GRAPH PLANARITY
and
RELATED TOPICS

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OUTLINE

- Euler's formula, Kuratowski's theorem
- Linear planarity tests
- Schnyder's theorem and drawing on a grid
- The two paths problem
- Pfaffian orientations
- Linkless embeddings
- The Four Color theorem
- Tutte's conjecture

A plane graph is a graph drawn in the plane with no crossings. A graph is planar if it can be drawn in the plane with no crossings.

Edges can be represented by
- •homeomorphic images of $[0,1]$
- •continuous images of $[0,1]$
- •piecewise-linear images of $[0,1]$
- •straight-line segments (Fáry's theorem)

EULER'S FORMULA For a connected plane graph G, $|V(G)| + |F(G)| = |E(G)| + 2$.

COROLLARY $|E(G)| \leq 3|V(G)| - 6$

KURATOWSKI'S THEOREM. A graph is planar $\Leftrightarrow$ it has no subgraph isomorphic to a subdivision of K_5 or $K_{3,3}$.

PF. $\Rightarrow$ K_5 and $K_{3,3}$ have too many edges.

$\Leftarrow$ (Thomassen) By induction. We may assume G is 3-connected, G/uv is 3-connected and planar.

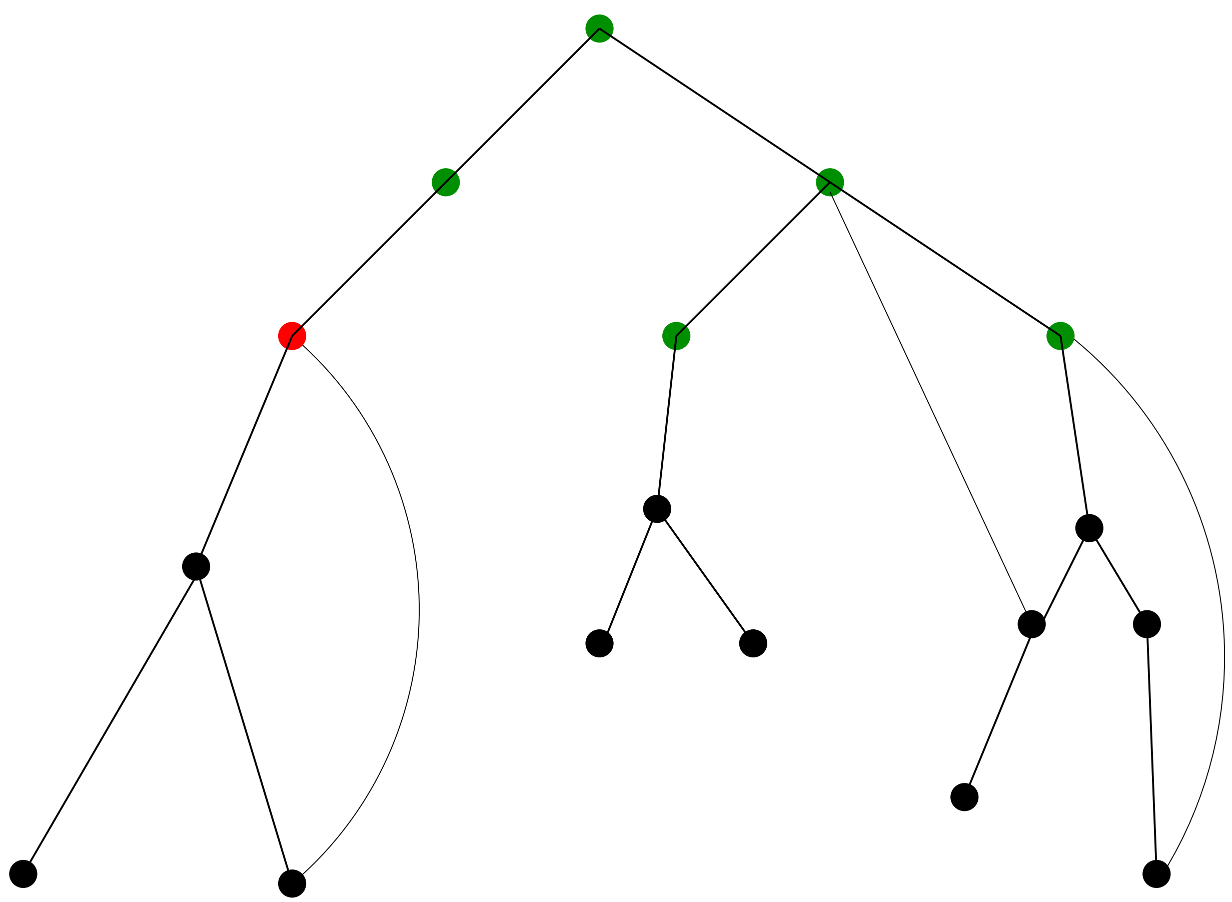
TESTING PLANARITY IN LINEAR TIME

•1974 Hopcroft and Tarjan

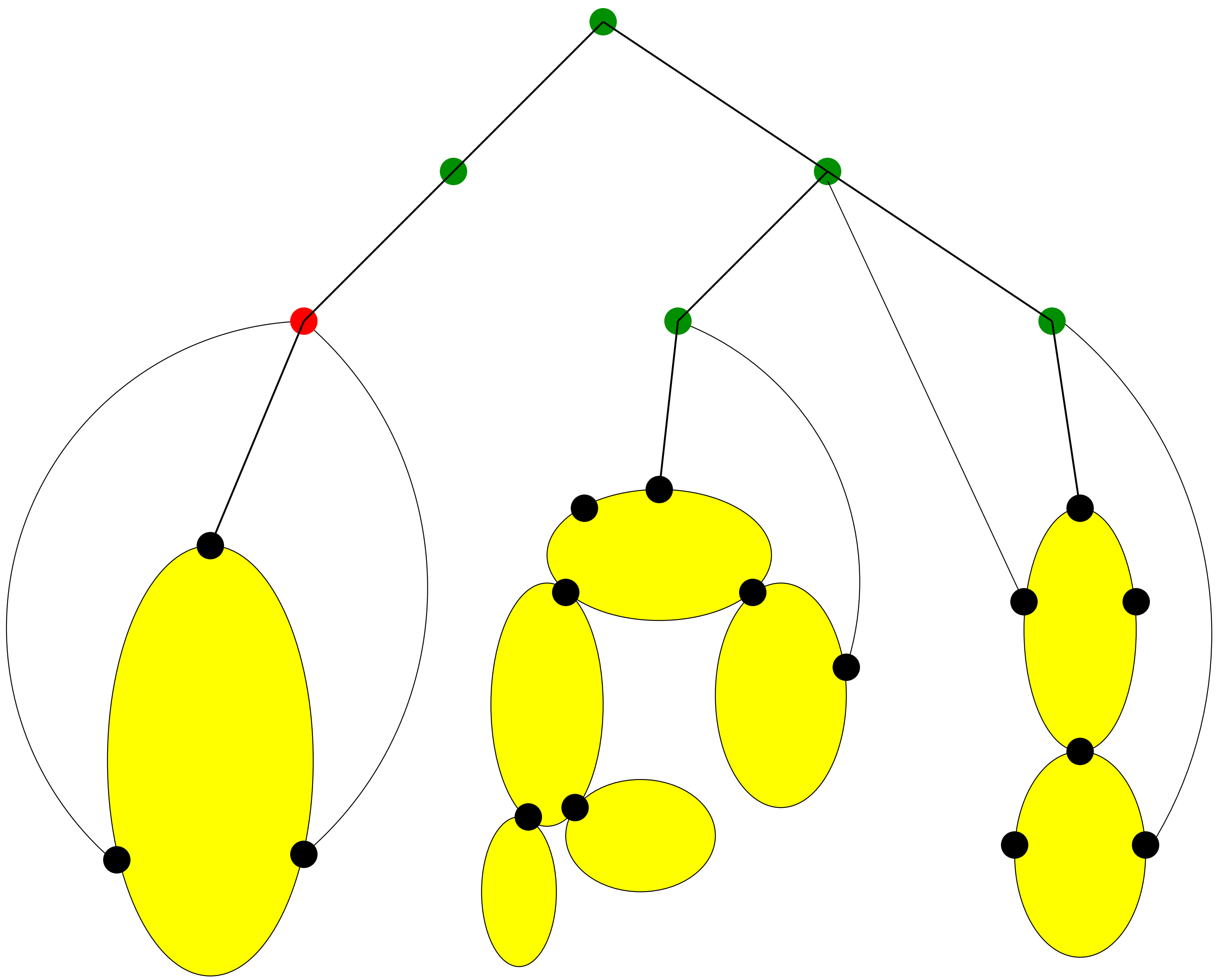
•1967 Lempel, Even and Cederbaum,
1976 Booth and Lueker

•Shih and Hsu, Boyer and Myrvold, RT class notes

Outline. Find a DFS tree, number vertices.



A recursive step



SCHNYDER'S THEOREM and DRAWING ON A GRID

THEOREM A graph is planar $\Leftrightarrow$ its vertex-edge poset has dimension at most three.

The latter is equivalent to: $\exists$ linear orderings $\leq_1, \leq_2, \leq_3$ satisfying:
$\forall uv \in E(G)\ \forall w \in V(G) - \{u, v\}\ \exists i$ s.t. $u \leq_i w$ and $v \leq_i w$.

Given $\leq_1, \leq_2, \leq_3$ there exists a $1\text{-}1$ map $v \in V(G) \to (v_1, v_2, v_3) \in \mathbf{R}^3$ s.t.

(i) $v_1 + v_2 + v_3 = 1$ for all $v \in V(G)$

(ii) $v_i \in [0, 2n-5]$ is an integer

(iii) for $uv \in E(G)$: $u \leq_i v \Leftrightarrow u_i \leq v_i$

It follows that this gives a straight-line embedding.

COLIN de VERDIERE'S PARAMETER

Let $\mu(G)$ be the maximum corank of a matrix M satisfying

(i) for $i \neq j$, $M_{ij} = 0$ if $ij \notin E$ and $M_{ij} < 0$ otherwise,

(ii) M has exactly one negative eigenvalue,

(iii) if X is a symmetric $n \times n$ matrix such that $MX = 0$ and $X_{ij} = 0$ whenever $i = j$ or $ij \in E$, then $X = 0$.

THEOREM $\mu(G) \leq 3 \Leftrightarrow G$ is planar.

SEPARATORS

Let G have n vertices. A separator in G is a set S such that every component of $G\backslash S$ has at most $2n/3$ vertices.

THEOREM (Lipton, Tarjan) Every planar graph has a separator of size at most $\sqrt{8n}$.

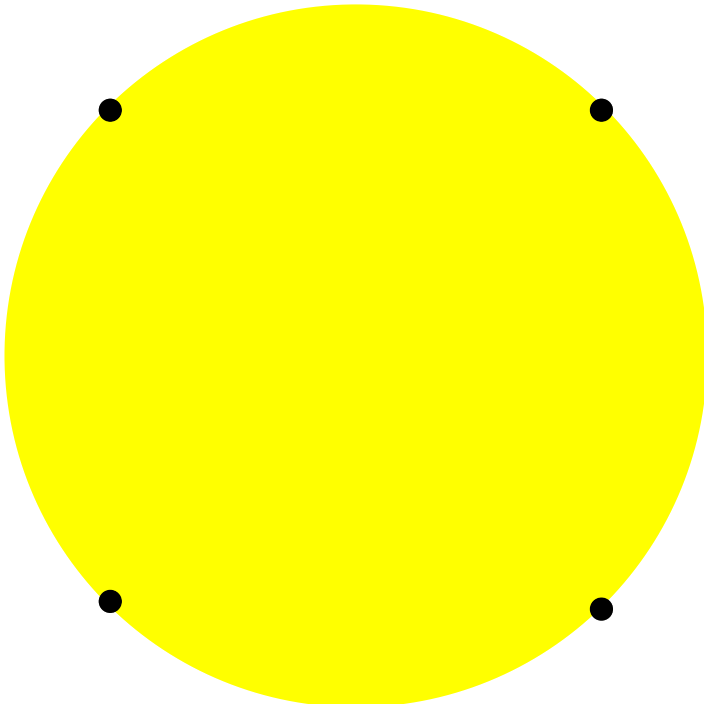
Alon, Seymour, RT improved to $\sqrt{4.5n}$, and proved that graphs not contractible to K_t have a separator of size at most $\sqrt{t^3 n}$.

THE TWO PATHS PROBLEM

Given G and $s_1, s_2, t_1, t_2 \in V(G)$ do there exist disjoint paths joining s_i to t_i?

A reduction

THEOREM If G is reduced, then the paths exist $\Leftrightarrow$ G cannot be drawn in a disk with s_1, s_2, t_1, t_2 drawn on the boundary of the disk in order.



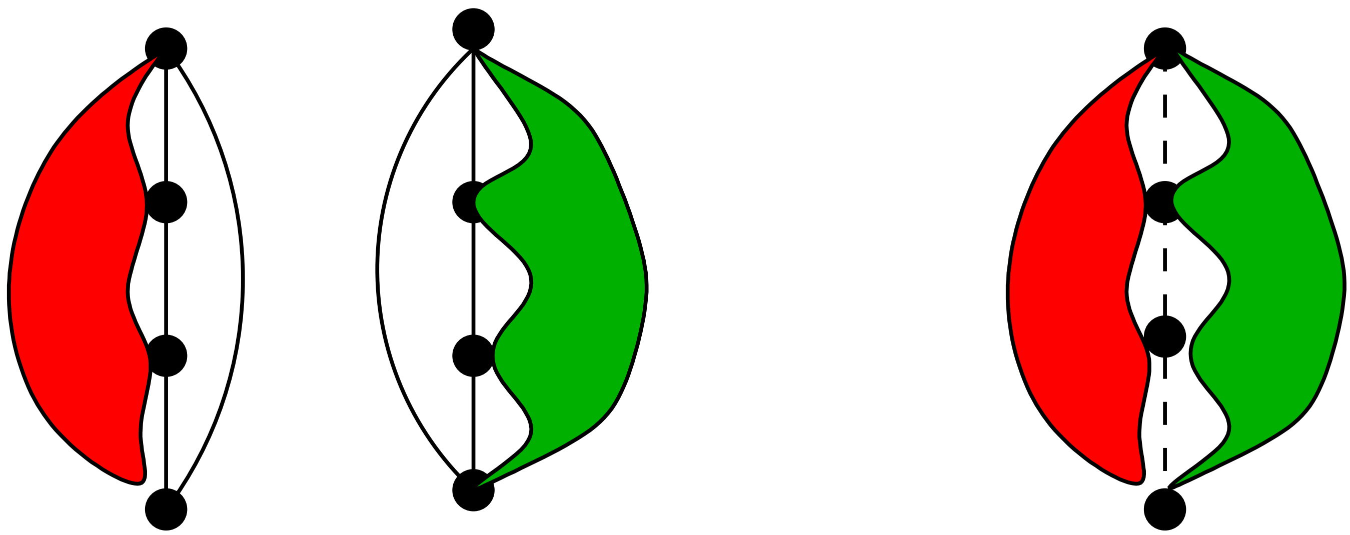
PFAFFIAN ORIENTATIONS

An orientation of a graph G is Pfaffian if every even cycle C such that $G\backslash V(C)$ has a perfect matching has an odd number of edges directed in either direction of the cycle. If a graph has a Pfaffian orientation, then the number of perfect matchings can be computed in polynomial time.

THEOREM (Kasteleyn) Every planar graph has a Pfaffian orientation.

For bipartite graphs the decision problem is equivalent to:
- Pólya's permanent problem
- even directed cycle problem
- a hypergraph problem
- the sign nonsingular matrix problem

THEOREM (McCuaig, Robertson, Seymour, RT)
A bipartite graph has a Pfaffian orientation $\Leftrightarrow$ it can be obtained from planar bipartite graphs and the Heawood graph by means of 0-, 2- and 4-sums.



COROLLARY There exists a cubic algorithm to solve the above-mentioned problems.

LINKLESS EMBEDDINGS

A (piecewise linear) embedding of a graph in $\mathbf{R}^3$ is linkless if every two disjoint cycles have zero linking number.

THEOREM (Robertson, Seymour, RT) For a graph G the following are equivalent:

(i) G has a linkless embedding

(ii) G has no subgraph that can be contracted onto one of 7 graphs (K_6, $K_{3,3}$+vrtx, $K_{4,4}\backslash e$, ..., Petersen)

(iii) G has a flat embedding

An embedding of a graph G in $\mathbf{R}^3$ is flat if every cycle C bounds a disk disjoint from the rest of G.

REMARK If G is planar, then $G \hookrightarrow \mathbf{R}^3$ is flat $\Leftrightarrow$ G is a subset of an embedded sphere.

THEOREM (RST) An embedding of a graph G in $\mathbf{R}^3$ is flat $\Leftrightarrow$ the fundamental group of the complement of every subgraph is free.

THEOREM (RST) A 4-connected flat graph has a unique flat embedding. More generally, any two flat embeddings of the same graph are related by "3-switches".

THEOREM (Lovász, Schrijver) A graph G has a flat embedding $\Leftrightarrow$ $\mu(G) \leq 4$.

An embedding of a graph G in $\mathbf{R}^3$ is knotless if every cycle of G bounds a disk.

No characterization of knotless graphs is known, but:

THEOREM There exists an algorithm to test whether an input graph has a knotless embedding.

THE FOUR COLOR THEOREM.

Every planar graph is 4-colorable.

Comments:

- Simple statement, yet proof is long.
- A 1976 proof by Appel and Haken.
- A simpler proof by Robertson, Sanders, Seymour, RT
- Proof is computer assisted.
- Over two dozen equivalent formulations (in terms of vector cross products, Lie algebras, divisibility, Temperley-Lieb algebras,...)
- Conjectured generalizations.
- See August 1998 Notices of the AMS for a survey.

Cross-product is not associative, and so

$$v_1 \times v_2 \times \cdots \times v_n \qquad (1)$$

is not well-defined if $n > 2$. If we put in enough brackets to make it well-defined, we get a bracketing.

EXAMPLE $((v_1 \times v_2) \times (v_3 \times v_4)) \times v_5$.

THEOREM (Kauffman) For every two bracketings of (1) there exists an assignment $v_1, \ldots, v_n \in \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ such that the evaluations of the two bracketings are equal and nonzero.

THEOREM (Kauffman) The above theorem is equivalent to the 4CT.

OUTLINE OF PROOF

Let G be a counterexample with $|V(G)|$ minimum. We may assume that G is an internally 6-connected triangulation. A configuration:

We exhibit a set $\mathcal{U}$ of 633 configurations such that

THEOREM 1 No member of $\mathcal{U}$ "appears" in a minimal counterexample to the 4CT

THEOREM 2 For every internally 6-connected triangulation T, some member of $\mathcal{U}$ "appears" in T.

A configuration K appears in a triangulation T is K is an induced subgraph of T and for every vertex of K its label equals its degree in T.

ABOUT THEOREM 2

In a triangulation $e = 3n - 6$ and so

$$d_1 + d_2 + \cdots + d_n = 2e = 6n - 12$$

or

$$(6 - d_1) + (6 - d_2) + \cdots + (6 - d_n) = 12.$$

Initially, a vertex of degree d will receive a charge of $6-d$. Thus the sum of the charges is 12. Charges will be redistributed according to certain rules, but the total sum will remain the same. Thus there is a vertex v of positive charge. We show that a reducible configuration appears in the second neighborhood of v.

COROLLARY A quadratic algorithm to 4-color planar graphs.

IS THIS REALLY A PROOF

PROGRAM $\rightarrow$ COMPILER $\rightarrow$ HARDWARE $\rightarrow$ RESULTS