SOME NOTES ON G.SPECTRAL THEORY.

Spectral Theory in Finite Dimensional Space:
$\sigma(T) = \{\lambda \in C : (\lambda I - T)$ is 1-1 $\}$ in finite dimensional Banach space. (spectrum)
$\lambda_0 \in \sigma(T)$, there exists $x_0 \neq 0$ s.t. $\big(T - \lambda_0 I\big)x_0 = 0.\lambda_0$ is eigenvalue of T and x_0 is eigenfunction(e.vector)
Corollary : The spectrum of an operator in a finite dimensional B.Space is non-void finite set of points.

FUNCTIONS OF AN OPERATOR :

$\rho(T) = \{\lambda \in C : (\lambda I - T)^{-1}$ exists and bounded operator with domain $\chi\}$ = *resolvent set*
$\sigma(T) = C - \rho(T)$ = *spectrum of T*
$R(\lambda;T) := \big(\lambda I - T\big)^{-1}$ = *resolvent of T*
Lemma: $\rho(T)$ is open. $R(\lambda;T)$ is analytic function on $\rho(T)$.
Corollary : If $d(\lambda)$ is the distance from λ to the spectrum $\sigma(T)$, then $|R(\lambda;T)| \geq \frac{1}{d(\lambda)}$. Thus, $|R(\lambda;T)| \to \infty$ as $d(\lambda) \to 0$. Here $\lambda \in \rho(T)$.
Lemma: The closed set $\sigma(T)$ is bounded and non-void. Moreover, $\sup|\sigma(T)| = \lim_{n\to\infty}|T^n|^{\frac{1}{n}} \leq |T|$. For $|\lambda| > \sup|\sigma(T)|$ the series

$$R(\lambda;T) = \sum_{n=0}^{\infty} \frac{T^n}{\lambda^{n+1}}$$

converges in the uniform operator topology.
$r(T) = \sup|\sigma(T)| = \lim_{n\to\infty}|T^n|^{\frac{1}{n}} \leq |T|$ is called the *spectral radius of T.*
Lemma : (Resolvent Equation)
$R(\lambda;T) - R(\mu;T) = (\lambda - \mu)R(\lambda;T)R(\mu;T)$.
Lemma : Spectrum of the adjoint T^* is identical with the spectrum of T. Further, $R(\lambda;T) = R(\lambda;T)^*$ for $\lambda \in \rho(T)$.
$F(T)$ = family of all functions f which are analytic on some neigborhood of $\sigma(T)$.

$$f(T) = \frac{1}{2\pi i}\int_B f(\lambda)R(\lambda;T)d\lambda,$$

here $f \in F(T), U-$ open set whose boundary B consists of finite number of rectifiable Jordan curves, $\sigma(T) \subset U, U \cup B \subset$ domain of analyticity of f.
*SPECTRAL MAPPING THEOREM:*If $f \in F(T)$, then $f(\sigma(T)) = \sigma(f(T))$.
A subset of $\sigma(T)$ which is both open and closed in $\sigma(T)$ is called a *spectral set*
Spectral Theory of COMPACT OPERATORS :
Recall: Let $T \in B(\chi, D)$, S = closed unit sphere in χ. T is compact if the strong closure of $T(S)$ is compact in the strong topology of D.
Lemma: T is compact operator, $0 \neq \lambda \in C$. $\lambda I - T$ is $1-1 \Rightarrow range(\lambda I - T)$ is closed.
Corollary: $T-$ compact operator, $0 \neq \lambda \in \sigma(T)$. Then $\exists$ either $0 \neq x \in \chi$ s.t. $Tx = \lambda x$

or $0 \neq x^* \in \chi^*$ s.t. $T^*x^* = \lambda x^*$.
Lemma: T is compact operator, $(\lambda_n)-$ sequence of distinct scalars, (x_n) sequence of non-zero vectors s.t. $(T - \lambda_n I)x_n = 0$ for all $n = 1, 2, ...$Then, $\lambda_n \to 0$.
PERTURBATION THEORY:
Lemma: The set G of elements in $B(\chi)$ which have inverses in $B(\chi)$ is an open set in the uniform topology of $B(\chi)$, containing with an operator A the sphere $\{B : |A - B| < |A^{-1}|^{-1}\}$. If B s in this sphere its inverse is given by the seies

$$B^{-1} = A^{-1}\sum_{n=0}^{\infty}[(A - B)A^{-1}]^n.$$

Corollary: Let $T, T_1 \in B(\chi), \lambda \in \rho(T)$ and $|T - T_1| < |R(\lambda; T)|^{-1}$. Then $\lambda \in \rho(T_1)$ and

$$R(\lambda; T_1) = R(\lambda; T)\sum_{n=0}^{\infty}[(T_1 - T)R(\lambda; T)]^n.$$

Lemma: Let T be in $B(\chi)$ and let $\epsilon > 0$.Then there is $\delta > 0$ such that if T_1 is in $B(\chi)$ and $|T_1 - T| < \delta$, then $\sigma(T_1) \subseteq S(\sigma(T), \epsilon)$ and $R(\lambda; T_1) - R(\lambda; T) < \epsilon$ for $\lambda \notin S(\sigma(T), \epsilon)$.
Lemma: Let $T(\mu)$ be an analytic operator valued function defined for $\mu < \gamma$, where $\gamma > 0$, and let U be an open set with $closure(U) \subset \rho(T(0))$. Then there exists a $\delta > 0$ such that if $|\mu| < \delta$ then $closure(U) \subset \rho(T(\mu))$ and $R(\lambda; T(\mu))$ is an analytic function of μ for each $\lambda \in E$.
Lemma: Let T be in $B(\chi), f$ be in $F(T)$ and $\epsilon > 0$. Then there is a $\delta > 0$ s.t. if $T_1 \in B(\chi)$ and $|T_1 - T| < \delta$, then $f \in F(T_1)$ and $|f(T) - f(T_1)| < \epsilon$.
Lemma: Let $f \in F(T(0))$, where $T(\mu)$ is an analytic operator valued function, defined for $|\mu| < \gamma$, where $\gamma > 0$. Then there is a positive $\delta < \gamma$ s.t. $f \in F(T(\mu))$ and s.t. $f(T(\mu))$ is an analytic operator valued function of μ, for $\mu < \delta$.
Lemma: Let E, E_1 be two projections in χ s.t. $|E - E_1| < min(|E|^{-1}, |E_1|^{-1})$. If one of them has finite dimensional range so does the other and $dim E\chi = dim E_1\chi$.
Lemma: Let $E(\mu)$ be an analytic projection valued function defined for $|\mu| < \gamma$ where $\gamma > 0$ and $E(0)\chi$ have the finite dimension m. Then, $\exists\delta > 0$ s.t. if $\{x_1, ..., x_m\}$ is a basis for $E(0)\chi$, the set $\{E(\mu)x_1, ..., E(\mu)x_m\}$ is a basis for $E(\mu)\chi$ when $|\mu| < \delta$.
THEOREM: Let $\gamma > 0$ and $T(\mu)$ be a $B(\chi)$ valued function defined and analytic for $\mu < \gamma$. Let λ_0 be an isolated point of $\sigma(T(0))$, and suppose that the subspace $E(\lambda_0; T(0))\chi$ has finite dimension m. Let U be an open set with $closure(U) \cap \sigma(T(0)) = \{\lambda_0\}$.Then $\exists\delta > 0$ with $\delta < \gamma$, an integer $k \leq m$ and an integer n, s.t. for $|\mu| < \delta, U \cap \sigma(T(\mu))$ is a finite set $\{\lambda_1(\mu), ..., \lambda_k(\mu)\}$. Each function $\lambda_i(\mu)$ depends analytically on the principal value of the fractional power $\mu^{1/n}$ of μ, and satisfies $\lambda_i(0) = \lambda_0$.Moreover, the projections $E(\lambda_i(\mu); T(\mu))$ can be expanded in fractional power Laurent series

$$E(\lambda_i(\mu); T(\mu)) = \sum_{j=-N}^{\infty} A_{ij}\mu^{j/n},$$

where A_{ij} are operators in $B(\chi)$.

RECALL :

The *uniform operator topology* in $B(X,Y)$ is the metric topology on $B(X,Y)$ induced by its norm, $|T| = \sup_{|x|\leq 1} |Tx|$.

The *strong operator topology* in $B(X,Y)$ is the topology defined by the basic set of neigborhoods

$$N(T; A, \epsilon) = \{R : R \in B(X,Y), |(T-R)x| < \epsilon, x \in A\}$$

where A is an arbitrary finite subset of X, and $\epsilon > 0$ arbitrary. Thus, in the strong operator topology, a generalized sequence (T_α) converges to T iff $(T_\alpha x)$ converges to $Tx, \forall x \in X$.

The *weak operator topology* in $B(X,Y)$ is the topology defined by the basic set of neigborhoods

$$N(T; A, B, \epsilon) = \{R : R \in B(X,Y), |y^*(T-R)x| < \epsilon, y^* \in B, x \in\}$$

where A and B are arbitrary finite sets of elements in X and Y^*, respectively and $\epsilon > 0$ is arbitrary. Thus, in the weak operator topology, a generalized sequence (T_α) converges to T iff $(y^* T_\alpha x)$ converges to $y^* Tx, \forall x \in X$ and $\forall y^* \in Y^*$.

THEOREM: Let S and N be commuting operators and let f be analytic function in a domain D, including the spectrum $\sigma(S)$ of S and every point within a distance of $\sigma(S)$ not greater than some positive number ϵ. Suppose the spectrum $\sigma(N)$ of N lies within the open circle of radius ϵ about the origin. Then f is analytic on a neighborhood of $\sigma(S+N)$, and

$$f(S+N) = \sum_{n=0}^{\infty} \frac{f^{(n)}(S)N^n}{n!},$$

the series converging in the uniform topology of operators.

Lemma: Let C be a set whose minimum distance from the spectrum $\sigma(T)$ of an operator T is greater than some positive number ϵ. Then, $\exists$ a constant K s.t.

$$|R(\lambda, t)^n| < K\epsilon^{-n}, n \geq 0, \lambda \in C.$$

Lemma: If $\lambda \to T(\lambda)$ is an analytic operator valued function defined on a domain D, then the function $\lambda \to T^{-1}(\lambda)$ is defined on an open subset of D and is analytic there. If $T(\lambda)$ is compact for eaxh $\lambda \in D$ and if D is connected, then either $I - T(\lambda)$ has a bounded inverse for no point in D or else this inverse exists except at a countable number of isolated points.

TAUBERIAN THEORY:

THEOREM: Let f, f_n be in $F(T)$ and let $(f(T)f_n(T))$ converge to 0 in the uniform operator topology. Let f vanish at a finite set of poles of $R(\lambda; T)$. Suppose that each root λ_0 of f on $\sigma(T)$ has finite order $\alpha(\lambda_0)$, that the sequences $(f_n^{(m)}(\lambda_0))$ converge for

$0 \leq m < \alpha(\lambda_0)$, and that $\lim_{n\to\infty} f_n(\lambda_0) \neq 0$. Then, the sequence $(f_n(T))$ converges in the uniform operator topology.

Corollary: Let $|T^n| = o(n)$, and let $\lambda = 1$ be a pole of $R(\lambda; T)$ of order 1. Then $(n^{-1}\sum_{j=0}^{n-1} T^j)$ converges in the uniform topology to $E(1)$.

THEOREM: Let f, f_n be in $F(T)$, and let $(f(T)f_n(T))$ converge to 0 in the weak operator topology. Suppose that $(f_n(T)x)$ is weakly sequentially compact for each $x \in X$, and that f vanishes at a finite set of points of $\sigma(T)$. If each root λ_0 has finite order $\alpha(\lambda_0)$, if the sequences $(f_n^{(m)}(\lambda_0))$ converge for $0 \leq m < \alpha(\lambda_0)$, and if $\lim_{n\to\infty} f_n(\lambda_0) \neq 0$ then $(f_n(T))$ converges in the weak operator topology.Moreover,

$$X = closure(f(T)X) \oplus \{x : x \in Xmf(T)x = 0\}.$$

THEOREM: Let f, f_n be in $F(T)$, and let $(f(T)f_n(T))$ converge to 0 in the strong operator topology. Suppose that $(f_n(T)x)$ is weakly sequentially compact for each $x \in X$, and that f vanishes at a finite set of points of $\sigma(T)$. If each root λ_0 has finite order $\alpha(\lambda_0)$, if the sequences $(f_n^{(m)}(\lambda_0))$ converge for $0 \leq m < \alpha(\lambda_0)$, and if $\lim_{n\to\infty} f_n(\lambda_0) \neq 0$ then $(f_n(T))$ converges in the strong operator topology.Moreover,

$$X = closure(f(T)X) \oplus \{x : x \in Xmf(T)x = 0\}.$$

*Corollary:*Let X be reflexive, λ_n be sequence in $\rho(T)$ which converges to 0, and let $\sup_n |\lambda_n R(\lambda_n; T)| < \infty$. Then $X = closure(TX) \oplus \{x : x \in X, Tx = 0\}$ and the sequence $(\lambda_n R(\lambda_n; T))$ converges in the strong operator toology to the projection E, whose null manifold is $closure(TX)$, and whose range is $\{x : Tx = 0\}$.