MATH 4032 Test 1 Solutions, February 27, 2008, WTT

Problem 1. Complete the statements:

Theorem 1. (Strong Gallai-Milgram) Let G be an oriented graph, let $k = \alpha(G)$, and let $\mathcal{Q} = \{Q_1, Q_2, \dots, Q_s\}$ be any path partition of G. Then there exists a path partition $\mathcal{P} = \{P_1, P_2, \dots, P_t\}$ so that (a) $t \leq k$, and (b) For each $i = 1, 2, \dots, t$, the terminating point of path P_i is also a terminating point of a path in $\mathcal{Q}$.

Theorem 2. (Dirac) Let G be a graph on n vertices. If every vertex of G has at least $\lceil n/2 \rceil$ neighbors, then G has a hamiltonian cycle.

Theorem 3. (P. Hall) Let $G = (X, Y, E)$ be a bipartite graph. If $|N(A)| \geq |A|$ for every subset $A \subseteq X$, then G has a complete matching.

Theorem 4. (Tutte) A simple graph G has a perfect matching if and only if for every subset S of the vertex set, the number of odd components in the graph that remains when the vertices in S are removed is at most $|S|$.

Theorem 5. (Dilworth) If $\mathbf{P} = (X, P)$ is a poset and $\text{width}(X, P) = w$, then there exists a partition $X = C_1 \cup C_2 \cup \cdots \cup C_t$ of X into chains.

Problem 2. Write out the proof of Theorem 1, the strong version of the Gallai-Milgram theorem.

Proof. We proceed by induction on n, the number of vertices in G, noting that the result holds trivially on a one point graph. Assume validity when G has at most n vertices and consider an oriented graph G with $n+1$ vertices. Let $k = \alpha(G)$, and let $\mathcal{Q} = \{Q_1, Q_2, \dots, Q_s\}$ be any path partition of G.

Of course, we may assume that $s > k$; else we could take $\mathcal{P} = \mathcal{Q}$. On the other hand, we may also assume that $s = k+1$. For if $s > k+1$, we can apply the inductive hypothesis to the graph G' obtained from G by deleting the points on Q_s and using the path partition $\mathcal{Q}' = \{Q_1, Q_2, \dots, Q_{s-1}\}$ of G'.

For each $i = 1, 2, \dots, k+1$, let y_i be the termination point of Q_i, and let $Y = \{y_1, y_2, \dots, y_{k+1}\}$. Since $\alpha(G) \leq k$, we know that Y is not an independent set of vertices. Without loss of generality, we may assume that (y_2, y_1) is an edge in G.

We may further assume that Q_1 is not a one point path consisting only of the terminal point y_1. Otherwise, we could replace $Q_1 \cup Q_2$ by the path Q' obtained by appending y_1 to the end of Q_2. This would result in a path partition of size k with all paths terminating at points from $Y - \{y_2\}$.

Now let z_1 be the point immediately before y_1 on path Q_1, and let Q_1'' be the path obtained from Q_1 by deleting the termination point y_1. Note that the path Q_1'' has z_1 as its termination point. Now apply the inductive hypothesis to the graph G'' obtained by deleting y_1 from G, together with the path partition $\mathcal{Q}'' = \{Q_1'', Q_2, Q_3, \dots, Q_{k+1}$. Consider the set T of termination points of the paths in the resulting path partition of G''. Suppose first that $z_1 \in T$. Then we may append y_1 to the end of the path terminating at z_1 to obtain the desired path partition of G.

So we may assume that $z_1 \notin T$. Now suppose that $y_2 \in T$. In this case, we may append y_1 to the end of the path terminating at y_2 to obtain the desired partition of G. Finally, if neither z_1 nor y_2 belong to T, then we have a path partition of G'' consisting of at most $k-1$ paths, all of which have termination points from $\{y_3, y_4, \dots, y_{k+1}\}$. And in this case, we can add $\{y_1\}$ as a one-point path to obtain the desired path partition of G. This completes the proof.

Problem 3. Write out the proof of either Dirac's theorem or Hall's theorem.

Proof of Dirac's theorem. First, let $P = (x_1, x_2, \ldots, x_m)$ be a maximum length path in G. Then all the neighbors of x_1 and x_m are on the path P. This requires that $m - 1 \geq \lceil n/2 \rceil$.

Now suppose that G contains a cycle $C = (y_1, y_2, \ldots, y_m)$ of size m. If $m = n$, then C is a hamiltonian cycle in G, so we may assume that $m < n$. Let z be any vertex of G that is not on the cycle C. Since $n - m - 1 < \lceil n/2 \rceil$, it follows that z has a neighbor on C. If zy_j is an edge in G, then

$$P' = (z, y_j, y_{j+1}, y_{j+2}, \ldots, y_m, y_1, y_2, \ldots, y_{j-1})$$

is a path on $m + 1$ vertices in G. The contradiction allows us to conclude that G does not have a cycle on m vertices.

Now suppose that for some integer i with $2 \leq i \leq m$, both x_1x_i and x_mx_{i-1} are edges of G. Then

$$C = (x_1, x_2, \ldots, x_{i-1}, x_m, x_{m-1}, x_{m-2}, \ldots, x_{i+1}, x_i)$$

is a cycle in G. It follows that whenever x_1x_i is an edge in G, then x_{i-1} is not adjacent to to x_m. Since x_1 has at least $\lceil n/2 \rceil$ neighbors on P, it follows that there are at least $\lceil n/2 \rceil$ vertices from $\{x_1, x_2, \ldots, x_{m-1}\}$ that are not adjacent to x_m. But this implies that the degree of x_m is at most $m - 1 - \lceil n/2 \rceil$, which is not true. The contradiction completes the proof.

Proof of Hall's theorem. The condition

$$|N(A)| \geq |A|, \text{ for every subset } A \subseteq X$$

is called the "matching condition." We show that any bipartite graph satisfying the matching condition has a complete matching. The argument is by induction on n, the total number of vertices in the graph. Note first that the theorem holds trivially when $n \leq 2$. Now assume that it holds whenever $n \leq k$, where $k \geq 2$, and consider the case where $n = k + 1$.

Suppose first that for every proper non-empty subset $A \subset X$, we have $|N(A)| > |A|$. Then let x be any vertex in X and let $y \in Y$ be any neighbor of x. Also, let H be the subgraph obtained by deleting x and y from F. Note that H satisfies the matching condition and has fewer vertices than G. So H has a complete matching, but this would imply that G also has a complete matching.

So we may assume without loss of generality that there there is some non-empty proper subset $A_0 \subset X$ with $|N(A_0)| = |A_0|$. Now, let $Y_0 = N(A_0)$, $E_0 = E \cap (A_0 \times Y_0)$, and let H_0 be the bipartite graph (A_0, Y_0, E_0). Note that H_0 satisfies the matching condition and thus has a complete matching.

Then let $A_1 = X - A_0$, $Y_1 = Y - Y_0$ and $E_1 = E \cap (A_1 \times Y_1$. Also, let H_1 be the bipartite graph (A_1, Y_1, E_1) We claim that H_1 satisfies the matching condition. To see that this statement holds, let B be a subset of A_1 and let $N = N(B) \cap Y_0$, i.e., N is the neighborhood of G in H_1. Then consider the subset $A = A_0 \cup B$. Note that $N(A) = N(A_0) \cup N$. Since G satisfies condition the matching condition, we know that $|N(A)| \geq |A|$, i.e.,

$$|N(A)| = |N(A_0) \cup N| = |N(A_0)| + |N| \geq |A| = |A_0 \cup B| = |A_0| + |B|$$

but since $|N(A_0)| = |A_0|$, this last inequality implies $|N| \geq |B|$. So H has a perfect matching, and thus so does G. This completes the proof.