100 MATHEMATICAL PROBLEMS

COMPILED BY YANG WANG

The following problems are compiled from various sources, particularly from

- D. J. Newman, *A Problem Seminar*, Springer-Verlag, 1982.
- Hugo Steinhaus, *One Hundred Problems in Elementary Mathematics*, Dover Publications, 1964.
- Piotr Biler and Alfred Witkowski, *Problems in Mathematical Analysis*, Marcel Dekker, Inc. 1990.
- Shan, Zun and Yan, Zhenjun, *Mathematics Olympiad – High School Edition*, Beijing University Press, 1993 (in Chinese).
- Problems from the Putnam Mathematical Competition.

Most of the problems listed here require no advanced mathematical background to solve, and they range from fairly easy to moderately difficult. I have deliberately avoided including very easy and very difficult problems. Nevertheless if you have not had experience solving mathematical problems you may find many of them challenging. The problems have *NOT* being sorted according to the degree of difficulty.

This list is not intended for commercial use.

Problem 1 Derive the operations $+$, $-$, $\times$, and $\div$ from $-$ and the reciprocal.

Problem 2 Invent a single (binary) operation that which $+$, $-$, $\times$, and $\div$ can be derived.

Problem 3 The multiplication of two complex numbers $(a+bi)\cdot(x+yi) = (ax-by)+(bx+ay)i$ appears to need 4 real multiplications, but does it really? If additions are free, can this be done using 3 real multiplications? In 2?

Problem 4 A micobe either splits into two perfect copies of itself or else die. If the probalility of splitting is p, what is the probability that one microbe will produce an everlasting colony?

Problem 5 Given any n distinct points in the plane, show that one of the angles determined by them is $\leq \pi/n$ (the 0 angle counts).

Problem 6 Prove that every infinite sequence (of real numbers) contains a monotone subsequence.

Problem 7 Device an experiment which uses only tosses of a fair coin, but which has success probability $1/3$. Do the same for any success probability p, $0 \leq p \leq 1$.

Problem 8 We alternate writing down binary digits after a decimal point, thereby producing a real number in the interval $[0, 1]$. You win if this number is transcendental. Can you force a win?

Problem 9 At a certain corner, the traffic light is green for 30 seconds and then red for 30 seconds. On the average, how much time is lost in this corner.

Problem 10 Prove that there is no equilateral triangle all of whose vertices are plane lattice points. (How about 3 dimensional lattice points?)

Problem 11 $x_{n+1} = (x_n + x_{n-1})/2$, x_0, x_1 given. Express $\lim_{n\to\infty} x_n$ explicitly.

Problem 12 If a set of psoitive integers has sum n, what is the biggest its product can be?

Problem 13 Prove that, at any party, there are two people who have the same number of friends present.

Problem 14 Maximize $2^{-x} + 2^{-1/x}$ over $(0, \infty)$.

Problem 15 N distinct non-collinear points are given. Prove that they determine at least N distinct lines.

Problem 16 Devise the smallest plane set such that no point is at a rational distance from all points of the set.

Problem 17 Given an infinite number of points in the plane with all the mutual distances integers, prove that the points are all collinear.

Problem 18 Given that $f(x, y)$ is a polynomial in x for each fixed y, and a polynomial in y for each fixed x, must $f(x, y)$ a polynomial in x and y?

Problem 19 Prove that the product of 3 consecutive integers is never a perfect power (i.e. a perfect square or a perfect cube, etc.).

Problem 20 I choose an integer from 0 through 15. You ask me 7 yes or no questions. I answer them all, but I'm allowed to *lie* once. (I needn't, but I am allowed to.) Determine my number.

Problem 21 Good coins weigh 10 gm, bad ones 9 gm. Given 4 coins and a scale (not a balance, but a true scale), detremine which are which in only three weighings.

Problem 22 Prove that the integer $[\sqrt{2} + 1)^n]$ are alternately even and odd. ($[x]$ denotes the largest integer not exceeding x.)

Problem 23 Split a beer *three ways.* To split a beer two ways you can let the first man divide into what he thinks are two equal parts and then let the second man choose one of them. Both are then satisfied. How can three do this?

Problem 24 Batter A has a higher batting average than batter B for the first half of the season and A also has a higher batting average than B for the second half of the season. Does it follow that A has a better batting average than B for the whole season?

Problem 25 Given that $f(x) + f'(x) \to 0$ as $x \to \infty$, prove that both $f(x) \to 0$ and $f'(x) \to 0$.

Problem 26 $F(x)$ is a positive increasing function on $[0, \infty)$ and y is any solution to the differential equation $y'' + F(x)y = 0$. Prove y remains bounded as $x \to \infty$.

Problem 27 Show that if $f(x)$ and $f''(x)$ are bounded, then so is $f'(x)$. (You can assume that $f''(x)$ is continuous.)

Problem 28 Prove that the equation $x^{x^{x^{\cdot^{\cdot^{\cdot}}}}} = 2$ is satisfied by $x = \sqrt{2}$, but the equation $x^{x^{x^{\cdot^{\cdot^{\cdot}}}}} = 4$ has no solution. What is the "break point"?

Problem 29 CRAZY DICE. Devise a pair of dice — cubes with positive integers on their faces, with exactly the same outcome as two ordinary dice (the sum 2 comes out once, the sum 3 comes out twice, etc.), but which are not ordinary dice.

Problem 30 At each plane lattice point there is placed a positive number in such a way that each is the average of its four nearest neighbors. Show that all the numbers are the same!

Problem 31 Let $x \geq 0$ be a real number and $m > 0$ be an integer. Simplify

$$\lfloor x \rfloor + \lfloor x + \tfrac{1}{m} \rfloor + \cdots + \lfloor x + \tfrac{m-1}{m} \rfloor$$

Problem 32 (Putnam 77) Determine all solutions of the system

$$\begin{aligned} x + y + z &= w \\ \frac{1}{x} + \frac{1}{y} + \frac{1}{z} &= \frac{1}{w}. \end{aligned}$$

Problem 33 Determine the triples of integers (x, y, z) satisfying the equation

$$x^3 + y^3 + z^3 = (x + y + z)^3.$$

Problem 34 Prove that every polynomial over the complex numbers has a nonzero polynomial multiple whose exponents are all divisible by 100.

Problem 35 Let $p_n(x) = 1+x+x^2+\cdots+x^{n-1}$. For which n is the $p_n(x^2)$ divisible by $p_n(x)$?

Problem 36 Factor $x^6 + x^4 + 3x^2 + 2x + 2$ over the integers into a product of irreducible factors.

Problem 37 If $n > 1$, show that $(x + 1)^n - x^n - 1 = 0$ has a multiple root if and only if $n - 1$ is divisible by 6.

Problem 38 Find all polynomials P with real coefficients and satisfying $P(P(x)) = P^2(x)$. Generalize.

Problem 39 Let u_n be the unique positive root of $x^n + x^{n-1} + \cdots + x - 1 = 0$. Find $\lim_{n\to\infty} u_n$.

Problem 40 Show that there exists no complex valued function $f(z)$ such that $f(f(z)) = az^2 + bz + c$, $a \neq 0$.

Problem 41 For what positive integer n is $x^n + 1/x^n$ expressible as a polynomial with real coefficients in $x - 1/x$?

Problem 42 Find all integral solutions to $x^2 + y^2 + z^2 = 2xyx$.

Problem 43 Determine all integral solutions of

$$a^2 + b^2 + c^2 = a^2b^2.$$

Problem 44 Prove that for any integer $n > 1$, the numbers $1, 2, 3, \ldots, n^2$ cannot be placed in an $n\times n$ square so that the product of the numbers in any row or column is the same.

Problem 45 For a positive integer n, show that the number of integral solutions (x, y) of

$$x^2 + xy + y^2 = n$$

is a multiple of 6.

Problem 46 23 people decide to play football — 11 people on each team plus one referee. To keep things fair, they agree that the total weight of each team must be the same. Everyone weighs an integer, and it turns out that no matter who is chosen to be the referee, it is always possible to construct two fair (sums of weights are equal) teams. Prove that everyone weighs the same.

Problem 47 Prove that for any $n > 1$ the sum $1 + \frac{1}{2} + \cdots + \frac{1}{n}$ is never an integer.

Problem 48 Prove that the number $5^{5k+1} + 4^{5k+2} + 3^{5k}$ is divisible by 11 for every natural k.

Problem 49 Let us divide a square of side length 1 into three parts A, B, C. Prove that whatever this division may be, there always exists at least one pair of points P, Q belonging both to the same part such that the distance PQ being greating than 1.00778.

Problem 50 Given an ellipse with major axis of length $2a$ and minor axis of length ab, draw a closed curve of the same length as the length of the ellipse such that the closed curve encloses an area greater than the area of the ellipse by $(a-b)^2$.

Problem 51 Let all plane sections of a certain surface be circles (a single point is taken to be a circle of radius 0). Show that this surface is a sphere.

Problem 52 We have 5 objects which differ in weight, and we wish to arrange them in a sequence of decreasing weights. We possess only a balance that allows us to compare the objects pairwise. How must we proceed in order to arrange the objects in the fastest possible way? How many comparisons will there be?

Problem 53 An ichthyologist wanted to estimate the number of fish in a pond which are suitable to be caught. He threw into the pond a net with regulation size mesh, and having taken the net out he found 30 fish in the net; he marked each of them with a suitable color, and threw them back into the pond. The next day he threw the same net and captured 40 fish, of which 2 had been marked. In what way did he compute (approximately) the number of fish in the pond?

Problem 54 A triangle has sides a, b, c and another has sides a', b', c'. Find necessary and sufficient conditions such that the first triangle can be put inside the second.

Problem 55 Find all functions $f:\ \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ with the property that $f(xf(y)) = yf(x)$.

Problem 56 The Fibonacci numbers $\{F_n\}$ are given by $F_1 = 1$, $F_2 = 2$ and $F_n = F_{n-1} + F_{n-2}$ for $n > 2$. Prove that every natural number m can be written uniquely as the sum of distinct Fibonacci numbers in which no two Fibonacci numbers are consecutive.

Problem 57 (Putnam 68) A is a subset of a finite group G (with group operation called multiplication), and A contains more than one half of the elements of G. Prove that each element is a product of two elements of A.

Problem 58 (Putnam 69) Let A abd B be matrices of size 3×2 and 2×3 respectively. Suppose that the product AB is given by

$$AB = \begin{bmatrix} 8 & 2 & -2 \\ 2 & 5 & 4 \\ -2 & 4 & 5 \end{bmatrix}.$$

Show that the product BA is gven by

$$BA = \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix}.$$

Problem 59 (Putnam 72) Let S be a set and let $*$ be a binary operator on S satisfying the laws

$$x * (x * y) = y \text{ and } (y * x) * x = y \text{ for all } x,\, y \in S.$$

Show that $*$ is commutative but not necessarily associative.

Problem 60 (Putnam 72) Let A and B be two elements in a group such that $ABA = BA^2B$, $A^3 = 1$ and $B^{2n-1} = 1$ for some positive integer n. Prove that $B = 1$.

Problem 61 (Putnam 76) Evaluate $\displaystyle\sum_{k=0}^{n} (-1)^k \binom{n}{k} (x-k)^n$.

Problem 62 (Putnam 77) Let H be a subgroup with h elements in a group G. Suppose that G gas an element a such that $(xa)^3 = 1$ for all $x \in H$. In G, let P be the subset of all products $x_1 a x_2 a \cdots x_n a$ with n a positive integer and the x_i in H.

(a) Show that P is a finite set.
(b) Show that, in fact, P has no more than $3h^2$ elements.

Problem 63 (Putnam 78) Let a, b, $p_1, p_2, \ldots, p_n$ be real numbers with $a \neq b$. Define $f(x) = \prod_{j=1}^{n}(p_j - x)$. Show that

$$\det \begin{bmatrix} p_1 & a & a & a & \cdots & a & a \\ b & p_2 & a & a & \cdots & a & a \\ b & b & p_3 & a & \cdots & a & a \\ b & b & b & p_4 & \cdots & a & a \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ b & b & b & b & \cdots & p_{n-1} & a \\ b & b & b & b & \cdots & b & p_n \end{bmatrix} = \frac{bf(a) - af(b)}{b-a}.$$

Problem 64 (Putnam 78) Express

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{m^2 n + mn^2 + 2mn}$$

as a rational number.

Problem 65 (Putnam 83) Let $\|u\|$ denote the distance from the real number u to the nearest integer. (For example, $\|2.8\| = 0.2 = \|3.2\|$.) For positive integers n let

$$a_n = \frac{1}{n} \int_1^n \left\| \frac{n}{x} \right\| dx.$$

Determine $\lim_{n\to\infty} a_n$. You may assume the identity

$$\frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot\frac{8}{7}\cdots=\frac{\pi}{2}.$$

Problem 66 Show that every polynomial p with rational coefficients such that $p^{-1}(\mathbb{Q}) \subseteq \mathbb{Q}$ must be linear.

Problem 67 Show that every complex polynomial p such that $p(\mathbb{R}) \subseteq \mathbb{R}$ and $p(\mathbb{C}\setminus\mathbb{R}) \subseteq \mathbb{C}\setminus\mathbb{R}$ must be linear.

Problem 68 Let $f(x) = (x - x_1)\cdots(x - x_n)$ where $x_i \neq x_j$ for $i \neq j$. Let $g(x) = x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_0$. Show that $\sum_{j=1}^{n} g(x_j)/f'(x_j) = 1$.

Problem 69 Show that every polynomial has a multiple which is a polynomial of x^{100000}.

Problem 70 Find the remainder when $(x + a)^n$ is divided by $(x + b)^m$.

Problem 71 Two polynomials $P(x)$ and $Q(x)$ are said to commute if $P(Q(x)) = Q(P(x))$. Prove that if two monic polynomials commute with a polynomial $a(x) \not\equiv x$ then they must commute.

Problem 72 The polynomials $P(x) = x^2 + ax + b$ and $Q(x) = x^2 + px + q$ have a common root. Find a quadratic polynomial whose roots coincide with the remaining roots of P and Q.

Problem 73 If $q = (\sqrt{5}+1)/2$ then for every natural n we have $[q[qn]]+1 = [q^2n]$. Here $[m]$ denote the greatest integer not exceeding m.

Problem 74 Prove the series $\displaystyle\sum_{n=2}^{\infty}(\log n)^{-\log\log n}$ diverges.

Problem 75 Show that $(\sin x)^2 \leq \sin(x^2)$ for $0 \leq x \leq \sqrt{\pi/2}$.

Problem 76 Solve the equation $\sqrt{\sin x} + \sqrt{\cos x} = t$.

Problem 77 Given a dense set in the plane does there always exist a line segment in which this set is dense?

Problem 78 Construct a smooth curve $g:\ \mathbb{R} \longrightarrow \mathbb{R}^2$ whose range is dense in $\mathbb{R}^2$.

Problem 79 Find all continuous functions f such that $f(2x + 1) = f(x)$ for all real x.

Problem 80 Let $A = [a_{ij}]$ be an $n \times n$ skew-symmetric matrix, i.e. $a_{ij} = -a_{ji}$. Suppose that $a_{ij} \in \mathbb{Z}$. Prove that $\det A$ is a perfect square.

Problem 81 Let A and B be positive semi-definite $n \times n$ matrices. Prove that $\det(A+B) \geq \det A$.

Problem 82 Let $f(x)$ be a strictly monotone function on $\mathbb{R}$ such that $f(x) + f^{-1}(x) = 2x$, where f^{-1} is the inverse of f. Find all such functions f.

Problem 83 Find a function $f(x)$ defined for $x > 1$ such that $\int_x^{x^2} f(t)dt = 1$ for all $x > 1$.

Problem 84 Compute $\sqrt{1+2\sqrt{1+3\sqrt{1+4\sqrt{\cdots}}}}$.

Problem 85 Fina all odd n such that $n|3^n - 1$.

Problem 86 Prove that there exists no integers b$n > 1$ such that $n|2^n - 1$.