TEMPERLEY-LIEB ALGEBRAS AND THE FOUR-COLOR THEOREM

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ABSTRACT

The Temperley-Lieb algebra T_n with parameter 2 is the associative algebra over $\mathbf{Q}$ generated by $1, e_0, e_1, \ldots, e_n$, where the generators satisfy the relations $e_i^2 = 2e_i$, $e_ie_je_i = e_i$ if $|i-j| = 1$ and $e_ie_j = e_je_i$ if $|i-j| \geq 2$. We use the Four Color Theorem to give a necessary and sufficient condition for certain elements of T_n to be nonzero. It turns out that the characterization is, in fact, equivalent to the Four Color Theorem.

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1. INTRODUCTION

Let $n \geq 1$ be an integer. The Temperley-Lieb algebra [11] T_n with parameter 2 is the associative algebra over $\mathbf{Q}$ generated by $1, e_0, e_1, \ldots, e_n$, where the generators satisfy the relations

$$e_i^2 = 2e_i \tag{1}$$

$$e_i e_j e_i = e_i \qquad \text{if } |i-j| = 1 \tag{2}$$

$$e_i e_j = e_j e_i \qquad \text{if } |i-j| \geq 2 \tag{3}$$

for all $i, j = 0, 1, \ldots, n$. For $i = 1, 2, \ldots, n$ we define

$$r_i = \begin{cases} 1 - \frac{1}{2}\, e_i & \text{if } i \text{ is odd} \\ 2e_i - 1 & \text{if } i \text{ is even.} \end{cases}$$

Notice that for $i, j = 0, 1, \ldots, n$ with $|i-j| \neq 1$ we have

$$e_i r_j = r_j e_i \tag{4}$$

and for $i, j = 0, 1, \ldots, n$ with $|i-j| = 1$ we have

$$r_i e_i = 0 \qquad \text{if } i \text{ is odd} \tag{5}$$

$$e_j r_i e_j = 0 \qquad \text{if } i \text{ is even.} \tag{6}$$

Let $\mathcal{W}_n$ denote the set of all finite words over the alphabet $\{e_0, e_1, \ldots, e_n, r_0, r_1, \ldots, r_n\}$. Following [6] we say that two words in $\mathcal{W}_n$ are *exterior equivalent* if the first can be transformed to the second using the relations (2), (3), (4), and the relations $e_i^2 = e_i$ for $i = 0, 1, \ldots, n$. Let $w \in \mathcal{W}_n$. We say that a word $w' \in \mathcal{W}_n$ is a *reduction* of w if either $w' = w$, or w' is obtained from w by repeatedly replacing occurrences of r_{2j} by e_{2j}, and/or deleting occurrences of r_{2j+1}. Thus for instance the word $e_3 e_2 r_3 e_2 r_2 e_1$ is a reduction of $e_3 r_2 r_3 e_2 r_2 r_3 e_1$. We say that $w' \in \mathcal{W}_n$ is a *complete reduction* of a word $w \in \mathcal{W}_n$ if either $w' = w$ and no r_i occurs in w, or w' is a reduction of w and exactly one symbol in w' is r_i for some $i = 0, 1, \ldots, n$. We say that a word $w \in \mathcal{W}_n$ is *loopless* if no complete reduction of w is exterior equivalent to a word containing $e_i r_i$ or $e_i r_j e_i$ (in consecutive positions) for some integers i, j, where $i, j \in \{0, 1, \ldots, n\}$, $|i-j| = 1$, and i is odd. We have the following result.

(1.1) *For every integer $n \geq 1$, a word $w \in \mathcal{W}_n$ represents a nonzero element of T_n if and only if it is loopless.*

We deduce (1.1) from the Four Color Theorem (4CT), and show that, conversely, (1.1) implies the 4CT. Thus (1.1) is equivalent to the 4CT. There are many other equivalent formulations of the Four Color Theorem, some of them rather puzzling. Therefore, (1.1) is not an isolated curiosity, but rather an addition to an already extensive list of equivalent formulations of the 4CT. We refer the reader to the excellent survey [10] and to [12] for a survey of the newer results [3, 5, 7].

The statement of (1.1) was inspired by [6]. Unfortunately, the authors of [6] overlooked the existence of large loops, and hence Theorem 6.4 of that paper is incorrect. Our result (1.1) remedies that problem.

2. PLANAR GRAPHS AND COLORING

A *graph* G consists of a set $V(G)$ of *vertices*, a set $E(G)$ of *edges*, and an *incidence relation* between vertices and edges such that every edge e is incident with precisely two vertices u, v, called the *ends* of e. If $u = v$, then e is called a *loop edge*. A graph is *loopless* if it has no loop edges. A graph is *planar* if it can be drawn in the plane without crossings. A 4-*coloring* of a graph G is a function $c : V(G) \to \{1, 2, 3, 4\}$ such that $c(u) \neq c(v)$ whenever u, v are the ends of some edge of G. A graph is 4-*colorable* if it admits at least one 4-coloring. The Four Color Theorem [1, 2, 8] asserts the following.

(2.1) *Every planar graph is* 4*-colorable if and only if it is loopless.*

The "only if" part is, of course, trivial—every 4-colorable (not necessarily planar) graph is loopless, but the converse is much much harder. So far, there are only two proofs of (2.1), and both are computer-assisted.

We need to restate the 4CT in terms of subgraphs of grid graphs with some edges contracted. For integers $n, t \geq 1$, the $n \times t$ *grid* is the graph G with vertex-set all pairs (i, j) for $i = 0, 1, \ldots, n$ and $j = 0, 1, \ldots, t$ in which (i, j) is is joined by an edge to (i', j') if and only if $|i - i'| + |j - j'| = 1$ and $j + j' \neq 2t$. We also say that G is a *grid*. See Figure 1

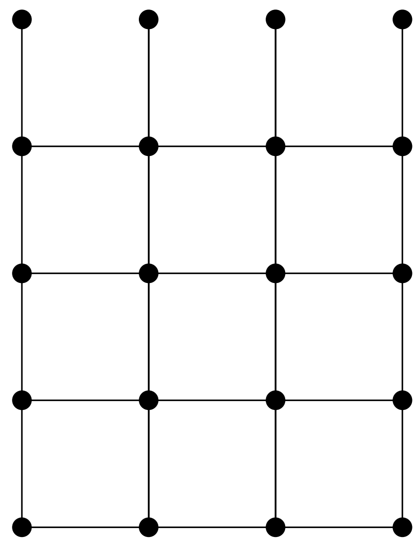

Figure 1. A grid.

for an example. Let G be a graph. A *coupling* on G is a function $\lambda : E(G) \to \{0, 1, \infty\}$. If G is a grid we call the pair (G, λ) a *coupled grid.* Given a graph G and a coupling λ on G we wish to define a new graph H, which we shall call the *realization* of (G, λ). The graph H is obtained from G by deleting all edges α with $\lambda(\alpha) = 1$ and contracting all edges α with $\lambda(\alpha) = \infty$. More precisely, let G_1 be the subgraph of G with vertex-set $V(G)$ and edge-set $\{e \in E(G) : \lambda(e) = \infty\}$. Then $V(H)$ is the set of all connected components of G_1, $E(H) = \{e \in E(G) : \lambda(e) = 0\}$, and $e \in E(H)$ is incident with $x \in V(H)$ if and only if an end of e belongs to $V(x)$, the vertex-set of the component x of G_1. This is illustrated in Figure 2, where we use the convention that edges α with $\lambda(\alpha) = \infty$ are drawn thicker and edges α with $\lambda(\alpha) = 1$ are not drawn. We need the following well-known result. A proof may be found in [9, Theorem (1.5)].

(2.2) *For every planar graph H there exists a grid G and a coupling λ on G such that H is isomorphic to the realization of* (G, λ).

Let G be a graph, and let λ be a coupling on G. A *loop* in (G, λ) is a cycle C in G such that for some edge $e \in E(C)$, $\lambda(e) = 0$ and $\lambda(f) = \infty$ for all $f \in E(C) - \{e\}$. We say that (G, λ) is *loopless* if it has no loop. A 4-*coloring* of (G, λ) is a function $c : V(G) \to \{1, 2, 3, 4\}$ such that $c(u) \neq c(v)$ whenever u, v are the ends of an edge $e \in E(G)$ with $\lambda(e) = 0$ and

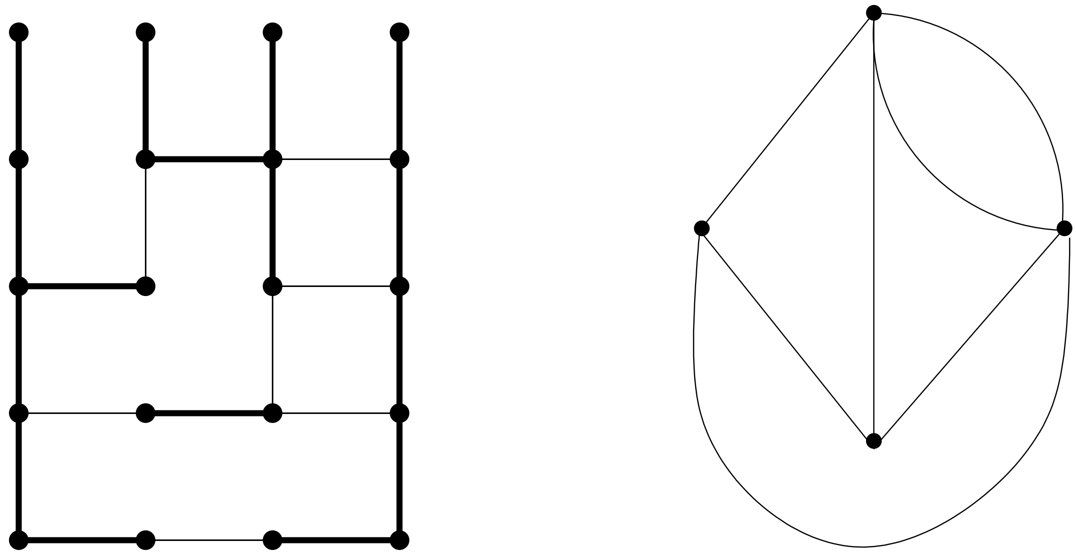

Figure 2. A coupled grid and its realization.

$c(u) = c(v)$ whenever u, v are the ends of an edge $e \in E(G)$ with $\lambda(e) = \infty$. We say that (G, λ) is 4-*colorable* if it admits at least one 4-coloring. The following is immediate.

(2.3) *Let G be a graph, let λ be a coupling on G, and let H be the realization of (G, λ). Then H is loopless if and only if (G, λ) is loopless, and H is 4-colorable if and only if (G, λ) is 4-colorable.*

3. THE POTTS MODEL REPRESENTATION

Let $n \geq 1$ be an integer, and let Σ be the set of all mappings $\{0, 1, \ldots, n\} \to \{1, 2, 3, 4\}$. We define $\Sigma \times \Sigma$ matrices $E_0, E_1, \ldots, E_{2n}$ as follows. For $i = 0, 1, \ldots, n$ we define

$$(E_{2i})_{\sigma\sigma'} = \begin{cases} 1/2 & \text{if } \sigma(j) = \sigma'(j) \text{ for all } j \in \{0, 1, \ldots, n\} - \{i\} \\ 0 & \text{otherwise} \end{cases}$$

and for $i = 1, 2, \ldots, n$ we define

$$(E_{2i-1})_{\sigma\sigma'} = \begin{cases} 2 & \text{if } \sigma = \sigma' \text{ and } \sigma(i) = \sigma(i-1) \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that the matrices $E_0, E_1, \dots, E_{2n}$ satisfy the relations (1), (2), (3) of T_{2n}. The matrix algebra generated by $E_0, E_1, \dots, E_{2n}$ is called the *Potts model representation*, and will be denoted by P_{2n}. Let $\Phi(1) = I$, the identity matrix, and for $i = 0, 1, \dots, 2n$ let $\Phi(e_i) = E_i$. Then Φ uniquely extends to a homomorphism $T_{2n} \to P_{2n}$. The following is a special case of [4, Theorem 2.8.5].

(3.1) *For every $n \geq 1$ there exists a mapping $Tr : T_n \to \mathbf{Q}$ such that*

(i) $Tr(x + y) = Tr(x) + Tr(y)$ *for all* $x, y \in T_n$,

(ii) $Tr(1) = 4^{n+1}$,

(iii) $Tr(xy) = Tr(yx)$ *for all* $x, y \in T_n$,

(iv) $Tr(we_i) = \frac{1}{2} Tr(w)$ *for all* $i = 0, 1, \dots, n$ *and all* w *in the subalgebra of* T_n *generated by* $1, e_0, e_1, \dots, e_{i-1}$, *and*

(v) *for every nonzero element* $x \in T_n$ *there exists* $y \in T_n$ *such that* $Tr(xy) \neq 0$.

We deduce

(3.2) *The homomorphism $\Phi : T_{2n} \to P_{2n}$ is an isomorphism.*

Proof. Let $tr(A)$ denote the usual trace of a matrix; then conditions (i)–(iv) of (3.1) imply that $Tr(x) = tr(\Phi(x))$ for every $x \in T_{2n}$. Now let x be a nonzero element of T_{2n}. It suffices to show that $\Phi(x)$ is a nonzero matrix. By (3.1) (v) there exists $y \in T_n$ such that

$$0 \neq Tr(xy) = tr(\Phi(xy)) = tr(\Phi(x)\Phi(y)),$$

which implies that $\Phi(x)$ is a nonzero matrix, as desired. □

4. GRIDS AND WORDS

Let $w \in \mathcal{W}_n$, and let us write w as $w = w_1 w_2 \dots w_{2t}$, a concatentation of $2t$ (possibly null) words, where for each $i = 1, 2, \dots, 2t$ and each $j = 0, 1, \dots, n$

(i) at most one of e_j, r_j occurs in w_i, and if it does, then it occurs at most once,

(ii) if e_j or r_j occurs in w_i, then i and j have the same parity.

We say that $w = w_1 w_2, \ldots, w_{2t}$ is a *decomposition* of w. Clearly every word $w \in \mathcal{W}_n$ has a decomposition.

Let $w = w_1 w_2 \ldots w_{2t}$ be a decomposition of a word $w \in \mathcal{W}_{2n}$. With this decomposition we associate the pair (G, λ), where G is the $n \times t$ grid, and λ is a coupling on G, defined as follows. Let $i \in \{1, 2, \ldots, n\}$ and $j \in \{0, 1, \ldots, t-1\}$, and let α denote the edge of G with ends $(i-1, j)$ and (i, j). We put

$$\lambda(\alpha) = \begin{cases} 0 & \text{if } r_{2i-1} \text{ occurs in } w_{2j+1} \\ \infty & \text{if } e_{2i-1} \text{ occurs in } w_{2j+1} \\ 1 & \text{otherwise.} \end{cases}$$

Now let $i \in \{0, 1, \ldots, n\}$ and $j \in \{1, 2, \ldots, t\}$, and let β denote the edge of G with ends $(i, j-1)$ and (i, j). We put

$$\lambda(\beta) = \begin{cases} 0 & \text{if } r_{2i} \text{ occurs in } w_{2j} \\ 1 & \text{if } e_{2i} \text{ occurs in } w_{2j} \\ \infty & \text{otherwise.} \end{cases}$$

We say that (G, λ) is *associated* with the decomposition $w = w_1 w_2 \ldots w_{2t}$. Conversely, for every coupled grid (G, λ) there exists a word w and a decomposition $w = w_1 w_2 \ldots w_{2t}$ of w such that (G, λ) is the associated coupled grid. Moreover, if $w = w'_1 w'_2 \ldots w'_{2t'}$ is another decomposition of the same word w such that (G, λ) is also associated with the decomposition $w'_1 w'_2 \ldots w'_{2t'}$, then $t = t'$, and for each $i = 1, 2, \ldots, 2t$, w_i and w'_i differ only by a permutation of their entries. Thus w and w' are exterior equivalent.

(4.1) *Let $n \geq 1$ be an integer, let $w \in \mathcal{W}_{2n}$, let $w = w_1 w_2 \ldots w_{2t}$ and $w = w'_1 w'_2 \ldots w'_{2t'}$ be two decompositions of w, and let (G, λ), (G', λ') be the respective associated coupled grids. Then the realizations of (G, λ) and (G', λ') are isomorphic.*

Proof. Let $w = w_1 w_2 \ldots w_{2t}$ and $w = w'_1 w'_2 \ldots w'_{2t'}$ be two decompositions of w such that either $t' = t - 2$ and for some integer $i \in \{1, 2, \ldots, 2t\}$ the words w_{i-1} and w_i are null, $w_j = w'_j$ for $j < i - 1$ and $w_j = w'_{j-2}$ for $j > i$, or $t = t'$ and for some integer $i \in \{2, 3, \ldots, 2t-1\}$, w_i and w'_i are null, and for some word $x \in \mathcal{W}_{2n}$ we have $w'_{i-1} = w_{i-1} x$, $w_{i+1} = x w'_{i+1}$ and $w_j = w'_j$ for all other j. In those circumstances we say that the two decompositions of w are *adjacent*. It is easy to see that

(i) if two decompositions of w are adjacent, then the realizations of the corresponding associated coupled grids are isomorphic, and

(ii) for any two decompositions d, d' of w there exist decompositions $d_0 = d, d_1, \ldots, d_k = d'$ such that d_{i-1} and d_i are adjacent for all $i = 1, 2, \ldots, k$.

The result follows from (i) and (ii). □

(4.2) *Let $n \geq 1$ be an integer, let $w_1, w_2 \in \mathcal{W}_{2n}$ be two exterior equivalent words, for $i = 1, 2$ let (G_i, λ_i) be the coupled grid associated with some decomposition of w_i, let H_i be the realization of (G_i, λ_i), and let H_i' be obtained from H_i by deleting isolated vertices. Then H_1' is isomorphic to H_2'.*

Proof. Let w_i, (G_i, λ_i), H_i and H_i' be as stated. It suffices to prove the lemma in the case when w_1 is obtained from w_2 by means of the relations (2), (3), (4), or the relations $e_i^2 = e_i$. We shall do so for the relation $e_{2i}^2 = e_{2i}$, leaving the other cases to the reader. (In the other cases H_1 and H_2 are isomorphic.)

Let us assume then that $w_1 = xe_{2i}e_{2i}y$ and $w_2 = xe_{2i}y$ for some $x, y \in \mathcal{W}_{2n}$. Let $w_1 = w_1''w_2'' \ldots w_{2t}''$ and $w_2 = w_1'w_2' \ldots w_{2t'}'$ be the decompositions of w that give rise to (G_1, λ_1) and (G_2, λ_2), respectively. By (4.1) we may assume that $t = t'$ and for some integer $j \in \{1, 2, \ldots, t-1\}$, $w_{2j}'' = w_{2j}' = w_{2j+2}'' = e_{2i}$, and w_{2j+1}'', w_{2j+1}' and w_{2j+2}' are null, and $w_{j'}'' = w_{j'}'$ for all other indices j'. Then it follows that the subgraph of G with vertex-set $\{(i, j)\}$ and no edges is an isolated vertex of H_1, and that the graph obtained from H_1 by deleting this vertex is isomorphic to H_2, as desired. □

We deduce

(4.3) *Let $n \geq 1$ be an integer, let $w_1, w_2 \in \mathcal{W}_{2n}$ be two exterior equivalent words, and for $i = 1, 2$ let (G_i, λ_i) be the coupled grid associated with some decomposition of w_i. Then (G_1, λ_1) is loopless if and only if (G_2, λ_2) is loopless.*

Proof. This follows immediately from (4.2). □

(4.4) *Let $n \geq 1$ be an integer, let $w \in \mathcal{W}_{2n}$, and let (G, λ) be the coupled grid associated with some decomposition of w. If (G, λ) has a loop of length four, then w is exterior equivalent to a word containing $e_{2j-1}r_{2j-1}$, or $e_{2j-1}r_{2j}e_{2j-1}$ or $e_{2j+1}r_{2j}e_{2j+1}$ for some j.*

Proof. Let C be a loop in (G, λ) of length four. Then the existence of C implies that w has one of the following forms: $w = xe_{2i-1}y_1r_{2i-1}z$, $w = xe_{2i-1}y_1r_{2i}y_2e_{2i-1}z$, $w = xe_{2i-1}y_1r_{2i-2}y_2e_{2i-1}z$, or $w = xr_{2i-1}y_1e_{2i-1}z$, where the words y_1 and y_2 include no terms with index within one from the indices of the entries immediately surrounding y_1 and y_2 in the expression for w. Using (3) and (4) it follows that w is exterior equivalent to one of the desired words. □

Let G be a grid, and let λ_1, λ_2 be two couplings on G such that for some edge α, $\lambda_1(\alpha) = 1$, $\lambda_2(\alpha) = \infty$ and $\lambda_1(\beta) = \lambda_2(\beta)$ for every edge $\beta \in E(G) - \{\alpha\}$. If u is a vertex of G incident with α such that $\lambda_1(\beta) = 1$ for every edge β incident with u, then we say that λ_1, λ_2 are 1-*similar*. If C is a cycle in G of length four such that $\alpha \in E(C)$ and $\lambda_2(\beta) = \infty$ for every $\beta \in E(C)$, then we say that λ_1 and λ_2 are 2-*similar*. We say that two couplings λ, λ' on G are *similar* if there exists a sequence $\lambda_0 = \lambda, \lambda_1, \ldots, \lambda_k = \lambda'$ of couplings on G such that λ_{i-1} and λ_i are 1-similar or 2-similar for all $i = 1, 2, \ldots, k$.

(4.5) *Let $n \geq 1$, let $w_1, w_2 \in \mathcal{W}_{2n}$, let G be a grid, and let λ_1, λ_2 be couplings on G such that for $i = 1, 2$ the coupled grid (G, λ_i) is associated with some decomposition of w_i. If λ_1 and λ_2 are similar, then w_1 and w_2 are exterior equivalent.*

Proof. It suffices to prove the statement in the case when λ_1 and λ_2 are 1-similar or 2-similar. Assume first that they are 1-similar, and let $u = (i, j) \in V(G)$ be as in the definition of 1-similar. Using relations (3) and (4) we may assume that w_1 has the form $w_1 = xe_{2i}e_{2i}y$ and that $w_2 = xe_{2i}e_{2i-1}e_{2i}y$ or $w_2 = xe_{2i}e_{2i+1}e_{2i}y$ or $w_2 = xe_{2i}y$. In each case we see that w_1 and w_2 are exterior equivalent, as desired.

Assume now that λ_1 and λ_2 are 2-similar, and let C be a cycle in G as in the definition of 2-similar. Then using relations (3) and (4) we may assume that w_2 has the form $w_2 = xe_{2i+1}e_{2i+1}y$ for some integer i, and that $w_1 = xe_{2i+1}y$ or $w_1 = xe_{2i+1}e_{2i}e_{2i+1}y$ or

$w_1 = xe_{2i+1}e_{2i+2}e_{2i+1}y$. In each case we see that w_1 and w_2 are exterior equivalent, as desired. $\square$

We need some terminology and a lemma. Let G be a grid. We say that a sequence $v_1, v_2, \ldots, v_k$ of distinct vertices of G is *diagonal* if there exist edge-disjoint cycles $D_1, D_2, \ldots, D_{k-1}$ in G, each of length four, such that for $i = 1, 2, \ldots, k-1$, v_{i+1} and v_i are diagonally opposite vertices of D_i. The cycles $D_1, D_2, \ldots, D_{k-1}$ are uniquely determined, and we say that $v_1, v_2, \ldots, v_k$ is a *diagonal sequence with cycles* $D_1, D_2, \ldots, D_{k-1}$. Now let C be a cycle in G. We say that a vertex v of C is a *corner* of C if the two edges of C incident with v belong to a cycle D of length four. In that case D is unique, and we say that v is a *convex corner* if the disk bounded by D is a subset of the disk bounded by C, and we say that it is a *concave corner* otherwise. Now let v be a convex corner and v' a concave corner of a cycle C, and assume that there is a diagonal sequence $v_1, v_2, \ldots, v_k$ with cycles $D_1, D_2, \ldots, D_{k-1}$ such that $v_1 = v$, $v_k = v'$, none of the vertices $v_2, v_3, \ldots, v_{k-1}$ belongs to C, the two edges of D_1 incident with v_1 belong to C, and so do the two edges incident with v_k that do not belong to D_{k-1}. It follows that $v_2, v_3, \ldots, v_{k-1}$ belong to the open disk bounded by C. In those circumstances we say that v *faces* v'. This is illustrated in Figure 3.

Assume now that $v_1, v_2, \ldots, v_k$ is a diagonal sequence with cycles $D_1, D_2, \ldots, D_{k-1}$, where the two edges of D_1 incident with v_1 belong to C (thus v_1 is a convex corner of C), and $v_2, v_3, \ldots, v_{k-1} \notin V(C)$. If $\alpha \in E(D_{k-1})$ is not incident with v_{k-1}, then we say that v_1 *faces* α. We need the following lemma.

(4.6) *Let G be a grid, let C be a cycle in G of length exceeding four, and let $\alpha_0 \in E(C)$. Then there exists a convex corner of C that faces no concave corner of C and does not face α_0.*

Proof. Since $\alpha_0 \in E(C)$, at most two convex corners of C face α_0. But the number of convex corners is equal to the number of concave corners plus four, and so the lemma follows. $\square$

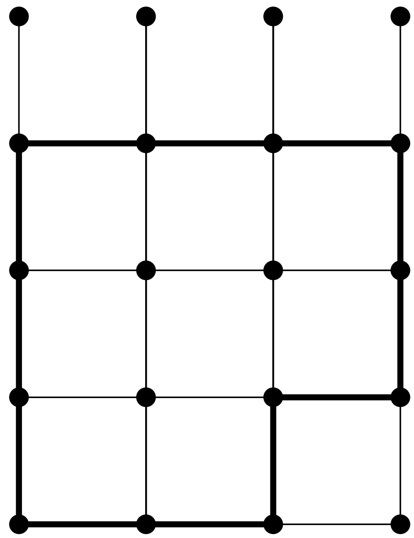

Figure 3. A convex corner facing a concave corner.

Now let λ be a coupling on G, let C be a cycle in G, and let v_1 be a convex corner of C. A (λ, C)-*stairway* based at v_1 is a diagonal sequence $v_1, v_2, \ldots, v_k$ with cycles $D_1, D_2, \ldots, D_{k-1}$ such that

(i) the two edges of D_1 incident with v_1 belong to C,

(ii) for $i = 1, 2, \ldots, k-1$, $\lambda(\alpha) = \infty$ if $\alpha \in E(D_i)$ is incident with v_i and $\lambda(\alpha) = 1$ if $\alpha \in E(D_i)$ is not incident with v_i, and

(iii) subject to (i) and (ii), k is maximum.

Thus for every convex corner v of C there is a unique (λ, C)-stairway based at v. See Figure 4.

(4.7) *Let $n, t \geq 1$, let G be the $n \times t$ grid, and let λ_0 be a coupling on G such that exactly one edge α_0 of G satisfies $\lambda(\alpha_0) = 0$. If (G, λ_0) has a loop, then λ_0 is similar to a coupling λ such that (G, λ) has a loop of length four.*

Proof. Let G, λ_0, α_0 be as stated. We choose λ and C such that

(i) λ is a coupling on G similar to λ_0,

(ii) C is a loop in (G, λ) bounding a disk Δ,

(iii) subject to (i) and (ii), the area of Δ is minimum,

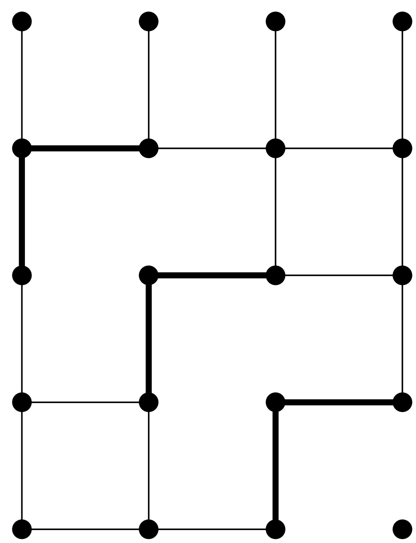

Figure 4. A (λ, C)-stairway.

(iv) subject to (i)–(iii), the number of edges $\alpha \in E(G)$ with $\lambda(\alpha) = \infty$ is minimum, and

(v) subject to (i)–(iv), $\eta(\lambda, C)$ is minimum,

where $\eta(\lambda, C) = 0$ if C has length four, and otherwise it is defined as the minimum length of a (λ, C)-stairway based at a convex corner v of C such that v does not face α_0 or a concave corner of C. It follows from (4.6) that this quantity is well-defined. The choice of λ and C is possible, because λ_0 and any loop in (G, λ_0) satisfy (i) and (ii). We shall prove that C has length four, which will complete the proof. We first notice the following.

($*$) *No vertex of $G \backslash V(C)$ is incident with exactly one edge $\alpha \in E(G)$ satisfying $\lambda(\alpha) = \infty$.*

To prove ($*$) suppose for a contradiction that $v \in V(G) - V(C)$ is incident with α, where $\lambda(\alpha) = \infty$, and α is the only such edge incident with v. Since $v \notin V(C)$ we see that $\lambda(\beta) = 1$ for every edge $\beta \neq \alpha$ incident with v. Let $\lambda'(\alpha) = 1$ and let $\lambda'(\beta) = \lambda(\beta)$ for all $\beta \in E(G) - \{\alpha\}$. Then λ' is similar to λ, the area of Δ has not changed, and yet (G, λ') has fewer edges β satisfying $\lambda(\beta) = \infty$, contrary to (iv). This proves ($*$).

Suppose for a contradiction that C has length strictly greater than four. Let v_1 be a convex corner of C not facing α_0 or any concave corner such that the (λ, C)-stairway based at v_1, say $v_1, v_2, \ldots, v_k$, satisfies $k = \eta(\lambda, C)$. Let $D_1, D_2, \ldots, D_{k-1}$ be the circuits of length four as in the definition of diagonal sequence. We claim that $v_2, v_3, \ldots, v_k \notin V(C)$. To see this, suppose for a contradiction that one of those vertices belongs to $V(C)$, and let $i \in \{2, 3, \ldots, k\}$ be the minimum integer such that $v_i \in V(C)$. Since $\lambda(\beta) = 1$ for both edges β of D_{i-1} incident with v_i, we deduce that the two edges of D_i incident with v_i belong to C (where D_k is defined in the natural way). But then v_i is a concave corner of C, and v_1 faces v_i, a contradiction. Thus $v_2, v_3, \ldots, v_k \notin V(C)$.

Next we claim that every edge α incident with v_k satisfies $\lambda(\alpha) = 1$. To prove this suppose otherwise. Since $v_k \notin V(C)$ and α_0 is the only edge with $\lambda(\alpha_0) = 0$, we see that v_k is incident with an edge α satisfying $\lambda(\alpha) = \infty$. By $(*)$ v_k is incident with at least two such edges, and hence it is incident with exactly two such edges, say α' and β'. Then α' and β' are the two edges incident with v_k that do not belong to D_{k-1}. Let D_k be the unique cycle of length four in G with $\alpha', \beta' \in E(D_k)$, and let v_{k+1} be the vertex of D_k not equal or adjacent to v_k. Let γ', δ' be the two remaining edges of D_k. Since v_1 does not face α_0 we deduce that γ' and δ' are not equal to α_0, and hence $\lambda(\gamma') \neq 0 \neq \lambda(\delta')$. The sequence $v_1, v_2, \ldots, v_{k+1}$ is not a (λ, C)-stairway by condition (iii) in the definition of (λ, C)-stairway. Thus one of $\lambda(\gamma')$, $\lambda(\delta')$ is not equal to one, and so we may assume that $\lambda(\gamma') = \infty$, because α_0 is the only edge with $\lambda(\alpha_0) = 0$. If $\lambda(\delta') = \infty$, then the coupling obtained from λ by changing the value of $\lambda(\alpha')$ to 1 is 2-similar to λ, contrary to (iv). Thus $\lambda(\delta') = 1$. Now let λ' be obtained from λ first by changing the value of $\lambda(\delta')$ to ∞, then changing the value of $\lambda(\alpha')$ to 1, and finally changing the value of $\lambda(\beta')$ to 1. The first two changes are done using 2-similarity, and the third is done using 1-similarity. Thus λ' is similar to λ, contrary to (iv). This proves that every edge α incident with v_k satisfies $\lambda(\alpha) = 1$.

Let $E(D_{k-1}) = \{\alpha, \beta, \gamma, \delta\}$, where α, β are incident with v_{k-1} and γ, δ are incident with v_k. Let λ_1 be the coupling on G defined by $\lambda_1(\alpha) = \lambda_1(\beta) = 1$, $\lambda_1(\gamma) = \infty$, $\lambda_1(\delta) = \infty$, and $\lambda_1(\epsilon) = \lambda(\epsilon)$ for all $\epsilon \in E(G) - E(D_{k-1})$. Then λ_1 is similar to λ. (To see this, first change $\lambda(\gamma)$ to ∞, then change $\lambda(\delta)$ to ∞, then change $\lambda(\beta)$ to 1, and finally change $\lambda(\alpha)$ to 1. The first and last changes are done using 1-similarity, and the other two are done using 2-similarity.) Now if $k \geq 2$, then λ_1 contradicts (v), and hence $k = 1$. Let C_1 be the cycle in G with edge-set $(E(C) - \{\alpha, \beta\}) \cup \{\gamma, \delta\}$; then the pair λ_1, C_1 contradicts condition (iii). This completes the proof of the fact that C has length four, and hence finishes the proof of the lemma. □

(4.8) *Let $n \geq 1$ be an integer, let $w \in \mathcal{W}_{2n}$, and let (G, λ) be the coupled grid associated with some decomposition of w. Then w is loopless if and only if (G, λ) is loopless.*

Proof. Let d be the decomposition of w that gives rise to (G, λ). Suppose first that w is not loopless. Then some complete reduction w' of w is exterior equivalent to a word w'' that contains $e_{2j-1}r_{2j-1}$ or $e_{2j-1}r_{2j}e_{2j-1}$ or $e_{2j+1}r_{2j}e_{2j+1}$. Then d induces a decomposition d' of w' in the natural way; let (G', λ') be the coupled grid associated with d'. Then $G = G'$. Let (G'', λ'') be the coupled grid associated with some decomposition of w''. Since w'' contains one of the words stated above, it follows that (G'', λ'') has a loop (of length four). Thus (G', λ') has a loop by (4.3). But a loop in (G', λ') is also a loop in (G, λ), as desired.

To prove the converse let (G, λ) have a loop, say C. We define a coupling λ' on G as follows

$$\lambda'(\alpha) = \begin{cases} 1 & \text{if } \alpha \notin E(C) \text{ and } \lambda(\alpha) = 0 \\ \lambda(\alpha) & \text{otherwise.} \end{cases}$$

It follows that (G, λ') is associated with a decomposition of some complete reduction w' of w. Notice that C is a loop in (G, λ'), and that there is exactly one edge, say $\alpha_0 \in E(G)$, such that $\lambda'(\alpha_0) = 0$.

By (4.7) there exists a coupling λ'' on G such that (G, λ') and (G, λ'') are similar, and such that (G, λ'') has a loop of length four. Let $w'' \in \mathcal{W}_{2n}$ be such that (G, λ'') is associated with some decomposition of w''; by (4.5) the words w' and w'' are exterior equivalent. But w'' is exterior equivalent to a word containing $e_{2j-1}r_{2j-1}$ or $e_{2j-1}r_{2j}e_{2j-1}$ or $e_{2j+1}r_{2j}e_{2j+1}$ by (4.4), and hence so is w', as desired. □

5. TRANSFER MATRICES

Let $n, t \geq 1$ be integers, let G be the $n \times t$ grid, and let λ be a coupling on G. Let Σ be as in Section 3, and let a $\Sigma \times \Sigma$ matrix M be defined by saying that $(M)_{\sigma\sigma'}$, the (σ, σ')-entry of M, is the number of 4-colorings c of (G,λ) such that $c((i,0))=\sigma(i)$ and $c((i,t))=\sigma'(i)$ for all $i=0,1,\ldots,n$. The matrix M is called the *transfer matrix* of (G,λ).

We now explain how transfer matrices can be calculated using the Potts model representation. Let n,t, and (G,λ) be as above. With each edge α of G we associate a matrix $X(\alpha)\in P_{2n}$ as follows. If α has ends $(i-1,j)$ and (i,j), then we define $X(\alpha)=I+\frac{1}{2}(\lambda(\alpha)-1)E_{2i-1}$ if $\lambda(\alpha)\neq\infty$ and $X(\alpha)=\frac{1}{2}E_{2i-1}$ otherwise. If α has ends $(i,j-1)$ and (i,j), then we define $X(\alpha)=(\lambda(\alpha)-1)I+2E_{2i}$ if $\lambda(\alpha)\neq\infty$ and $X(\alpha)=I$ otherwise. For $j=1,2,\ldots,t$ let ρ_{2j-1} be the set of all edges of G with ends $(i-1,j-1)$ and $(i,j-1)$ for some $i=1,2,\ldots,n$, and let ρ_{2j} be the set of all edges of G with ends $(i,j-1)$ and (i,j) for some $i=0,1,\ldots,n$. Since the matrices E_i satisfy the relations (3) we see that $X(\alpha)$ and $X(\beta)$ commute whenever $\alpha,\beta\in\rho_j$ for some j, and hence the expression

$$\prod_{\alpha\in\rho_1}X(\alpha)\prod_{\alpha\in\rho_2}X(\alpha)\cdot\ldots\cdot\prod_{\alpha\in\rho_{2t}}X(\alpha). \tag{7}$$

is well-defined.

(5.1) *The transfer matrix of a coupled grid* (G,λ) *is given by formula* (7).

Proof. We proceed by induction on t. If $t=0$, then the graph G consists of $n+1$ isolated vertices, and hence M is the identity matrix. Since $t=0$ the expression (7) consists of the empty product, and hence the result holds.

Let now $t>0$, and assume that the result holds for all smaller integers. Let G' be the $n\times(t-1)$ grid; then G' is a subgraph of G. Let λ' be the restriction of λ to $E(G')$, let J be the subgraph of G with vertex-set $\{(i,j): i\in\{0,1\ldots,n\}, j\in\{t-1,t\}\}$ and edge-set $E(G)-E(G')$ and let μ be the restriction of λ to $E(J)$. Let M,M' be the transfer matrices of G,G', respectively. By the induction hypothesis

$$M'=\prod_{\alpha\in\rho_1}X(\alpha)\prod_{\alpha\in\rho_2}X(\alpha)\cdot\ldots\cdot\prod_{\alpha\in\rho_{2n-2}}X(\alpha),$$

and hence it suffices to show that $M=M'X$, where

$$X=\prod_{\alpha\in\rho_{2n-1}}X(\alpha)\prod_{\alpha\in\rho_{2n}}X(\alpha).$$

So we need to show that for all $\sigma_1, \sigma_2 \in \Sigma$,

$$(M)_{\sigma_1\sigma_2} = \sum_{\sigma\in\Sigma} (M')_{\sigma_1\sigma}(X)_{\sigma\sigma_2}.$$

This is indeed true, and follows from the following fact. Let $c : V(G) \to \{1,2,3,4\}$ be defined by $c((i,t-1)) = \sigma(i)$ and $c((i,t)) = \sigma_2(i)$. Then $(X)_{\sigma\sigma_2} = 1$ if c is a 4-coloring of (J,μ), and $(X)_{\sigma\sigma_2} = 0$ otherwise.

Thus $M = M'X$, and the result follows. □

(5.2) *Let $n \geq 1$ be an integer, let $w \in \mathcal{W}_{2n}$, and let (G,λ) be the coupled grid associated with some decomposition $w = w_1 w_2 \dots w_{2t}$ of w. Then (G,λ) is 4-colorable if and only if w represents a nonzero element of T_{2n}.*

Proof. It follows from the definition of the associated coupled grid and from the definition of $X(\alpha)$ that $\prod_{\alpha\in\rho_i} X(\alpha) = \Phi(w_i)$ for all $i = 1,2,\dots,2t$, where Φ is defined prior to (3.1). From (3.2) and (5.1) we deduce that $w \neq 0$ in T_{2n} if and only if the transfer matrix M of (G,λ) is a nonzero matrix. But M is a nonzero matrix if and only if (G,λ) is 4-colorable, as desired. □

6. PROOF OF THE EQUIVALENCE

We are now ready to prove the equivalence of (1.1) and (2.1).

(6.1) *Theorem* (2.1) *implies* (1.1).

Proof. Let $n \geq 1$ be an integer. Since T_n is a subalgebra of T_{n+1}, we may assume that n is even. Let $w \in \mathcal{W}_n$, and let (G,λ) be the coupled grid associated with some decomposition of w. By (4.8) the word w is loopless if and only if (G,λ) is loopless. Let H be the realization of (G,λ). By (2.3), (G,λ) is loopless if and only if H is loopless. By (2.1) the graph H is loopless if and only if it is 4-colorable. By (2.3) H is 4-colorable if and only if (G,λ) is 4-colorable. By (5.2) (G,λ) is 4-colorable if and only if w is nonzero in T_n. Thus w is loopless if and only if it represents a nonzero element in T_n, as desired. □

(6.2) *Theorem* (1.1) *implies* (2.1).

Proof. Let H be a planar graph. Clearly, if H has a loop, then it is not 4-colorable. Conversely, suppose that H is loopless. By (2.2) there exists a coupled grid (G, λ) such that H is isomorphic to the realization of (G, λ). By (2.3) the coupled grid (G, λ) is loopless. Let n be such that G is an $n \times t$ grid for some integer t. Let $w \in \mathcal{W}_{2n}$ be such that (G, λ) is the coupled grid associated with some decomposition of w. By (4.8) the word w is loopless, and hence represents a nonzero element of T_{2n} by (1.1). From (5.2) we deduce that (G, λ) is 4-colorable. By (2.3) H is 4-colorable, as desired. □

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