DIGRAPH MINORS AND ALGORITHMS

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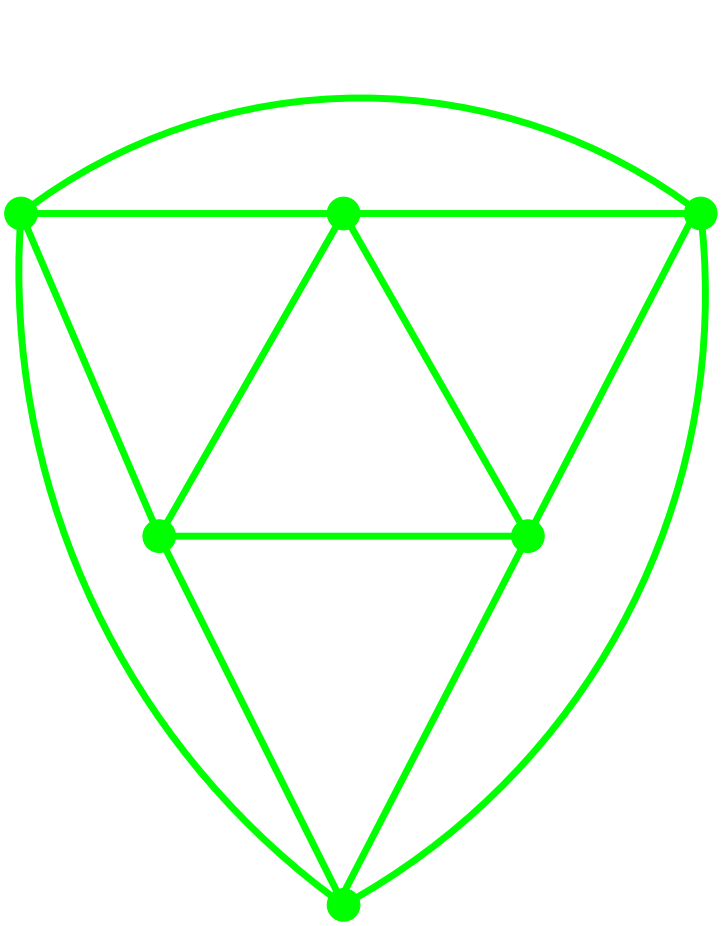
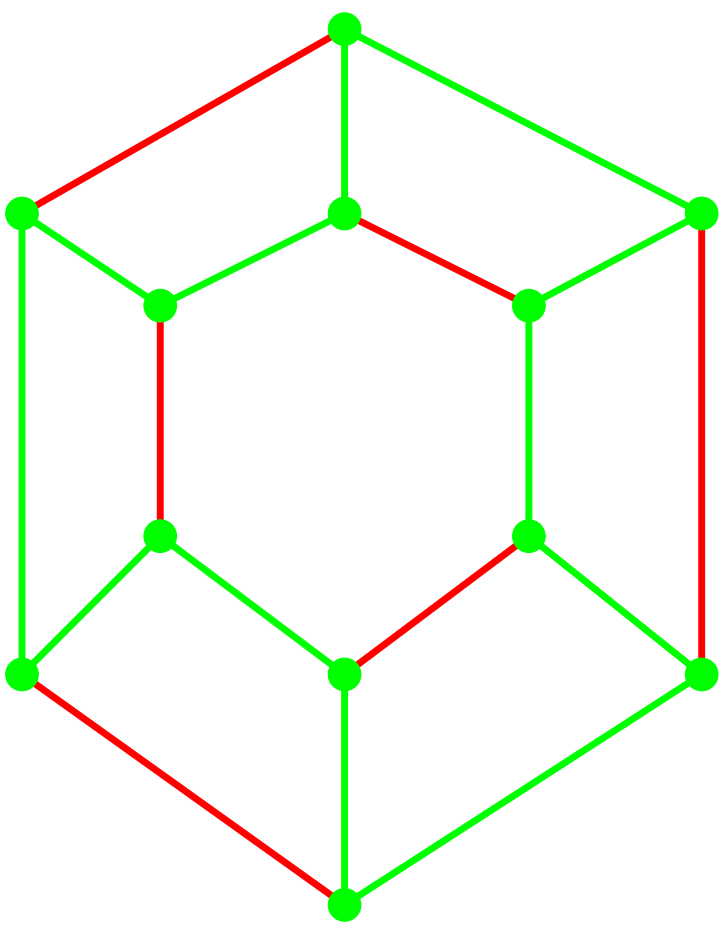
OUTLINE

- Review of graphs minors
- Applications of graph minors
- Digraph minors
- Algorithms
- Open problems

DEFINITION A graph H is a minor of a graph G if H can be obtained from a subgraph of G by contracting edges.

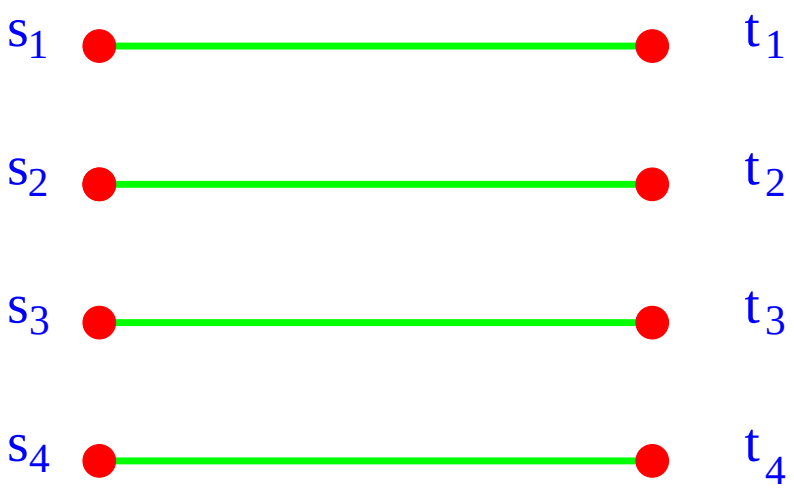




THEOREM A (Robertson and Seymour) For every infinite set of graphs, one is a minor of another.

THEOREM B (Robertson and Seymour) For every fixed graph H there is an $O(n^3)$ algorithm to test if an input graph has an H minor.

THEOREM B' (Robertson and Seymour) For every fixed integer k there is an $O(n^3)$ algorithm to solve the k DISJOINT PATHS PROBLEM.


s_1
t_1
s_2
t_2
s_3
t_3
s_4
t_4

COROLLARY For every minor-closed family $\mathcal{F}$ there is an $O(n^3)$ membership test.

PROOF Let $\mathcal{L}$ be the minimal non-members of $\mathcal{F}$. No member of $\mathcal{L}$ is a minor of another. By Theorem A $\mathcal{L}$ is finite. By Theorem B membership in $\mathcal{F}$ can be tested by testing the non-containment of members of $\mathcal{L}$.

NOTE Guarantees the existence of an algorithm; gives no clue how to construct one.

COROLLARY There is an $O(n^3)$ algorithm to test if a graph has a knotless embedding.

NOTE No explicit algorithm known.

A tree-decomposition of a graph G is (T, W), where T is a tree and $W = (W_t : t \in V(T))$ satisfies

(T1) $\bigcup_{t \in V(T)} W_t = V(G)$,

(T2) if $t' \in T[t, t'']$, then $W_t \cap W_{t''} \subseteq W_{t'}$,

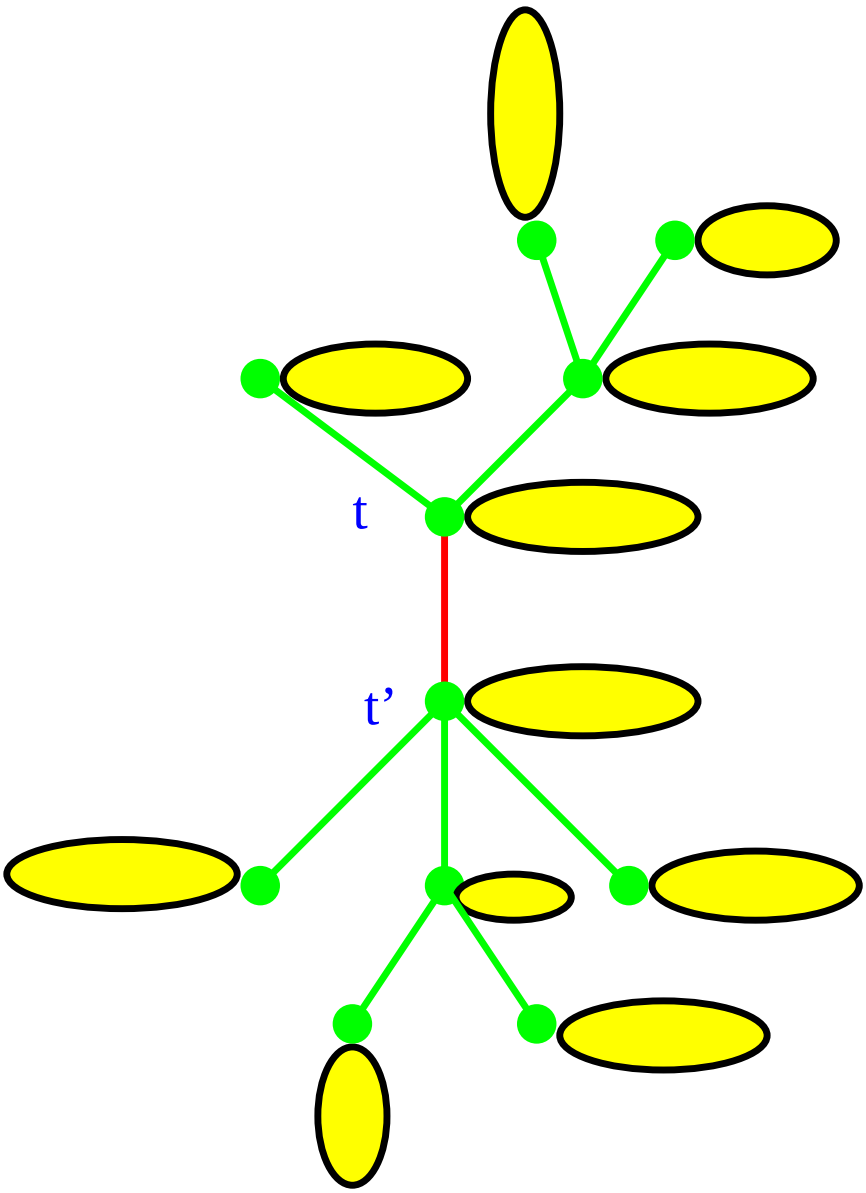
(T3) $\forall uv \in E(G) \exists t \in V(T)$ s.t. $u, v \in W_t$.

The width is $\max(|W_t| - 1 : t \in V(T))$.

The tree-width of G is the minimum width of a tree-decomposition of G.

A haven of order k in G assigns to every $X \in [V(G)]^{<k}$ the vertex-set of a component of $G \backslash X$ such that

(H) $X \subseteq Y \in [V(G)]^{<k} \Rightarrow \beta(Y) \subseteq \beta(X)$.


t
t’

A tree-decomposition of a graph G is (T, W), where T is a tree and $W = (W_t : t \in V(T))$ are subsets of $V(G)$ such that

(T) for every edge $tt' \in E(T)$ and every component H of $G \backslash (W_t \cap W_{t'})$, the components of $T \backslash tt'$ can be numbered R_1 and R_2 so that $V(H) \subseteq \bigcup_{r \in R_1} W_r - \bigcup_{r \in R_2} W_r$.

The width is $\max(|W_t| - 1 : t \in V(T))$.

The tree-width of G is the minimum width of a tree-decomposition of G.

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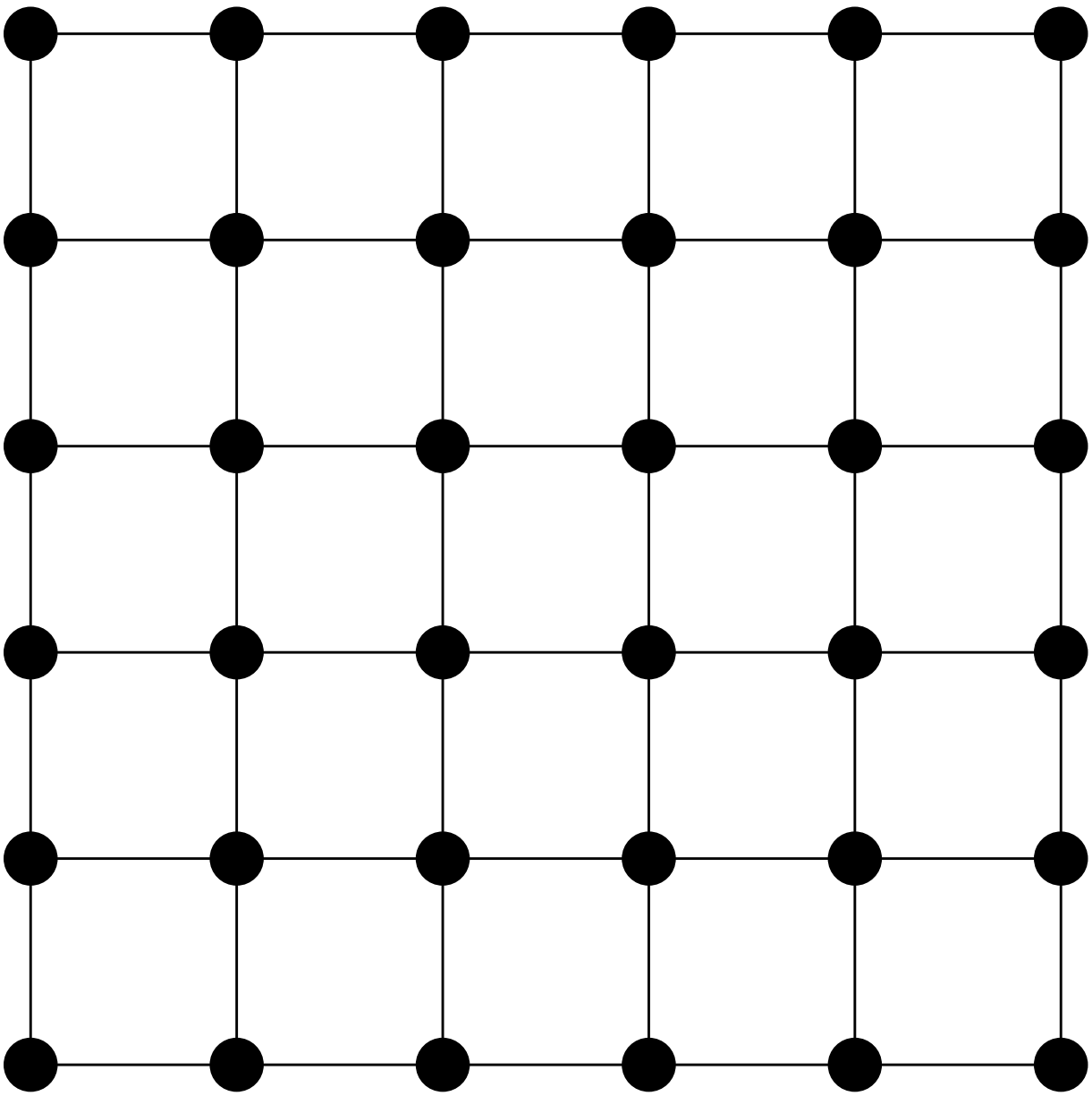
Cops and robbers. Fix a graph G and an integer k. There are k cops, they move slowly in helicopters. There is a robber, who moves infinitely fast along cop-free paths. He can see a helicopter landing, and can run to a safe place before the chopper lands.

Fact. A tree-decomposition of width $k-1$ gives a search strategy for k cops.

Fact. A haven gives an escape strategy for the robber.

THEOREM (Seymour, RT) G has a haven of order $k \Leftrightarrow G$ has tree-with at least $k-1$

COROLLARY Search strategy $\Rightarrow$ monotone search strategy.



THEOREM (Robertson, Seymour, RT)
Every graph of tree-width at least 20^{2g^5} has a $g \times g$ grid minor.

THEOREM (Bodlaender) For every k there is a linear-time algorithm to decide whether $\mathsf{tw}(G) \leq k$.

THEOREM (Arnborg, Proskurowski, ...)
Many problems can be solved in linear time when restricted to graphs of bounded tree-width.

Tree-width is useful in

- theory
- design of theoretically fast algorithms
- practical computations

FEEDBACK VERTEX-SET FOR FIXED k

INSTANCE A graph G

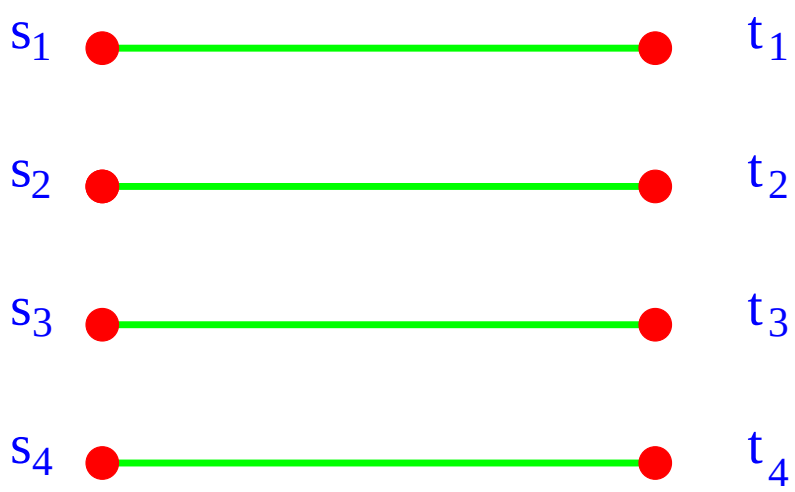
QUESTION Is there a set $X \subseteq V(G)$ such that $|X| \leq k$ and $G \backslash X$ is acyclic?

ALGORITHM If $\mathsf{tw}(G)$ is small use bounded tree-width methods. Otherwise answer "no". That's correct, because big tree-width $\Rightarrow$ big grid $\Rightarrow$ $k+1$ disjoint cycles $\Rightarrow$ X does not exist.

k DISJOINT PATHS IN PLANAR GRAPHS

INSTANCE A planar graph G, vertices $s_1, s_2, \ldots, s_k, t_1, t_2, \ldots, t_k$ of G

QUESTION Are there disjoint paths $P_1, .., P_k$ such that P_i has ends s_i and t_i?



ALGORITHM tw(G) small $\Rightarrow$ bounded tree-width methods. Otherwise big grid minor $\Rightarrow$ big grid minor with the terminals outside. The middle vertex of this grid minor can be deleted, without affecting the feasibility of the problem.

SEPARATORS

Let G have n vertices. A separator in G is a set S such that every component of $G \backslash S$ has at most $2n/3$ vertices.

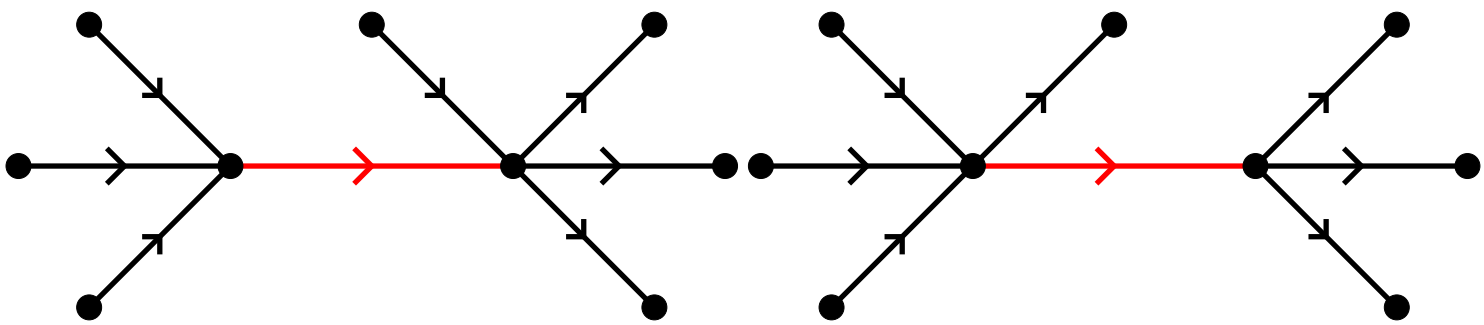
THEOREM (Lipton, Tarjan) Every planar graph has a separator of size at most $\sqrt{8n}$.

Alon, Seymour, RT found a short proof and improved to $\sqrt{4.5n}$.

THEOREM (Alon, Seymour, RT)
Every graph with no minor isomorphic to K_t has a separator of size at most $\sqrt{t^3 n}$.

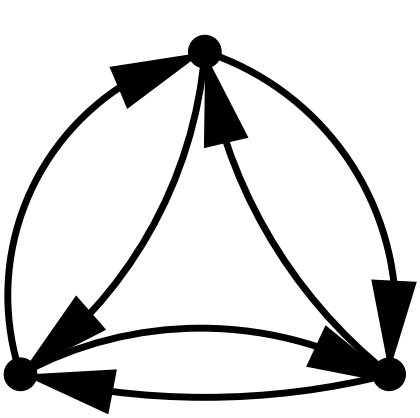
DIGRAPH MINORS

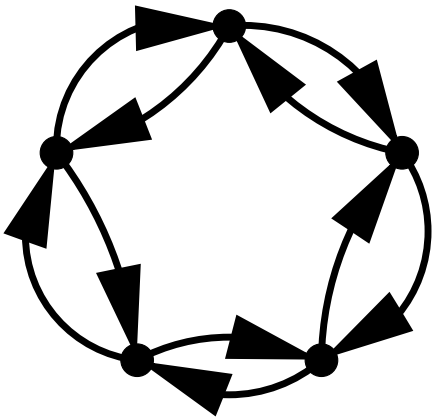
An edge in a digraph is contractible if either it is the only edge leaving its tail, or it is the only edge entering its head.



A digraph D is a butterfly minor of a digraph D' if D can be obtained from a subdigraph of D' by contracting contractible edges.

A digraph is even if every subdivision has an even directed cycle. An odd double cycle, O_{2k+1}:





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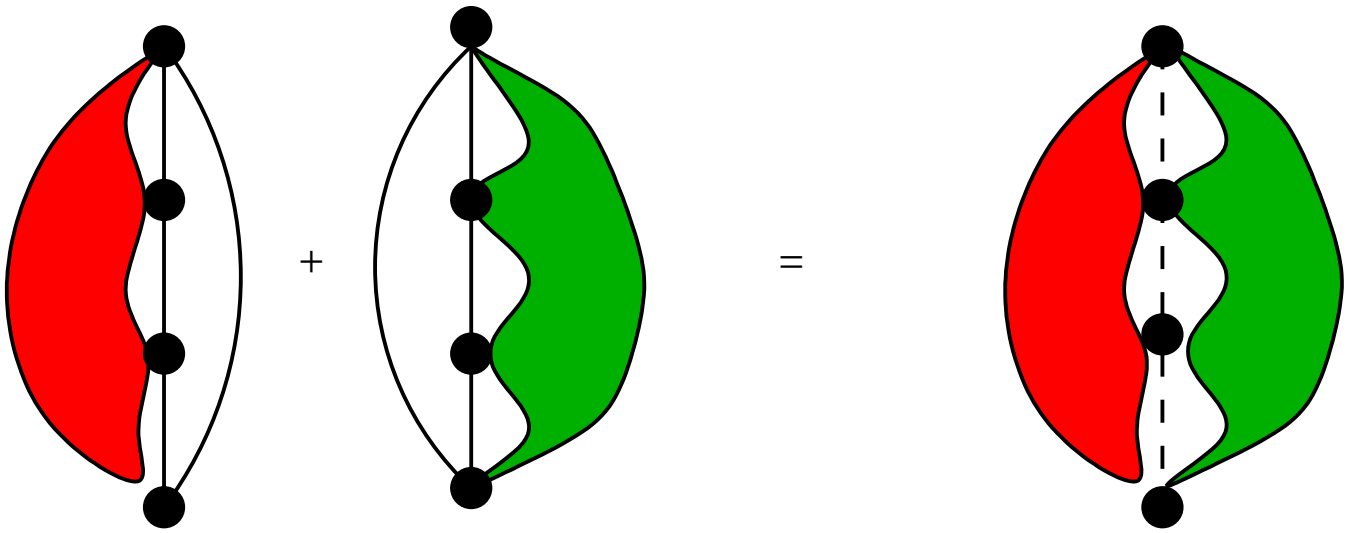
THEOREM (Seymour, Thomassen) A digraph is not even $\Leftrightarrow$ it has no odd double cycle minor.

THEOREM (McCuaig; Robertson, Seymour, RT) A digraph is not even $\Leftrightarrow$ it can be obtained from strongly planar digraphs and F_7 by means of 0-, 1-, 2-, 3-, and 4-sums.

digraph $D \rightarrow$ bipartite graph G, p.m. M

THEOREM (McCuaig; Robertson, Seymour, RT) A strongly 2-connected digraph D is not even $\Leftrightarrow D \cong F_7$ or G can be obtained from planar bipartite graphs by means of C_4-sums.



COROLLARY There is an $O(n^3)$ algorithm to decide if a directed graph has an even directed cycle.

OPEN QUESTION Can strong 2-connectivity be tested in linear time?

THEOREM (McCuaig) Let D be a digraph. D has no two disjoint cycles $\Rightarrow$ D has a set X of at most three vertices such that $D \backslash X$ is acyclic.

THEOREM (Reed, Robertson, Seymour, RT) There is a function f such that:
D has no k disjoint cycles $\Rightarrow$ D has a set X of at most $f(k)$ vertices such that $D \backslash X$ is acyclic.

COROLLARY A polynomial-time algorithm to solve the DIRECTED SUBGRAPH HOMEOMORPHISM PROBLEM for digraphs with no k disjoint cycles.

Fortune, Hopcroft and Wyllie found an algorithm for acyclic graphs.

DIGRAPH MINORS

QUESTION A For an infinite set of digraphs, is one a minor of another?
HOPELESSLY FALSE

QUESTION B For every fixed digraph H is there a polynomial-time algorithm to test if an input digraph has an H minor?
OPEN
TRUE for arborescences, disjoint loops.
Suspected to be NP-hard for some H.

Both may be true for Eulerian digraphs of bounded degree. A theory of digraph minors may lead to an algorithm for the DISJOINT PATHS PROBLEM in Eulerian digraphs.

An arboreal decomposition of D is (R, X, W), where R is an arborescence, and $X = (X_e : e \in E(R))$ and $W = (W_r : r \in V(R))$ satisfy

(D1) $(W_r : r \in V(R))$ partitions $V(D)$

(D2) $\bigcup_{r>e} W_r$ induces a strong component of $D \backslash X_e$ for every $e \in E(R)$

The width is the min, over all $r \in V(R)$, of $|W_r \cup \bigcup_{e \sim r} X_e| - 1$.

The tree-width of D is the minimum width of an arboreal decomposition of D.

FACT Tree-width is minor-monotone.

A haven of order k in D assigns to every $X \in [V(D)]^{<k}$ the vertex-set of a strong component of $D \backslash X$ such that
(H) $X \subseteq Y \in [V(D)]^{<k} \Rightarrow \beta(Y) \subseteq \beta(X)$.

FACT Haven of order $k \Rightarrow \mathsf{tw}(D) \geq k-1$.

QUESTION Converse?

THEOREM (Johnson, Robertson, Seymour, RT)

tw$(D) \geq 3k - 1 \Rightarrow$ haven of order k

COPS-AND-ROBBER GAME Same as for undirected graphs, except that robber must stay within strongly connected components of the cop-free subdigraph.

A haven of order k gives an escape strategy for the robber agains $k-1$ cops, and an arboreal decomposition of width $k-1$ gives a search strategy for k cops.

REMARK The search strategy need not be monotone.

PROOF Choose an arboreal decomposition (R, X, W) of D such that

(1) $|W_r \cup \bigcup_{e \sim r} X_e| \leq 3k - 1$ for all $r \in V(R)$ of out-degree at least one,

(2) $|X_e| \leq 2k - 1$ for every $e \in E(R)$,

(3) subject to (1) and (2), the number of vertices $v \in V(D)$ in W_r of size $\geq k + 1$ is minimum.

Claim. (R, X, W) has width $\leq 3k - 2$.

ALGORITHMS

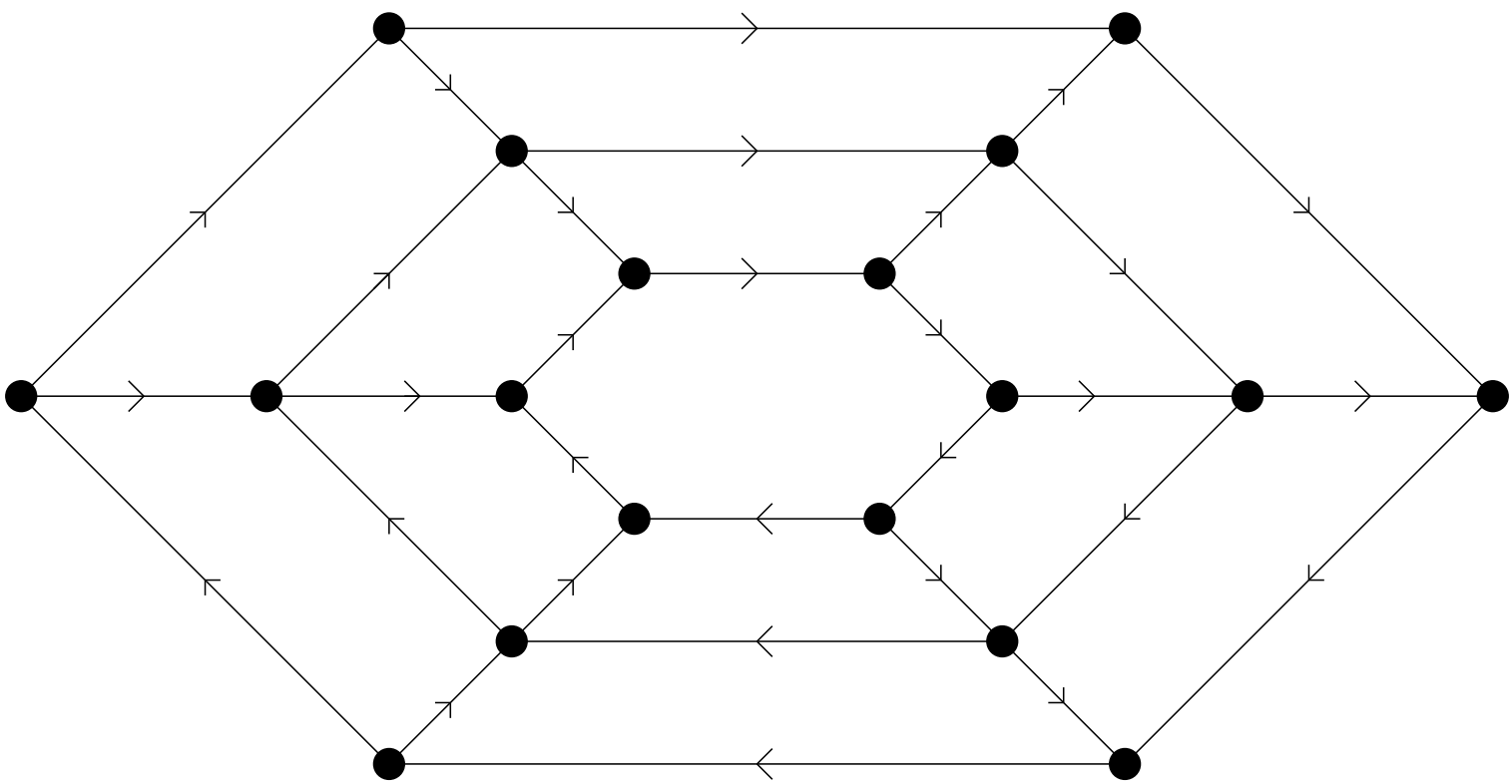
Let $Z \subseteq V(D)$, and let $S_1, \ldots, S_t$ be the strong components of $D \backslash Z$ such that no edge goes from S_j to S_i for $j > i$. Then $S = S_i \cup S_{i+1} \cup \ldots \cup S_j$ is Z-normal. If $|Z| \leq k$, then S is k-protected.

For some k-protected sets A we will compute an itinerary for A.

AXIOM 1 $A, B \subseteq V(D)$ disjoint, no edge of D has head in A and tail in B. Then an itinerary for $A \cup B$ can be computed from itineraries of A and B in time $O((|A| + |B|)^\alpha)$.

AXIOM 2 $A, B \subseteq V(D)$ disjoint sets, A is k-protected and $|B| \leq k$. Then an itinerary for $A \cup B$ can be computed from itineraries of A and B in time $O((|A| + 1)^\alpha)$.

THEOREM (Johnson, Robertson, Seymour, RT)
There is a polynomial-time algorithm for:
INPUT A digraph D with an arboreal decomposition of bounded width.
OUTPUT An itinerary for $V(D)$

Thus HAMILTON PATH, HAMILTON CIRCUIT, k-DISJOINT PATHS (k fixed) and other problems can be solved in polynomial time for digraphs of bounded tree-width.

CONJECTURE There is a function f such that every digraph of tree-width at least $f(k)$ has a cylindrical $k \times k$ grid minor.