CHAPTER 11: GENERATING FUNCTIONS

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Abstract. In your calculus courses, you studied power series representations of functions having the form $f(x) = \sum_{n=0}^{\infty} a_n x^n$. In a combinatorial setting, we consider such power series of this type as just another way of encoding the values of a sequence $\{a_n : n \geq 0\}$ indexed by the non-negative integers. Power series can be manipulated just like ordinary functions, i.e., they can be added, subtracted and multiplied. And when we find it convenient to do so, we will use the familiar techniques from calculus and differentiate or integrate them term by term.

1. Basic Notation and Terminology

With a sequence $\sigma = \{a_n : n \geq 0\}$ of real numbers, we associate a "function" $F(x)$ defined by

$$F(x) = \sum_{n=0}^{\infty} a_n x^n.$$

The word "function" is put in quotes as we do not necessarily care about substituting a value of x and obtaining a specific value for $F(x)$. In other words, we consider $F(x)$ as a formal power series and frequently ignore issues of convergence.

It is customary to refer to $F(x)$ as the *generating function* of the sequence σ. As we have already remarked, we are not necessarily interested in calculating $F(x)$ for specific values of x. However, by convention, we take $F(0) = a_0$.

Example 1. Consider the constant sequence $\sigma = \{a_n : n \geq 0\}$ with $a_n = 1$ for every $n \geq 0$. Then the generating function $F(x)$ of σ is given by

$$F(x) = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + \dots.$$

This last equation is the MacLaurin Series for the function $F(x) = 1/(1-x)$; note that the series converges when $|x| < 1$.

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + \cdots = \sum_{n=0}^{\infty} x^n \quad \text{when } |x| < 1.$$

Example 2. MacLaurin series can be differentiated and integrated term by term inside the interval of convergence—although the behavior at the end points of the interval can change. Therefore

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 + 7x^6 + \cdots = \sum_{n=1}^{\infty} nx^{n-1} \quad \text{when } |x| < 1.$$

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and (after a bit of algebraic manipulation)

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \cdots = \sum_{n=1}^{\infty} \frac{x^n}{n} \quad \text{when } -1 < x \leq 1.$$

Example 3. The exponential function e^x has the following well-known expansion:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

This series converges for all real x, as do the familiar series for $\sin x$ and $\cos x$:

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \quad \text{and} \cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}.$$

On the other hand, we can talk about the formal power series

$$F(x) = \sum_{n=0}^{\infty} n! x^n.$$

even though it has radius of convergence 0, i.e., the series $F(x)$ converges only for $x = 0$, so that $F(0) = 1$. Nevertheless, it makes sense to speak of the formal power series $F(x)$ as the generating function for the sequence $\{a_n : n \geq 0\}$, $a_0 = 1$ and a_n is the number of permutations of $\{1, 2, \ldots, n\}$ when $n \geq 1$.

2. The General Form of the Binomial Theorem

Earlier in our course, you learned that the following formula holds for all integers $p \geq 1$:

$$(1+x)^p = \sum_{k=0}^{p} \binom{p}{n} x^n. \tag{1}$$

Also, recall that the binomial coefficient $\binom{p}{k}$ was defined recursively for $p \geq k \geq 0$ by setting (1) $\binom{p}{p} = \binom{p}{0} = 1$ for all $p \geq 0$; and (2) $\binom{p}{k} = \binom{p-1}{k-1} + \binom{p-1}{k}$, when $p > k > 0$. Also, we used the notation $P(p, k)$ for the number of permutations of p things taken k at a time. Of course, $P(p, k)$ is defined only for $p \geq k \geq 0$. The recursive definition is (1) $P(p, 0) = 1$ for all $p \geq 0$, and (2) $P(p, k) = pP(p-1, k-1)$ when $p \geq k > 0$.

Definition 4. We will extend the definition of a binomial coefficent so that $\binom{p}{k}$ is defined for all real p and non-negative integer values of k. This is done by first defining $P(p, k)$ exactly as above, i.e., first set $P(p, 0) = 1$ for all real numbers p. Then set $P(p, k+1) = pP(p-1, k)$ (Note that we drop the requirement that $p \geq k$ in making this definition). Then set

$$\binom{p}{k} = \frac{P(p, k)}{k!}.$$

Note that $P(p, k) = \binom{p}{k} = 0$ when p and k are integers with $0 \leq p < k$. On the other hand, we have some interesting new concepts such as $P(-5, 4) = (-5)(-6)(-7)(-8)$ and

$$\binom{-7/2}{5} = \frac{(-7/2)(-9/2)(-11/2)(-13/2)(-15/2)}{5!}.$$

Now here is the general form of the binomial theorem valid for all real non-zero values of p. You can find the proof in most advanced calculus books.

Theorem 5. *For all real p with $p \neq 0$,*

$$(1+x)^p = \sum_{n=0}^{\infty} \binom{p}{n} x^n.$$

Note that the general form reduces to the original version of the binomial theorem when p is a positive integer.

3. An Application of the Binomial Theorem

Lemma 6. *For each $k \geq 0$, $P(p, k+1) = P(p, k)(p-k)$.*

Proof. When $k = 0$, both sides evaluate to p. Now assume validity when $k = m$ for some non-negative integer m. Then

$$\begin{aligned} P(p, m+2) &= pP(p-1, m+1) \\ &= p[P(p-1, m)(p-1-m)] \\ &= [pP(p-1, m)](p-1-m) \\ &= P(p, m+1)[p-(m+1)]. \end{aligned}$$

□

Lemma 7. *For each $k \geq 0$,*

$$\binom{-1/2}{k} = (-1)^k \frac{\binom{2k}{k}}{2^{2k}}.$$

Proof. We proceed by induction on k. Both sides reduce to 1 when $k = 0$. Now assume validity when $k = m$ for some non-negative integer m. Then

$$\begin{aligned} \binom{-1/2}{m+1} &= \frac{P(-1/2, m+1)}{(m+1)!} \\ &= \frac{P(-1/2, m)(-1/2-m)}{(m+1)m!} \\ &= \frac{-1/2-m}{m+1} \binom{-1/2}{m} \\ &= (-1)\frac{2m+1}{2(m+1)}(-1)^m \frac{\binom{2m}{m}}{2^{2m}} \\ &= (-1)^{m+1} \frac{\binom{2m+2}{m+2}}{2^{2m+2}}. \end{aligned}$$

□

Theorem 8. *The generating functon of the sequence $\{\binom{2n}{n} : n \geq 0\}$ is $f(x) = (1-4x)^{-1/2}$.*

Proof. By the general form of the binomial theorem and Lemma 7, we know that

$$\begin{aligned}(1-4x)^{-1/2} &= \sum_{n=0}^{\infty}\binom{-1/2}{n}(-4x)^n\\ &= \sum_{n=0}^{\infty}(-1)^n 2^{2n}\binom{-1/2}{n}x^n\\ &= \sum_{n=0}^{\infty}\binom{2n}{n}x^n. \qquad \square\end{aligned}$$

The next result is very elementary and is stated primarily to emphasize the form of a product of two power series.

Proposition 9. *Let* $A(x) = \sum_{n=0}^{\infty} a_n x^n$ *and* $B(x) = \sum_{n=0}^{\infty} b_n x^n$ *be generating functions. Then* $A(x)B(x)$ *is the generating function of the sequence whose* n^{th} *term is given by*

$$a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \cdots + a_n b_0 = \sum_{k=0}^{n} a_k b_{n-k}.$$

The result below now follows immediately from Theorem 8 and Proposition 9 using the function $f(x) = (1-4x)^{-1/2}$.

Corollary 10. *For all* $n \geq 0$*,*

$$2^{2n} = \sum_{k=0}^{n}\binom{2k}{k}\binom{2n-2k}{k}.$$

4. Partitions of an Integer

By a partition P of an integer, we mean a collection of (not necessarily distinct) positive integers such that $\sum_{i\in P} i = n$. For example, $2+2+1$ is a partition of 5. For each $n \geq 0$, let p_n denote the number of partitions of the integer n (with $p_0 = 1$ by convention). Note that $p_8 = 22$ as evidenced by the list in Table 1. Note that there are 6 partitions of 8 into *distinct* parts. Also there are 6 partitions of 8 into *odd* parts. As we will soon see, these numbers are in fact always equal.

Theorem 11. *For each* $n \geq 1$*, the number of partitions of* n *into distinct parts is equal to the number of partitions of* n *into odd parts.*

Proof. The generating function $P(x)$ for the sequence p_n, the number of partitions of the integer n is given by

$$P(x) = \prod_{n=1}^{\infty}\frac{1}{1-x^n} = \left(\sum_{n=0}^{\infty}x^n\right)\left(\sum_{n=0}^{\infty}x^{2n}\right)\cdots\left(\sum_{n=0}^{\infty}x^{kn}\right)\cdots,$$

since an x^n term in the product arises for eqch partition by picking the $(x^k)^j$ term from the k^{th} factor in the product, where j is the number of k's appearing in the partition in question. Also, the generating function $D(x)$ for the number of partitions of n into distinct parts is given by

$$D(x) = \prod_{n=1}^{\infty}(1+x^n),$$

$$\begin{aligned}
8 &= 8 \quad \text{distinct parts}\\
&= 7+1 \quad \text{distinct parts, odd parts}\\
&= 6+2 \quad \text{distinct parts}\\
&= 6+1+1\\
&= 5+3 \quad \text{distinct parts, odd parts}\\
&= 5+2+1 \quad \text{distinct parts}\\
&= 5+1+1+1 \quad \text{odd parts}\\
&= 4+4\\
&= 4+3+1 \quad \text{distinct parts}\\
&= 4+2+2\\
&= 4+2+1+1\\
&= 4+1+1+1+1\\
&= 3+3+2\\
&= 3+3+1+1 \quad \text{odd parts}\\
&= 3+2+2+1\\
&= 3+2+1+1+1\\
&= 3+1+1+1+1+1 \quad \text{odd parts}\\
&= 2+2+2+2\\
&= 2+2+2+1+1\\
&= 2+2+1+1+1+1\\
&= 2+1+1+1+1+1+1\\
&= 1+1+1+1+1+1+1+1 \quad \text{odd parts}
\end{aligned}$$

TABLE 1. THE PARTITIONS OF 8, NOTING THOSE INTO DISTINCT PARTS AND THOSE INTO ODD PARTS.

since you can either have a particular positive integer appear once in the distinct parts partition or not have it at all. Finally, the generating function $O(x)$ for the number of partitions of n into odd parts is

$$O(x) = \prod_{n=1}^{\infty} \frac{1}{1-x^{2n-1}}$$

by the same argument as used for $P(x)$. To see that $D(x) = O(x)$, we note that $1 - x^{2n} = (1-x^n)(1+x^n)$ for all $n \geq 1$. Therefore

$$\begin{aligned}
D(x) &= \prod_{n=1}^{\infty}(1+x^n) = \prod_{n=1}^{\infty} \frac{1-x^{2n}}{1-x^n} = \frac{\prod_{n=1}^{\infty}(1-x^{2n})}{\prod_{n=1}^{\infty}(1-x^n)}\\
&= \frac{\prod_{n=1}^{\infty}(1-x^{2n})}{\prod_{n=1}^{\infty}(1-x^{2n-1})\prod_{n=1}^{\infty}(1-x^{2n})} = \prod_{n=1}^{\infty}\frac{1}{1-x^{2n-1}}\\
&= O(x). \qquad \square
\end{aligned}$$

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