}$ in G_1. Now let T be the tree obtained from a path with vertices $r_1, r_2, \dots, r_{n-1}$ in order by adding, for each $i = 2, 3, \dots, n-2$, a new vertex t_i and joining it by an edge to r_i. Let $t_1 = r_1$ and $t_{n-1} = r_{n-1}$, and let $\eta(t_i) = v_i$. Since $\overline{Z}_1$ has length $k-1$ and each W_i has length k we deduce that (T,η) is a carving decomposition of G_1 of width at most $2k-1$, as desired. □

Proof of Theorem 1.1. Let $k \geq 1$ be an integer, let H be a minor-minimal k-representative graph in $\mathbb{P}$, and let G be the double cover of H. If $k=1$, then H consists of one vertex and one edge, and hence G has two vertices and two edges between them. It follows that the theorem holds in that case, and hence we may assume that $k \geq 2$.

We first show that G has branch width at least $2k$. Let M be the medial graph of G. By Lemma 2.3 the graph M has an antipodality of range $\geq 4k$, and hence has carving-width at least $4k$ by Theorem 2.2. It follows from Theorem 2.1 that G has branch-width at least $2k$, as desired.

Let G_1 be obtained from G by deleting or contracting an edge e. To complete the proof we show that G_1 has branch-width at most $2k-1$; that will imply that the branch-width of G is exactly $2k$ and that it is minor-minimal. To this end let M_1 be the medial graph of G_1; then M_1 is obtained from M by opening at the vertex of M that correponds to e. Let N be the medial graph of H; then N is $2k$-tight by Lemma 3.1 and M is the planar double cover of N. By Lemma 3.6 the graph M_1 has carving-width at most $4k-2$, and hence G_1 has branch-width at most $2k-1$ by Theorem 2.1, as desired. □

4. Minor-minimal planar graphs which are not double covers

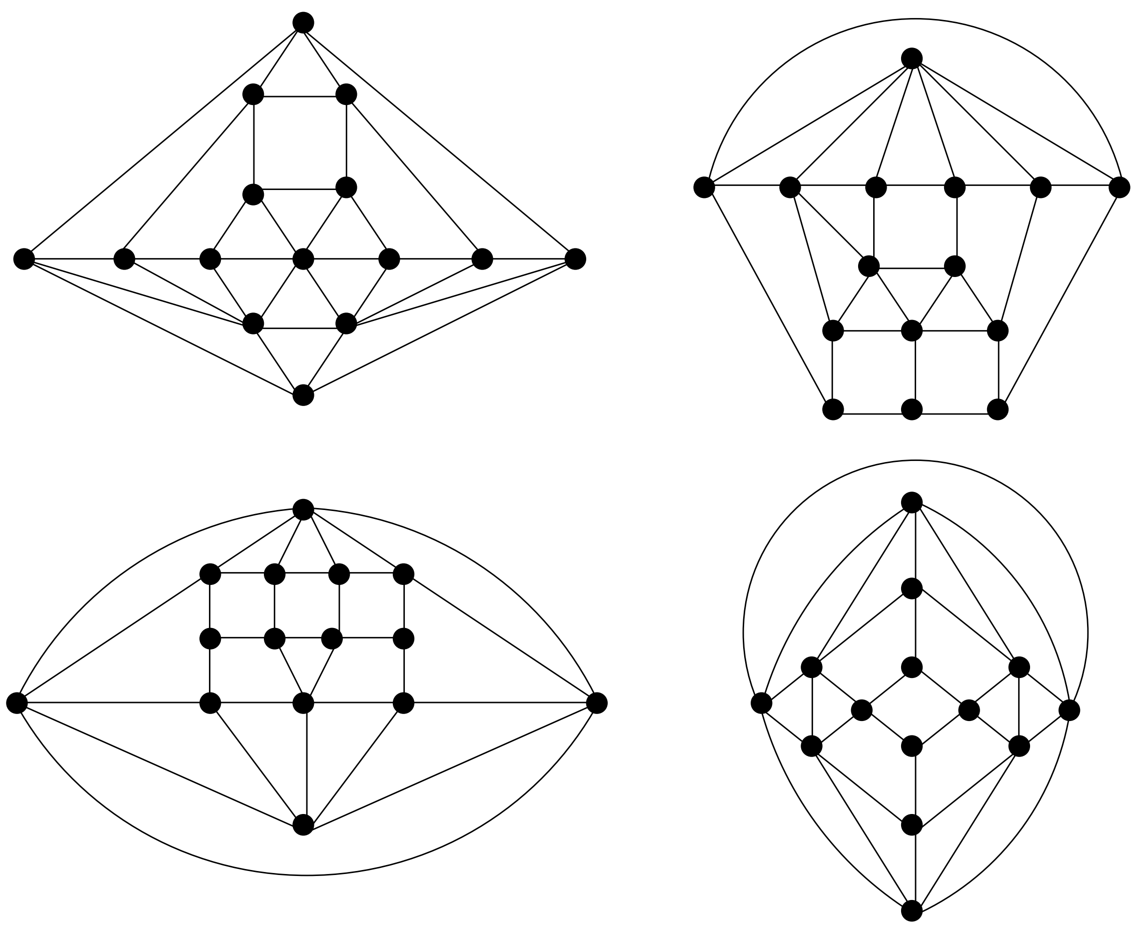

Figure 2. Minor-minimal graphs of branch-width 6 which are not double covers

There are seven non-isomorphic minor-minimal planar graphs of branch-width 6 which arise as planar double covers of the seven minor-minimal 3-representative embeddings of graphs in the projective plane (the latter were determined in [1] and [14]). However, Figure 2 shows four additional minor-minimal planar graphs of branch-width 6, and their geometric duals provide four additional examples. We have generated those graphs using a computer program.

Finally, here is an example of a minor-minimal planar graph of odd branch-width. For every $k \geq 2$ consider the planar $k \times (2k+1)$ circular grid with k concentric cycles and $2k+1$ paths joining the cycles, and add a new vertex that is connected to all $2k+1$ vertices on the innermost cycle to obtain a graph G_k. The graph G_3 is depicted in Figure 3. It can be shown that G_k has branch-width exactly $2k+1$ and is minor-minimal with that property.

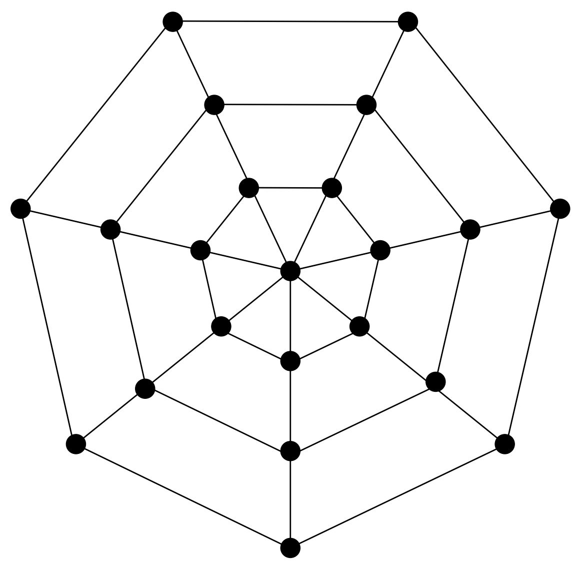

FIGURE 3. The graph G_3

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