DIRECTED TREE-WIDTH

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joint work with

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OUTLINE

- Tree-width and havens for undirected graphs
- Even directed circuits
- Packing directed circuits
- Path-width of directed graphs
- Tree-width of directed graphs
- Havens in directed graphs
- Algorithms

A tree-decomposition of a graph G is (T, W), where T is a tree and $W = (W_t : t \in V(T))$ satisfies

(T1) $\bigcup_{t \in V(T)} W_t = V(G)$,

(T2) if $t' \in T[t, t'']$, then $W_t \cap W_{t''} \subseteq W_{t'}$,

(T3) $\forall uv \in E(G) \exists t \in V(T)$ s.t. $u, v \in W_t$.

The width is $\max(|W_t| - 1 : t \in V(T))$.

The tree-width of G is the minimum width of a tree-decomposition of G.

A haven of order k in G assigns to every $X \in [V(G)]^{<k}$ the vertex-set of a component of $G \backslash X$ such that

(H) $X \subseteq Y \in [V(G)]^{<k} \Rightarrow \beta(Y) \subseteq \beta(X)$.

Consider all functions ϕ mapping graphs into integers such that

(1) $\phi(K_n) = n - 1$,

(2) G minor of $H \Rightarrow \phi(G) \leq \phi(H)$,

(3) If $G \cap H$ is a clique, then

$$\phi(G \cup H) = \max\{\phi(G), \phi(H)\}.$$

Order such functions by $\phi \leq \psi$ if $\phi(G) \leq \psi(G)$ for all G.

THEOREM (Halin) Tree-width is the maximum element in the above poset.

Cops and robbers. Fix a graph G and an integer k. There are k cops, they move slowly in helicopters. There is a robber, who moves infinitely fast along cop-free paths. He can see a helicopter landing, and can run to a safe place before the chopper lands.

Fact. A tree-decomposition of width $k-1$ gives a search strategy for k cops.

Fact. A haven gives an escape strategy for the robber.

THEOREM (Seymour, RT) G has a haven of order $k \Leftrightarrow G$ has tree-with at least $k-1$

COROLLARY Search strategy $\Rightarrow$ monotone search strategy.

THEOREM (Robertson, Seymour, RT) Every graph of tree-width at least 20^{2g^5} has a $g \times g$ grid minor.

THEOREM (Bodlaender) For every k there is a linear-time algorithm to decide whether $\mathrm{tw}(G) \leq k$.

THEOREM (Arnborg, Proskurowski, ...) Many problems can be solved in linear time when restricted to graphs of bounded tree-width.

Tree-width is useful in

- theory
- design of theoretically fast algorithms
- practical computations

FEEDBACK VERTEX-SET FOR FIXED k
INSTANCE A graph G
QUESTION Is there a set $X \subseteq V(G)$ such that $|X| \leq k$ and $G \backslash X$ is acyclic?

ALGORITHM If $\text{tw}(G)$ is small use bounded tree-width methods. Otherwise answer "no". That's correct, because big tree-width $\Rightarrow$ big grid $\Rightarrow$ $k+1$ disjoint circuits $\Rightarrow$ X does not exist.

An edge in a digraph is contractible if either it is the only edge leaving its tail, or it is the only edge entering its head.

A digraph D is a butterfly minor of a digraph D' if D can be obtained from a subdigraph of D' by contracting contractible edges.

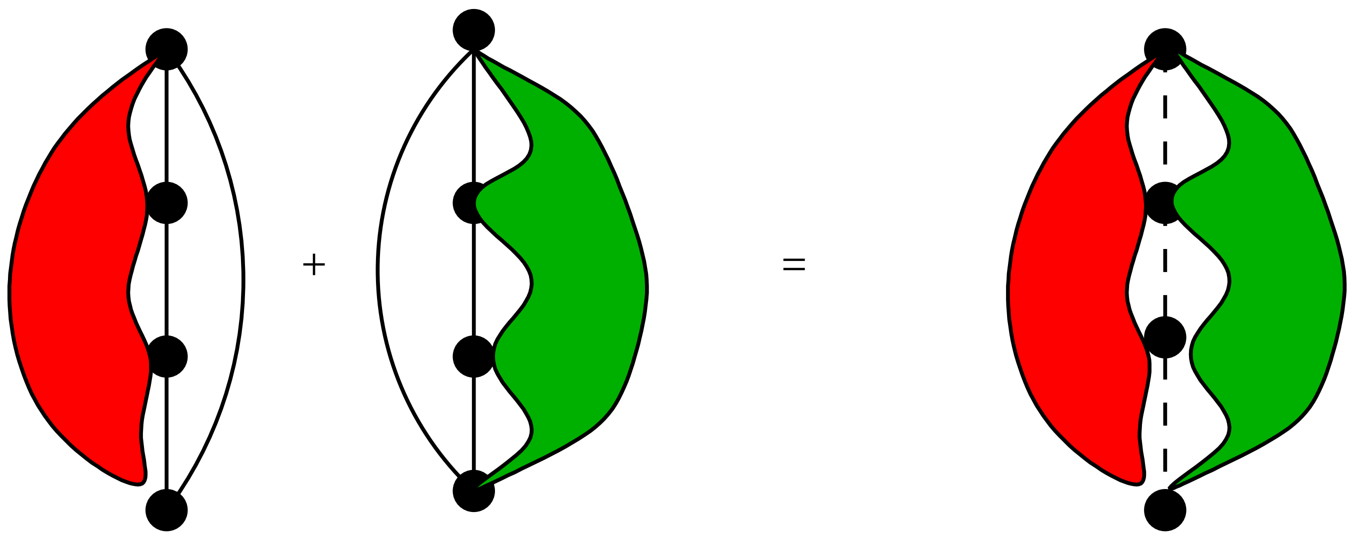
A digraph is even if every subdivision has an even directed circuit. An odd double cycle, O_{2k+1}:

THEOREM (Seymour, Thomassen) A digraph is not even $\Leftrightarrow$ it has no odd double cycle minor.

THEOREM (McCuaig; Robertson, Seymour, RT) A digraph is not even $\Leftrightarrow$ it can be obtained from strongly planar digraphs and F_7 by means of 0-, 1-, 2-, 3-, and 4-sums.

digraph $D \rightarrow$ bipartite graph G, p.m. M

THEOREM (McCuaig; Robertson, Seymour, RT) A strongly 2-connected digraph D is not even $\Leftrightarrow$ $D \cong F_7$ or G can be obtained from planar bipartite graphs by means of C_4-sums.



OPEN QUESTION Can strong 2-connectivity be tested in linear time?

$\tau(D) = \min\{|X| \subseteq V(D) : D \backslash X \text{is acyclic}\}$
$\nu(D) =$ max number of disjoint circuits

THEOREM (Guenin, RT) $\tau(D') = \nu(D')$ for every subdigraph D' of D $\Leftrightarrow$ D has no O_{2k+1} or F_7 minor.

THEOREM (McCuaig)

$\nu(D) \leq 1 \Rightarrow \tau(D) \leq 3$

THEOREM (Reed, Robertson, Seymour, RT)

There is a function f such that $\tau(D) \leq f(\nu(D))$ for every D.

An arboreal decomposition of D is (R, X, W), where R is an arborescence, and $X = (X_e : e \in E(R))$ and $W = (W_r : r \in V(R))$ satisfy

(D1) $(W_r : r \in V(R))$ partitions $V(D)$

(D2) $\bigcup_{r>e} W_r$ induces a strong component of $D \backslash X_e$ for every $e \in E(R)$

The width is the min, over all $r \in V(R)$, of $|W_r \cup \bigcup_{e \sim r} X_e| - 1$.

The tree-width of D is the minimum width of an arboreal decomposition of D.

FACT Tree-width is minor-monotone.

A haven of order k in D assigns to every $X \in [V(D)]^{<k}$ the vertex-set of a strong component of $D\backslash X$ such that

(H) $X \subseteq Y \in [V(D)]^{<k} \Rightarrow \beta(Y) \subseteq \beta(X)$.

FACT Haven of order $k \Rightarrow \text{tw}(D) \geq k - 1$.

QUESTION Converse?

THEOREM (Johnson, Robertson, Seymour, RT)

$\mathsf{tw}(D) \geq 3k - 1 \Rightarrow$ haven of order k

COPS-AND-ROBBER GAME Same as for undirected graphs, except that robber must stay within strongly connected components of the cop-free subdigraph.

A haven of order k gives an escape strategy for the robber agains $k-1$ cops, and an arboreal decomposition of width $k-1$ gives a search strategy for k cops.

REMARK The search strategy need not be monotone.

PROOF Choose an arboreal decomposition (R, X, W) of D such that

(1) $|W_r \cup \bigcup_{e \sim r} X_e| \leq 3k - 1$ for all $r \in V(R)$ of out-degree at least one,

(2) $|X_e| \leq 2k - 1$ for every $e \in E(R)$,

(3) subject to (1) and (2), the number of vertices $v \in V(D)$ in W_r of size $\geq k + 1$ is minimum.

Claim. (R, X, W) has width $\leq 3k - 2$.

ALGORITHMS

Let $Z \subseteq V(D)$, and let $S_1, \ldots, S_t$ be the strong components of $D \backslash Z$ such that no edge goes from S_j to S_i for $j > i$. Then $S = S_i \cup S_{i+1} \cup \ldots \cup S_j$ is Z-normal. If $|Z| \leq k$, then S is k-protected.

For some k-protected sets A we will compute an itinerary for A.

AXIOM 1 $A, B \subseteq V(D)$ disjoint, no edge of D has head in A and tail in B. Then an itinerary for $A \cup B$ can be computed from itineraries of A and B in time $O((|A| + |B|)^\alpha)$.

AXIOM 2 $A, B \subseteq V(D)$ disjoint sets, A is k-protected and $|B| \leq k$. Then an itinerary for $A \cup B$ can be computed from itineraries of A and B in time $O((|A| + 1)^\alpha)$.

THEOREM (Johnson, Robertson, Seymour, RT)
There is a polynomial-time algorithm for:
INPUT A digraph D with an arboreal decomposition of bounded width.
OUTPUT An itinerary for $V(D)$

Thus HAMILTON PATH, HAMILTON CIRCUIT, k-DISJOINT PATHS (k fixed) and other problems can be solved in polynomial time for digraphs of bounded tree-width.

CONJECTURE There is a function f such that every digraph of tree-width at least $f(k)$ has a cylindrical $k \times k$ grid minor.