SYMMETRIC ERROR ESTIMATES FOR MOVING MESH MIXED METHODS FOR ADVECTION-DIFFUSION EQUATIONS

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Abstract. A mixed method allowing a general class of mesh movements is proposed for an advection-diffusion equation in either conservative or non-conservative form. Various symmetric error estimates are derived for the method under certain conditions. In one space dimension (1-d), optimal order L^2 convergence and superconvergence are proved as a corollary of the symmetric estimates.

Keywords. mixed methods, parabolic equations, finite elements, moving mesh

AMS subject classification. 65M60, 65M12

1. Introduction.

Moving mesh finite element methods have been widely studied; in [9, 8] methods based on Galerkin formulations were given. In [5, 2] error analysis was provided for related classes of moving mesh finite element methods which allow piece-wise time continuous mesh movements. In this work, we examine moving mesh methods for mixed methods that incorporate some of the ideas in [4], where a procedure for including characteristics within finite element methods for advection-diffusion equations was proposed.

A symmetric error estimate is, to within a constant, a best approximation result. That is, if the error *can be* made small in the given norm then it *is* small in that norm. Somewhat more precisely, there is a norm $\|\cdot\|$, and a constant, C, such that

$$\|\text{error}\| \leq C\|\text{best approximation error}\|.$$

Dupont [5], Bank and Santos [2], Dupont and Liu [6], and Section 5 of this work establish bounds of this type. In [6] and this paper, the constant C does not increase as the advective term increases in size, provided the mesh movement approximates the advective term sufficiently well. These results thus make it clear that the mesh movement is actually modeling the advection. Also, the norms in Section 5 involve the convective derivative instead of the partial with respect to time, and as Douglas and Russel pointed out in [3], for advection dominated problems the convective derivative will typically be much smoother, and therefore easier to approximate well. While symmetric error estimates for parabolic equations have a certain attractiveness in the simplicity of the statement that they make, it is sometimes hard to see the precise meaning of the result because the norms involved are made up of several parts. We exploit the idea of [6] to weaken some of these parts to "concentrate" the norm on certain terms.

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Characteristics-type mixed methods have been studied in several papers, see e.g. Yang [11] and Arbogast and Wheeler [1], but the analytical understanding of mixed methods in combination with moving meshes is far from complete. Unlike Galerkin methods using conforming finite element spaces, moving mesh methods using mixed formulations and discontinuous approximation spaces can develop singularities in the time derivative at the edges between elements. Therefore it is critical to use directional time derivatives along the mesh movement direction throughout the analysis. In this paper, we first introduce our method and prove our symmetric error estimate. Next, an optimal order L^2 error estimate and a superconvergence result are proved for one space dimension as a corollary of the symmetric error estimate. The error bound is made as local as possible, which along with the error constant C, fully describes the effectiveness of a given mesh movement. In particular, while accuracy in aligning the mesh movement with the characteristics may be difficult in some circumstances, it may not be necessary as long as the difference between the advection velocity and the velocity of mesh movement remains bounded. Furthermore, the locality in the error bound shows that a second factor motivating mesh movement should be to provide a finer mesh where the solution has larger second order (or above) derivatives.

The remainder of this paper is organized as follows. In Section 2, we discuss the advection-diffusion equation in conservative form, introduce several notations, and formulate the mixed method for general mesh movements. In Section 3, we introduce a pseudo inverse operator "A" of "div", which plays a critical role in the symmetric error analysis in Section 5. In Section 4 we develop the basic properties of the directional derivative "$\frac{D}{Dt}$" which are important to the energy type analysis. Optimal order error bounds are proved in Section 6. In Section 7, we consider a mixed method for an advection-diffusion equation in non-conservative form, allowing general mesh movements. Symmetric error analysis and 1-d applications are derived in a manner that parallels the earlier analysis.

2. Model Problem and Mixed Method. Consider the following advection-diffusion model problem on $Q = \Omega \times (0,T)$,

$$\begin{cases} \partial_t u - \nabla \cdot (a \nabla u + bu) = f, & \text{on } Q, \\ u = 0, & \text{on } \partial\Omega \times (0,T), \\ u = u_0, & \text{for } t = 0, \end{cases} \tag{2.1}$$

where $a(x)$, $b(x)$, and $f(x,t)$ are smooth and bounded and $a_1 \geq a(x) \geq a_0 > 0$ for some constants a_0, a_1. Ω is a bounded domain in R^n. For simplicity, we assume Ω is a fixed polyhedron.

We use $\|\cdot\|_s$ to denote the $H^s(\Omega)$ norm. When $s = 0$, we usually use $\|\cdot\|$. If we use domains other than Ω we will use $\|\cdot\|_{H^s(\Omega_i)}$ or $\|\cdot\|_{L^2(\Omega_i)}$. The norm for the dual space of $H_0^1(\Omega)$ is denoted $\|\cdot\|_{-1}$, and $\|\xi\|_{L^p(0,T;X)}$ denotes the $L^p(0,T)$ norm of $\|\xi(\cdot,t)\|_X$. We will use $(\cdot,\cdot)$ as the inner product on $L^2(\Omega)$ and on $(L^2(\Omega))^n$, and rely on context to show which.

We will study methods that approximate the solution u of (2.1) on a moving mesh which is given as a time-dependent image of a fixed reference mesh. Suppose that $\bar{D} = \cup D_i$ is a fixed polyhedron where D_i's are closed sets with nonvoid disjoint interiors. We need few assumptions on the D_i's for much of the argument, but to keep the discussion simple, we suppose that each D_i is a simplex and that they form a tessellation of $\bar{D}$. Further, we suppose that there is a continuous mapping $\mathcal{G}$ from $\bar{D} \times [0,T]$ onto $\bar{\Omega} \times [0,T]$ such that:

1. for each t, $\mathcal{G}(\cdot,t)$ is a one-to-one piecewise linear mapping (with respect to $\{D_j\}$) of $\bar{D}$ onto $\bar{\Omega}$;
2. $\mathcal{G}$ is continuously differentiable on each D_i; and
3. $\partial\Omega = \mathcal{G}(\partial D, t)$.

Let $\Omega_i(t) = \mathcal{G}(D_i,t)$, $h_i(t)$ be the diameter of $\Omega_i(t)$ and $h(t) = \max_i\{h_i(t)\}$. Then $\Omega_i(t)$ is also a simplex and $\{\Omega_i(t)\}$ becomes the moving partition of Ω. It is sometimes convenient to think of this moving mesh as being generated by a mapping of Ω onto itself. Let $\mathcal{G}^{-1}$ denote the inverse of $\mathcal{G}$ as a map of D onto Ω; so this function is defined on Q. Let $\mathcal{G}_t$ be the partial of $\mathcal{G}$ with respect to t. The finite element mesh is advected with a flow that is given by

$$\dot{x}(t) = \mathcal{G}_t(\mathcal{G}^{-1}(x,t),t).$$

Given the assumptions on $\mathcal{G}$, the function $\dot{x}$ is a continuous piecewise linear function over the partition $\{\Omega_i\}$ of Ω. Let $\tilde{V}_h$ be a finite dimensional subspace of $L^2(D)$. Then the finite element space on Ω is defined by

$$V_h(t) = \{\phi(x,t) : \phi(\mathcal{G}(\cdot,t),t) \in \tilde{V}_h\}.$$

We will take $H_h(t)$ to be a finite dimensional subspace of $H(div,\Omega)$ so that $div\, H_h = V_h$ for any t. In particular, we will take V_h to be the space of discontinuous polynomials of total degree at most m and the H_h to be the Raviart-Thomas flux space. Let P_h denote the L^2 projection onto V_h. Let Π_h be the linear operator $H(div,\Omega) \to H_h$ satisfying $(div\,(W - \Pi_h W), r) = 0, \forall r \in V_h$ and $div\,\Pi_h = P_h div$ as defined by Raviart and Thomas in [10].

Let $\underline{h}(x,t)$ denote the function that has the value $h_i(t)$ on each $\Omega_i(t)$. For a function φ such that its restriction to Ω_i is in $H^s(\Omega_i)$, let

$$\|\varphi\|^2_{\underline{H}^s} = \sum_i \|\varphi\|^2_{H^s(\Omega_i)}.$$

We denote a particular directional derivative, DF/Dt, as follows:

$$\frac{DF(x,t)}{Dt} = \frac{\partial F(x,t)}{\partial t} + \dot{x}\cdot\nabla_x F(x,t).$$

Note that if $F(\cdot,t) \in V_h(t)$ is differentiable on each Ω_i then DF/Dt is also in V_h. Even though it might seem that both $\partial F/\partial t$ and $\nabla_x F$ are singular on the boundaries $\partial\Omega_i$, the directions involved in DF/Dt never cross the boundary of any Ω_i.

The first mixed method we consider uses a mesh movement induced flux across subdomain boundaries. Let $\sigma = -(a\nabla u + bu + \dot{x}u)$ and $\alpha = 1/a, \beta = b/a$. The exact solution u satisfies

$$\frac{Du}{Dt} + div\,\sigma + (\nabla\cdot\dot{x})u = f.$$

This leads to the following mixed formulation,

$$\begin{cases} (\alpha\sigma + (\beta + \alpha\dot{x})u, \mathcal{X}) - (u, div\,\mathcal{X}) = 0, & \forall \mathcal{X} \in H(div,\Omega), \\ \left(\dfrac{Du}{Dt} + div\,\sigma + (\nabla\cdot\dot{x})u, r\right) = (f,r), & \forall r \in L^2(\Omega). \end{cases} \tag{2.2}$$

We define the mixed approximation to be functions $u_h : [0,T] \to V_h$ and $\sigma_h : [0,T] \to H_h$, such that $u_h(0) = P_h u(0)$ and

$$\begin{cases} (\alpha\sigma_h + (\beta + \alpha\dot{x})u_h, \mathcal{X}) - (u_h, div\,\mathcal{X}) = 0, & \forall \mathcal{X} \in H_h, \\ \left(\dfrac{Du_h}{Dt} + div\,\sigma_h + (\nabla \cdot \dot{x})u_h, r\right) = (f, r), & \forall r \in V_h. \end{cases} \tag{2.3}$$

Note that this method is *locally conservative*, because the rate of change of the integral of u over each subdomain is given by the integral around the boundary of the normal component of σ, and the normal component of σ is continuous across subdomain boundaries. (If this is less than clear, please see the proof of Lemma 7.)

In proving the symmetric error estimates we don't need specific approximation properties, but we will need such properties in order to obtain a priori error bounds based on the mesh size and the smoothness of the solution u. We summarize these additional conditions here.

CONDITION 1 (Approximation). *There exists a constant C_1 such that for any $w \in H^{s_1}(\Omega)$, $s_1 \geq 0$, and any $t \in [0,T]$,*

$$\|w - P_h w\| \leq C_1 \|\underline{h}^{\min\{m+1, s_1\}} w\|_{\underline{H}^{s_1}};$$

and for any $W \in (H^{s_2}(\Omega))^n$, $s_2 \geq 1$, and any $t \in [0,T]$,

$$\|W - \Pi_h W\| \leq C_1 \|\underline{h}^{\min\{m_1+1, s_2\}} W\|_{\underline{H}^{s_2}},$$

where $m_1 = m + 1$ in 1-d and $m_1 = m$ in higher space dimension. This condition holds for for the Raviart-Thomas spaces, where C_1 depends on m and on a bound for $h_i/\tilde{h}_i$, where $\tilde{h}_i$ is the diameter of the largest ball in R^n contained in Ω_i.

CONDITION 2 (Stability of Π_h). *There exists a constant C_2 such that any for any $W \in (H^1(\Omega))^n$, and any $t \in [0,T]$,*

$$\|\Pi_h W\| \leq C_2 \|W\|_1.$$

If Condition 1 holds then C_2 can be taken to be $1 + C_1 h$. But this condition is strictly weaker than Condition 1; it allows controlled degeneracy in the elements as the mesh size decreases.

CONDITION 3 (H^2 Regularity). *The domain Ω is regular enough such that there exists a C_3 such that, for any $\xi \in L^2(\Omega)$, the boundary value problem*

$$\begin{cases} \Delta g = \xi, & \text{in } \Omega, \\ g = 0, & \text{on } \partial\Omega, \end{cases} \tag{2.4}$$

has a unique solution and $\|g\|_2 \leq C_3 \|\xi\|$.

3. A Pseudo-Inverse of *div*. In this section we define and explore the properties of a smoothing mapping that appears naturally in the symmetric error estimates. Let $A : L^2(\Omega) \to H_h$ be the pseudoinverse of div in the sense that

$$\varphi - div\,(A\varphi) \perp V_h,$$

$$\|A\varphi\| \text{ is minimal.}$$

Note that $A(\varphi) = A(P_h\varphi)$, thus we can factor A as $A_{V_h}P_h$, where A_{V_h} is A restricted to V_h. Note that this factorization gives that A^* maps H_h into V_h. Let $H_h = \mathcal{O} \oplus \mathcal{O}^\perp$ where $\mathcal{O} = \{\mathcal{X} \in H_h : div\, \mathcal{X} = 0\}$, and $\mathcal{O}^\perp$ is its orthogonal complement with respect to the $(L^2(\Omega))^n$ inner product. Then div is a one-to-one mapping from $\mathcal{O}^\perp$ onto V_h, and A_{V_h} is its inverse. In the case of 1-d with $m = 0$ the operator A can be explicitly described: $A\varphi$ is the piecewise linear interpolant of a constant plus the integral of φ. The following result shows that in more general situations A behaves as a smoothing operator.

THEOREM 4. *If Conditions 1 and 3 hold, then there is a $C = C(C_1, C_3)$ such that for any $\xi \in L^2(\Omega)$,*

$$\|A\xi\| \le C\{h\|\xi\| + \|\xi\|_{-1}\},$$
$$\|A\xi\| \le C\{h\|P_h\xi\| + \|P_h\xi\|_{-1}\}.$$

Proof. Let g be the solution of (2.4) and set $W = \nabla g$. Take $\rho \in H_h$ and $\nu \in V_h$ to the the mixed method approximation of W and g:

$$\begin{cases} (\rho, \mathcal{X}) + (\nu, div\, \mathcal{X}) = 0, & \forall \mathcal{X} \in H_h, \\ (div\, \rho, r) = (\xi, r), & \forall r \in V_h. \end{cases} \tag{3.1}$$

We want to show that $\rho = A\xi$. In fact, the second equation of (3.1) implies $div\, \rho = P_h\xi$ and the first one implies $(\rho, \mathcal{X}) = 0, \forall \mathcal{X} \in \mathcal{O}$, which in turn implies that $\|\rho\|$ is minimal among all elements in H_h whose divergence is $P_h\xi$.

Next we need an approximation result for mixed methods (see e.g. [7]) to see that

$$\begin{aligned} \|A\xi\|^2 &= (\rho, \rho) \\ &= (\rho, \rho - W) + (\rho, W) \\ &\le \|\rho\|\{Ch\|g\|_2 + \|W\|\} \\ &\le \|A\xi\|\{Ch\|\xi\| + \|W\|\}. \end{aligned} \tag{3.2}$$

It follows from (2.4) that

$$\|W\| = \|\nabla g\| \le C\|\xi\|_{-1}.$$

From this and (3.2) the first result of this theorem follows. The second follows since $A\xi = AP_h\xi$. □

Note that even if Ω fails to have the assumed H^2-regularity, the result may still be proved in some cases. Suppose that Ω can be expanded to $\tilde{\Omega}$ which has H^2-regularity and the function spaces can be extended to $\tilde{\Omega}$ with the approximation properties still holding. Then extending ξ to be zero on $\tilde{\Omega} - \Omega$ and a slight modification of the above proof gives the conclusions of the theorem. For example, if Ω were a L-shaped region in two space dimensions, H^2-regularity would fail, but the extension to a square might be possible.

On H_h, the operator $A\ div$ does not increase the L^2-norm. Suppose that $\rho \in H_h$ and let $\psi = A\ div\, \rho$. Then $div\, \rho - div\, \psi \perp V_h$. Hence $\psi = \rho + z$, where $z \in \mathcal{O}$. Because $\|\psi\|$ is taken to be minimal and $z \equiv 0$ is possible, we see that

$$\|A\ div\, \rho\| = \|\psi\| \le \|\rho\|. \tag{3.3}$$

In 1-d the choice of discontinuous piecewise polynomial spaces allows a more local version of Theorem 4. In fact, let $\Omega = (x_0, x_N)$ and $\Omega_i = (x_{i-1}, x_i)$, then $A\xi = \int_{x_0}^{x} P_h\xi(s)ds + C$.

THEOREM 5. *If Condition 2 holds, then there is a C such that for any $\xi \in L^2(\Omega)$,*

$$\|A\xi\| \le C\|\xi\|.$$

Proof. Take $\rho \in H_h$ and $\nu \in V_h$ to be defined by (3.1); so we know that $A\xi = \rho$. From (3.1) with $\chi = A\xi$ and $r = \nu$ we see that

$$\|A\xi\|^2 = -(\xi, \nu) \le \|\xi\| \, \|\nu\|.$$

Let B be a cube that contains Ω and take φ be the extension of ν to B by zero outside Ω. Take $g \in H_0^1(B)$ such that, on B, $\Delta g = \varphi$. Then, because the cube has H^2 regularity for the Laplacian, we see that ∇g is bounded in $(H^1(B))^2$ by $C\|\varphi\|_{L^2(B)} = C\|\nu\|$. Note that

$$\|\nu\|^2 = (\nu, div\nabla g) = (\nu, div\Pi_n\nabla g) = (A\xi, \Pi_h\nabla g).$$

The operator Π_h is bounded as a map of H^1 into L^2 by Condition 2. Thus it follows that

$$\|\nu\|^2 \le \|A\xi\| \, \|\Pi_h\nabla g\| \le C\|A\xi\| \, \|g\|_2 \le C\|A\xi\| \, \|\nu\|.$$

The two displayed inequalities then give the desired result. □

4. Properties of $\frac{D}{Dt}$. From the definition of directional derivative we have the following basic relations which we use later in energy-type arguments.

LEMMA 6.

$$\nabla_x \cdot \dot{x} = \frac{\partial|\det(\nabla_s\mathcal{G}(s,t))|/\partial t}{|\det(\nabla_s\mathcal{G})|}.$$

Proof. Take $D_0 \subset D$ to be an arbitrary small ball and let $\Omega_0(t) = \mathcal{G}(D_0, t)$. Then, with n as the outward normal to Ω_0,

$$\frac{\partial}{\partial t}\int_{\Omega_0(t)} dx = \int_{\partial\Omega_0(t)} \dot{x} \cdot n d\sigma = \int_{\Omega_0(t)} \nabla_x \cdot \dot{x} dx.$$

On the other hand,

$$\begin{aligned}\frac{\partial}{\partial t}\int_{\Omega_0(t)} dx &= \frac{\partial}{\partial t}\int_{D_0} |\det(\nabla_s\mathcal{G}(s,t))|ds = \int_{D_0} \frac{\partial}{\partial t}|\det(\nabla_s\mathcal{G}(s,t))|ds \\ &= \int_{\Omega_0(t)} \frac{\partial|\det(\nabla_s\mathcal{G}(s,t))|/\partial t}{|\det(\nabla_s\mathcal{G})|}dx.\end{aligned}$$

The result follows from the arbitrary choice of D_0. □

We will say that a function ξ on Q is piecewise C^1 if when it is pulled back by $\mathcal{G}$ to $D_i^o \times (0,T)$ it can be extended to be C^1 on $D_i \times [0,T]$. A function that is the limit in $H^1(D_i \times [0,T])$ of piecewise C^1 functions will be called piecewise smooth on Q. We will usually operate formally on piecewise smooth functions without going through the step of approximating them by smooth functions and taking limits, since this is routine.

LEMMA 7. *Suppose that ξ, is piecewise smooth on Q, then with $\mathcal{R} = \Omega$ or Ω_i,*

$$\frac{d}{dt}\int_{\mathcal{R}} \xi dx = \int_{\mathcal{R}} \frac{D\xi}{Dt} dx + \int_{\mathcal{R}} \xi(\nabla_x \cdot \dot{x}) dx.$$

Proof. It suffices to show the result for Ω_i. Note that

$$\begin{aligned} \frac{d}{dt}\int_{\Omega_i} \xi dx &= \frac{d}{dt}\int_{D_i} \xi |\det(\nabla \mathcal{G})| ds \\ &= \int_{D_i} \frac{\partial \xi}{\partial t} |\det(\nabla \mathcal{G})| ds + \int_{D_i} \xi \frac{\partial}{\partial t} |\det(\nabla \mathcal{G})| ds \\ &= \int_{\Omega_i} \frac{D\xi}{Dt} dx + \int_{\Omega_i} \xi \left(\frac{\partial |\det(\nabla_s \mathcal{G}(s,t))| / \partial t}{|\det(\nabla_s \mathcal{G})|} \right) dx. \end{aligned}$$

Using Lemma 6, the proof is complete. □

$\frac{D}{Dt}$ also has the following properties for any piecewise smooth functions ξ, η:

$$\begin{aligned} \frac{D}{Dt}(\xi\eta) &= \eta \frac{D\xi}{Dt} + \xi \frac{D\eta}{Dt}; \\ \frac{D}{Dt}\nabla_x \xi &= \nabla_x \frac{D\xi}{Dt} - (\nabla_x \dot{x})^T \nabla_x \xi, \end{aligned}$$

where $\nabla_x \xi$ is a column vector and $\nabla_x \dot{x}$ is the Jacobian of $\dot{x}$ with respect to x.

It easily follows from this and Lemma 7 that

$$\left(\frac{D\xi}{Dt}, \xi \right) = \frac{1}{2}\frac{d}{dt}\|\xi\|^2 - \frac{1}{2}(\xi, \xi(\nabla_x \cdot \dot{x})). \tag{4.1}$$

We denote the pseudo derivative of ξ by

$$D_t \xi = \frac{D\xi}{Dt} + (\nabla \cdot \dot{x})\xi,$$

and now show that D_t commutes with P_h.

LEMMA 8. *For function ξ that is piecewise smooth on Q $P_h D_t \xi = D_t P_h \xi$.*

Proof. Let $\psi = P_h \xi$, then $(\xi - \psi, r) = 0$ for any $r \in V_h$. Given $t_0 \in [0, T]$, let $\phi(x)$ be any function in $V_h(t_0)$. Let $r(x,t) = \phi(\mathcal{G}(\mathcal{G}^{-1}(x,t), t_0))$. Then $r(x, t_0) = \phi(x)$, $r(\cdot, t) \in V_h(t)$ and $\frac{Dr}{Dt} = 0$ for any $t \in [0, T]$. Thus at t_0,

$$\begin{aligned} 0 &= \frac{d}{dt}(\xi - \psi, r) \\ &= \left(\frac{D}{Dt}(\xi - \psi), \phi \right) + \left(\xi - \psi, \frac{Dr}{Dt} \right) + (\xi - \psi, (\nabla_x \cdot \dot{x})\phi). \end{aligned}$$

That is,

$$0 = (D_t(\xi - \psi), \phi) = (P_h D_t \xi - D_t P_h \xi, \phi).$$

The proof is completed by observing $D_t P_h \xi \in V_h$. □

5. Symmetric Error Estimates. In this section, we prove four symmetric error estimates.

Let F_h be a linear operator $V_h(t) \to H_h(t)$ such that for any $v_h \in V_h(t)$,

$$(\alpha F_h(v_h) + (\beta + \alpha \dot{x}) v_h, \mathcal{X}) - (v_h, div\, \mathcal{X}) = 0, \ \forall \mathcal{X} \in H_h.$$

Thus F_h is the flux operator associated with the space V_h. Using F_h and the norms $|||(\cdot,\cdot)|||$ and $|||(\cdot,\cdot)|||_*$ defined by

$$|||(\eta,\psi)|||^2 = \|\eta\|^2_{L^\infty(0,T:L^2(\Omega))} + \left\| A\frac{D\eta}{Dt}\right\|^2_{L^2(0,T:L^2(\Omega))} + \|A(div\,\psi)\|^2_{L^2(0,T:L^2(\Omega))},$$

$$|||(\eta,\psi)|||_*^2 = \|P_h\eta\|^2_{L^\infty(0,T:L^2(\Omega))} + \left\| A\frac{D\eta}{Dt}\right\|^2_{L^2(0,T:L^2(\Omega))} + \|A(div\,\psi)\|^2_{L^2(0,T:L^2(\Omega))},$$

we have the following pair of symmetric error estimates.

THEOREM 9. *Suppose Condition 2 holds, and there exist constants c_1, c_2 such that for all $(x,t) \in Q$,*

$$|\nabla_x \cdot \dot{x}| \le c_1, \qquad and \qquad |\beta + \alpha \dot{x}| \le c_2.$$

Then there exists a constant $C > 0$, depending only on C_2, c_1, c_2, T, the bounds of coefficient a, and Ω, such that for any piecewise smooth function v_h with $v_h(\cdot,t) \in V_h(t)$,

$$|||(u-u_h, \sigma-\sigma_h)||| \le C|||(u-v_h, \sigma - F_h(v_h))|||,$$
$$|||(u-u_h, \sigma-\sigma_h)|||_* \le C|||(u-v_h, \sigma - F_h(v_h))|||_*.$$

Proof. Take v_h to be a piecewise C^1 function such that $v_h(\cdot,t) \in V_h(t)$. With $\mathcal{S}_h = F_h(v_h)$, adopt the notation

$$\nu = u_h - v_h, \qquad \rho = \sigma_h - \mathcal{S}_h,$$
$$\eta = u - v_h, \qquad \psi = \sigma - \mathcal{S}_h.$$

Subtracting (2.2) from (2.3), we obtain the following orthogonalities:

$$\begin{aligned} &(\alpha\rho + (\beta+\alpha\dot{x})\nu, \mathcal{X}) - (\nu, div\,\mathcal{X}) = 0, && \forall \mathcal{X} \in H_h, \\ &\left(\frac{D\nu}{Dt} + div\,\rho + (\nabla\cdot\dot{x})\nu, r\right) = \left(\frac{D\eta}{Dt} + div\,\psi + (\nabla\cdot\dot{x})\eta, r\right), && \forall r \in V_h. \end{aligned} \tag{5.1}$$

With $\mathcal{X} = \rho$ and $r = \nu$, these and (4.1) give

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\nu\|^2 + (\alpha\rho + (\beta+\alpha\dot{x})\nu, \rho) \\ &\qquad = \left(\frac{D\eta}{Dt} + div\,\psi, \nu\right) + ((\nabla\cdot\dot{x})\eta, \nu) - \frac{1}{2}\int_\Omega \nu^2(\nabla\cdot\dot{x})dx \\ &\qquad = \left(div\,A\left(\frac{D\eta}{Dt} + div\,\psi\right), \nu\right) + ((\nabla\cdot\dot{x})\eta, \nu) - \frac{1}{2}\int_\Omega \nu^2(\nabla\cdot\dot{x})dx \\ &\qquad = \left(\alpha\rho + (\beta+\alpha\dot{x})\nu, A\left(\frac{D\eta}{Dt} + div\,\psi\right)\right) + ((\nabla\cdot\dot{x})\eta, \nu) \\ &\qquad\quad - \frac{1}{2}\int_\Omega \nu^2(\nabla\cdot\dot{x})dx. \end{aligned} \tag{5.2}$$

Therefore

$$\frac{d}{dt}\|\nu\|^2 + \alpha_1\|\rho\|^2 \le C\left\{\|\nu\|^2 + \left\|A\left(\frac{D\eta}{Dt} + div\,\psi\right)\right\|^2 + \|\eta\|^2\right\}, \tag{5.3}$$

where $\alpha_1 = 1/a_1$. It follows from Gronwall's inequality that

$$\|\nu\|^2_{L^\infty(0,T:L^2(\Omega))} + \|\rho\|^2_{L^2(0,T:L^2(\Omega))} \le C\{\|\nu(0)\|^2 + \|\!|(\eta,\psi)|\!\|^2\}.$$

The choice of $u_h(0) = P_h u(0)$ shows $\|\nu(0)\| \le \|\eta(0)\|$ so the $\|\nu(0)\|$-term is bounded by $\|\!|(\eta,\psi)|\!\|$. Combining these results with (3.3) we see that

$$\|\nu\|^2_{L^\infty(0,T:L^2(\Omega))} + \|A\;div\,\rho\|^2_{L^2(0,T:L^2(\Omega))} \le C\|\!|(\eta,\psi)|\!\|^2.$$

Note that $\nu = P_h\nu$ and $((\nabla\cdot\dot{x})\eta,\nu) = ((\nabla\cdot\dot{x})P_h\eta,\nu)$, since $\nabla\cdot\dot{x}$ is constant on each Ω_i and V_h has no continuity between subdomains. Therefore we can replace $\|\nu\|$ by $\|P_h\nu\|$, $\|\eta\|$ by $\|P_h\eta\|$ in (5.3) to obtain

$$\|P_h\nu\|^2_{L^\infty(0,T:L^2(\Omega))} + \|A\;div\,\rho\|^2_{L^2(0,T:L^2(\Omega))} \le C\|\!|(\eta,\psi)|\!\|^2_*.$$

It remains to estimate $\|A(\frac{D\nu}{Dt})\|^2$. Using (5.1) and Theorem 5

$$\begin{aligned}
\left(A\frac{D\nu}{Dt}, A\frac{D\nu}{Dt}\right) &= \left(\frac{D\nu}{Dt}, A^*A\frac{D\nu}{Dt}\right)\\
&= -\left(div\,\rho + (\nabla\cdot\dot{x})\nu, A^*A\frac{D\nu}{Dt}\right)\\
&\qquad\qquad + \left(\frac{D\eta}{Dt} + div\,\psi + (\nabla\cdot\dot{x})\eta, A^*A\frac{D\nu}{Dt}\right)\\
&= -\left(A\;div\,\rho + A(\nabla\cdot\dot{x})\nu, A\frac{D\nu}{Dt}\right)\\
&\qquad\qquad + \left(A\frac{D\eta}{Dt} + A\;div\,\psi + A(\nabla\cdot\dot{x})\eta, A\frac{D\nu}{Dt}\right)\\
&\le C\left\|A\frac{D\nu}{Dt}\right\|\left\{\|A\;div\,\rho\| + \|\nu\| + \left\|A\frac{D\eta}{Dt}\right\| + \|A\;div\,\psi\| + \|\eta\|\right\}.
\end{aligned} \tag{5.4}$$

Therefore we have

$$\left\|A\frac{D\nu}{Dt}\right\|^2_{L^2(0,T:L^2(\Omega))} \le C\|\!|(\eta,\psi)|\!\|^2.$$

Since

$$\left(A(\nabla\cdot\dot{x})\eta, A\frac{D\nu}{Dt}\right) = \left(AP_h(\nabla\cdot\dot{x})\eta, A\frac{D\nu}{Dt}\right) = \left(A(\nabla\cdot\dot{x})P_h\eta, A\frac{D\nu}{Dt}\right),$$

we also have

$$\left\|A\frac{D\nu}{Dt}\right\|^2_{L^2(0,T:L^2(\Omega))} \le C\|\!|(\eta,\psi)|\!\|^2_*.$$

Hence,

$$\|(\nu,\rho)\| \le C\|(u-v_h,\sigma-\mathcal{S}_h)\|$$
$$\|(\nu,\rho)\|_* \le C\|(u-v_h,\sigma-\mathcal{S}_h)\|_*.$$

Applying the triangle inequality completes the proof. □

Next we define two additional norms $\|(\cdot,\cdot)\|_{D_t}$, $\|(\cdot,\cdot)\|_{D_t^*}$ by

$$\|(\eta,\psi)\|_{D_t}^2 = \|\eta\|^2_{L^\infty(0,T:L^2(\Omega))} + \|AD_t\eta\|^2_{L^2(0,T:L^2(\Omega))} + \|A(div\,\psi)\|^2_{L^2(0,T:L^2(\Omega))},$$
$$\|(\eta,\psi)\|_{D_t^*}^2 = \|P_h\eta\|^2_{L^\infty(0,T:L^2(\Omega))} + \|AD_t\eta\|^2_{L^2(0,T:L^2(\Omega))} + \|A(div\,\psi)\|^2_{L^2(0,T:L^2(\Omega))},$$

and use them to get the following pair of symmetric error estimates.

THEOREM 10. *Suppose there exist constants $c_1, c_2 > 0$ such that*

$$-\nabla_x\cdot\dot{x} \le c_1 \quad \textit{and} \quad |\beta+\alpha\dot{x}| \le c_2$$

for all $(x,t)\in Q$. Then there exists a constant $C>0$, depending only on c_1, c_2, T, the bounds of coefficient a, and Ω, such that, for any piecewise smooth function v_h with $v_h(\cdot,t)\in V_h(t)$,

$$\|(u-u_h,\sigma-\sigma_h)\|_{D_t} \le C\|(u-v_h,\sigma-F_h(v_h))\|_{D_t},$$
$$\|(u-u_h,\sigma-\sigma_h)\|_{D_t^*} \le C\|(u-v_h,\sigma-F_h(v_h))\|_{D_t^*}.$$

Proof. We slightly modify the proof of Theorem 9. The inequality (5.2) becomes

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nu\|^2 + (\alpha\rho+(\beta+\alpha\dot{x})\nu,\rho)\\
&\quad= (D_t\eta + div\,\psi,\nu) - \frac{1}{2}\int_\Omega \nu^2(\nabla\cdot\dot{x})dx\\
&\quad= (div\,A(D_t\eta+div\,\psi),\nu) - \frac{1}{2}\int_\Omega \nu^2(\nabla\cdot\dot{x})dx \qquad (5.5)\\
&\quad= (\alpha\rho+(\beta+\alpha\dot{x})\nu, A(D_t\eta+div\,\psi)) - \frac{1}{2}\int_\Omega \nu^2(\nabla\cdot\dot{x})dx.
\end{aligned}$$

Therefore

$$\frac{d}{dt}\|\nu\|^2 + \alpha_1\|\rho\|^2 \le C\{\|\nu\|^2 + \|A(D_t\eta+div\,\psi)\|^2\}. \qquad (5.6)$$

It then follows from Gronwall's inequality and (3.3) that

$$\|\nu\|^2_{L^\infty(0,T:L^2(\Omega))} + \|A\ div\,\rho\|^2_{L^2(0,T:L^2(\Omega))} \le C\|(\eta,\psi)\|^2_{D_t},$$

and, since $P_h\nu=\nu$,

$$\|P_h\nu\|^2_{L^\infty(0,T:L^2(\Omega))} + \|A\ div\,\rho\|^2_{L^2(0,T:L^2(\Omega))} \le C\|(\eta,\psi)\|^2_{D_t^*}.$$

It remains to estimate $\|AD_t\nu\|^2$.

$$\begin{aligned}
(AD_t\nu, AD_t\nu) &= (D_t\nu, A^*AD_t\nu)\\
&= -(div\,\rho, A^*AD_t\nu) - (D_t\eta + div\,\psi, A^*AD_t\nu)\\
&= -(A\ div\,\rho, AD_t\nu) - (AD_t\eta + A\ div\,\psi, AD_t\nu)\\
&\le C\|AD_t\nu\|\{\|A\ div\,\rho\| + \|AD_t\eta\| + \|A\ div\,\psi\|\}.
\end{aligned}$$

Therefore

$$\|AD_t\nu\|^2_{L^2(0,T:L^2(\Omega))} \le C(\|\nu(0)\|^2 + \|(\eta,\psi)\|^2_{D_t}),$$

and

$$\|AD_t\nu\|^2_{L^2(0,T:L^2(\Omega))} \le C(\|P_h\nu(0)\|^2 + \|(\eta,\psi)\|^2_{D_t^*}).$$

As before, the triangle inequality completes the proof. □

Note that Theorem 10 uses A but does not rely on Theorem 5; hence it does not require Condition 2 to hold.

6. Optimal Order and Superconvergent $L^2(\Omega)$ Bounds in One Space Dimension. In 1-d, Ω is an interval. Let c_4 be a constant satisfying $c_4 \ge \frac{1}{2}(a_0 + \frac{\tilde{c}_2}{a_0})$, where $\tilde{c}_2 = \|b + \dot{x}\|_{L^\infty([0,T],L^\infty(\Omega))}$. Assume a, b are sufficiently regular such that for any $g \in L^2(\Omega)$, the elliptic equation

$$\begin{cases} -\partial_x(a\partial_x w) + (b+\dot{x})\partial_x w + c_4 w = g, in\ \Omega, \\ w|_{\partial\Omega} = 0, \end{cases} \tag{6.1}$$

has a unique solution w satisfying $\|w\|_2 \le C\|g\|$.

We have the following optimal order $L^2(\Omega)$ error estimate.

THEOREM 11. *Suppose Condition 1 holds, and there exist constants c_1, c_2, c_3 such that, for any $t \in [0,T]$, $\|\partial_x\dot{x}\|_\infty, \|\partial_x b\|_\infty \le c_1$; $\|\beta + \alpha\dot{x}\|_\infty, \|\frac{D}{Dt}(\beta+\alpha\dot{x})\|_\infty \le c_2$; $\|\partial_x a\|_\infty, \|\frac{D\alpha}{Dt}\|_\infty \le c_3$; Then there exists a constant C, depending on C_1, c_1, c_2, c_3, Ω, T, and the bounds of coefficient a such that, for h sufficiently small*

$$\begin{aligned} \|u - u_h\| \le C \Big\{ &\|\underline{h}^{\min\{m+1,s\}} u\|_{L^\infty[0,T,\underline{H}^s]} + \|h\underline{h}^{\min\{m+1,s-1\}} \frac{Du}{Dt}\|_{L^2[0,T;\underline{H}^{s-1}]} \\ &+ \|\underline{h}^{\min\{m+2,s\}}\sigma\|_{L^2[0,T;\underline{H}^s]} + \|h^2\underline{h}^{\min\{m+1,s-2\}}\sigma\|_{L^2[0,T;\underline{H}^{s-1}]} \\ &+ \|h^2\underline{h}^{\min\{m+1,s-2\}}\frac{D\sigma}{Dt}\|_{L^2[0,T;\underline{H}^{s-1}]} \Big\}. \end{aligned} \tag{6.2}$$

Proof. This is an application of Theorem 10 using $\|\cdot\|_{D_t}$. Since $\|(u-u_h, \sigma-\sigma_h)\|_{D_t}$ dominates the term we want to bound, it suffices to show that $\|(u-v_h, \sigma - F_h(v_h))\|_{D_t}$ can bounded by terms on the right- hand side of (6.2) for suitable choice of v_h.

At each time we take the elliptic projection (v_h, S_h) of (u,σ) into $V_h \times H_h$ to satisfy

$$\begin{cases} (\alpha(S_h - \sigma) + (\beta+\alpha\dot{x})(v_h - u), \mathcal{X}) - (v_h - u, \partial_x\mathcal{X}) = 0, & \forall \mathcal{X} \in H_h, \\ (\partial_x(S_h - \sigma) + c_4(v_h - u), r) = 0, & \forall r \in V_h, \end{cases} \tag{6.3}$$

Notice that $S_h = F_h(v_h)$.

Differentiating (6.3) with respect to time, using Lemma 7 and properties of $\frac{D}{Dt}$, we have

$$\begin{cases} \left(\alpha\frac{D}{Dt}(S_h - \sigma) + (\beta+\alpha\dot{x})\frac{D}{Dt}(v_h - u), \mathcal{X}\right) - \left(\frac{D}{Dt}(v_h - u), \partial_x\mathcal{X}\right) & \\ \qquad = (E_1(S_h - \sigma), \mathcal{X}) + (E_2(v_h - u), \mathcal{X}), & \forall \mathcal{X} \in H_h, \\ \left(\partial_x\frac{D}{Dt}(S_h - \sigma) + c_4\frac{D}{Dt}(v_h - u), r\right) = (E_3(v_h - u), r), & \forall r \in V_h, \end{cases} \tag{6.4}$$

where

$$E_1 = -\left(\frac{D}{Dt}\alpha + \alpha\partial_x\dot{x}\right),$$
$$E_2 = -\left(\frac{D}{Dt}(\beta + \alpha\dot{x}) + (\beta + \alpha\dot{x})\partial_x\dot{x}\right),$$
$$E_3 = -\, c_4\partial_x\dot{x}.$$

Here we are also using the fact that for any given $t_0 \in [0,T]$, $\mathcal{X}(x) \in H_h(t_0)$ and $r(x) \in V_h(t_0)$, we can define $\tilde{\mathcal{X}}(x,t) = \mathcal{X}(\mathcal{G}(\mathcal{G}^{-1}(x,t),t_0)) \in H_h(t)$ and $\tilde{r}(x,t) = r(\mathcal{G}(\mathcal{G}^{-1}(x,t),t_0)) \in V_h(t)$ for any $t \in [0,T]$, so that $\tilde{\mathcal{X}}(x,t_0) = \mathcal{X}(x)$, $\tilde{r}(x,t_0) = r(x)$ and $\frac{D}{Dt}\tilde{\mathcal{X}} = \frac{D}{Dt}\tilde{r} = 0$.

Because of (6.1), using the duality lemma in [3], for any h sufficiently small, we have

$$\|v_h - P_h u\| \leq C\{h\|S_h - \sigma\| + h\|P_h u - u\| + h^2\|\partial_x(S_h - \sigma)\|\}. \tag{6.5}$$

From the second equation of (6.3) we have

$$\|P_h\partial_x(S_h - \sigma)\| \leq C\|v_h - P_h u\|. \tag{6.6}$$

Therefore using the triangle inequality

$$\|\partial_x(S_h - \sigma)\| \leq C\|v_h - P_h u\| + \|P_h\partial_x\sigma - \partial_x\sigma\|. \tag{6.7}$$

Also from the first equation of (6.3)

$$\begin{aligned}\|S_h - \Pi_h\sigma\|^2 &\leq C(\alpha(S_h - \sigma), S_h - \Pi_h\sigma) + C(\alpha(\sigma - \Pi_h\sigma), S_h - \Pi_h\sigma) \\ &= C(v_h - u, \partial_x(S_h - \Pi_h\sigma)) - C((\beta + \alpha\dot{x})(v_h - u), S_h - \Pi_h\sigma) \\ &\qquad + C(\alpha(\sigma - \Pi_h\sigma), S_h - \Pi_h\sigma).\end{aligned} \tag{6.8}$$

Note that

$$(v_h - u, \partial_x(S_h - \Pi_h\sigma)) = (v_h - P_h u, P_h\partial_x(S_h - \sigma)) \leq C\|v_h - P_h u\|^2,$$

therefore

$$\begin{aligned}\|S_h - \Pi_h\sigma\|^2 &\leq C\{\|v_h - P_h u\|^2 + \|u - P_h u\|^2 + \|\sigma - \Pi_h\sigma\|^2\}, \\ \|S_h - \sigma\|^2 &\leq C\{\|v_h - P_h u\|^2 + \|u - P_h u\|^2 + \|\sigma - \Pi_h\sigma\|^2\}.\end{aligned} \tag{6.9}$$

Substituting into (6.5) we have

$$\|v_h - P_h u\| \leq C\{h\|u - P_h u\| + h\|\sigma - \Pi_h\sigma\| + h^2\|P_h\partial_x\sigma - \partial_x\sigma\|\}. \tag{6.10}$$

Using the triangle inequality

$$\|v_h - u\| \leq C\{\|u - P_h u\| + h\|\sigma - \Pi_h\sigma\| + h^2\|P_h\partial_x\sigma - \partial_x\sigma\|\}. \tag{6.11}$$

Substitute (6.10) into (6.9)

$$\|S_h - \sigma\| \leq C\{\|u - P_h u\| + \|\sigma - \Pi_h\sigma\| + h^2\|P_h\partial_x\sigma - \partial_x\sigma\|\}. \tag{6.12}$$

Similarly applying the duality lemmas in [3] to (6.4), noting that $\|E_1\|_\infty$, $\|E_2\|_\infty$, $\|E_3\|_\infty \le C$, we have for h sufficiently small,

$$\left\|\frac{D}{Dt}v_h - P_h\frac{D}{Dt}u\right\| \le C\left\{h\left\|\frac{D}{Dt}S_h - \frac{D}{Dt}\sigma\right\| + h\left\|P_h\frac{D}{Dt}u - \frac{D}{Dt}u\right\|\right.$$
$$\left. + h^2\left\|\partial_x\left(\frac{D}{Dt}S_h - \frac{D}{Dt}\sigma\right)\right\| + \|S_h - \sigma\| + \|v_h - u\|\right\}. \quad (6.13)$$

From the second equation of (6.4), we have

$$\left\|P_h\partial_x\left(\frac{D}{Dt}S_h - \frac{D}{Dt}\sigma\right)\right\| \le C\left\{\left\|\frac{D}{Dt}v_h - P_h\frac{D}{Dt}u\right\| + \|v_h - P_h u\|\right\}.$$

Therefore a triangle inequality yields

$$\left\|\partial_x\left(\frac{D}{Dt}S_h - \frac{D}{Dt}\sigma\right)\right\|$$
$$\le C\left\{\left\|\frac{D}{Dt}v_h - P_h\frac{D}{Dt}u\right\| + \|v_h - P_h u\| + \left\|P_h\partial_x\frac{D}{Dt}\sigma - \partial_x\frac{D}{Dt}\sigma\right\|\right\}.$$

Also, from the first equation of (6.4)

$$\left\|\frac{D}{Dt}S_h - \Pi_h\frac{D}{Dt}\sigma\right\|^2 \le C\left\{\left\|\frac{D}{Dt}v_h - P_h\frac{D}{Dt}u\right\|^2 + \|v_h - P_h u\|^2 + \right.$$
$$\left.\left\|P_h\frac{D}{Dt}u - \frac{D}{Dt}u\right\|^2 + \left\|\frac{D}{Dt}\sigma - \Pi_h\frac{D}{Dt}\sigma\right\|^2 + \|S_h - \sigma\|^2 + \|v_h - u\|^2\right\},$$

so is $\|\frac{D}{Dt}S_h - \frac{D}{Dt}\sigma\|^2$. Substitute these into (6.13)

$$\left\|\frac{D}{Dt}v_h - P_h\frac{D}{Dt}u\right\| \le C\left\{h\|v_h - P_h u\| + h\left\|P_h\frac{D}{Dt}u - \frac{D}{Dt}u\right\|\right.$$
$$\left. + h\left\|\frac{D}{Dt}\sigma - \Pi_h\frac{D}{Dt}\sigma\right\| + \|S_h - \sigma\| + \|v_h - u\| + h^2\left\|P_h\partial_x\frac{D}{Dt}\sigma - \partial_x\frac{D}{Dt}\sigma\right\|\right\} \quad (6.14)$$

Choosing v_h in Theorem 10 to be the solution of (6.3), and noticing that S_h of (6.3) is equal to $F_h(v_h)$, $\partial_x\dot{x}$ is piece-wise constant and therefore commutes with P_h, we have $\|P_h(u - v_h)\| = \|P_h u - v_h\|$, and

$$\begin{aligned}
\|AD_t(u - v_h)\| &= \|AP_h D_t(u - v_h)\| \\
&= \left\|A\left(P_h\frac{D}{Dt}u - \frac{D}{Dt}v_h\right) + A(\partial_x\dot{x})P_h(u - v_h)\right\| \\
&\le C\left\|P_h\frac{D}{Dt}u - \frac{D}{Dt}v_h\right\| + C\|P_h u - v_h\| \\
&\le C\Big\{h\left\|P_h\frac{D}{Dt}u - \frac{D}{Dt}u\right\| + h\left\|\frac{D}{Dt}\sigma - \Pi_h\frac{D}{Dt}\sigma\right\| \\
&\quad + h^2\left\|P_h\partial_x\frac{D}{Dt}\sigma - \partial_x\frac{D}{Dt}\sigma\right\| + \|u - P_h u\| \\
&\quad + \|\sigma - \Pi_h\sigma\| + h^2\|P_h\partial_x\sigma - \partial_x\sigma\|\Big\},
\end{aligned}$$

and

$$\begin{aligned}
\|A\partial_x(\sigma - F_h(v_h))\| &\le C\|P_h\partial_x\sigma - \partial_x S_h)\| \\
&\le C\{h\|u - P_h u\| + h\|\sigma - \Pi_h\sigma\| + h^2\|P_h\partial_x\sigma - \partial_x\sigma\|\}.
\end{aligned}$$

Using approximation properties of P_h and Π_h the proof of is complete. □

With more restrictions on the coefficients and the mesh movement we can have the following superconvergence result.

THEOREM 12. *Suppose the conditions of Theorem 11 hold and that there exist constants* $c_5, c_6, c_7 > 0$ *such that* $\|\partial_x(\frac{D}{Dt}(\beta+\alpha\dot{x}))\|_\infty \le c_5$, $\|\partial_x \frac{D\alpha}{Dt}\|_\infty \le c_6$, $|\partial_x\dot{x}(x_i-) - \partial_x\dot{x}(x_i+)| \le c_7 \min\{h_i, h_{i+1}\}$ *for all* i. *Then there exists a constant* C, *depending on* $C_1, c_1, c_2, c_3, c_5, c_6, c_7$, Ω, T *and the bounds of coefficient* a *such that for any* h *sufficiently small*

$$\begin{aligned}\|P_h u - u_h\| \le & C\Big\{ \|h\underline{h}^{\min\{m+1,s\}} u\|_{L^\infty[0,T;\underline{H}^s]} + \|h\underline{h}^{\min\{m+1,s-1\}}\frac{Du}{Dt}\|_{L^2[0,T;\underline{H}^{s-1}]} \\ & + \|h\underline{h}^{\min\{m+2,s-1\}}\sigma\|_{L^2[0,T;\underline{H}^{s-1}]} + \|h^2\underline{h}_i^{\min\{m+1,s-2\}}\sigma\|_{L^2[0,T;\underline{H}^{s-1}]} \\ & + \|h^2\underline{h}^{\min\{m+1,s-2\}}\frac{D\sigma}{Dt}\|_{L^2[0,T;\underline{H}^{s-1}]}\Big\}.\end{aligned}$$

Proof. We modify slightly the proof of Theorem 11. First we apply the duality argument in [3] to (6.3) to get

$$\|S_h - \sigma\|_{-1} \le C\{h^2\|\partial_x(S_h - \sigma)\| + \|v_h - P_h u\| + h\|u - P_h u\|\}. \tag{6.15}$$

Let ω be a piecewise linear continuous function on $\{\Omega_i\}$ such that $\omega(x_i) = \{\partial_x\dot{x}(x_i-) + \partial_x\dot{x}(x_i+)\}/2$ for any i. Then it is easy to see that $\|\omega - \partial_x\dot{x}\|_\infty \le Ch$ and $\|\omega\|_1 \le C$. From the right hand side of (6.4) we have

$$\begin{aligned}(E_3(v_h - u), r) &= (E_3(v_h - P_h u), r), \\ (E_2(v_h - u), \mathcal{X}) &= -\left(v_h - u, \frac{D}{Dt}(\beta + \alpha\dot{x})\ \mathcal{X}\right) \\ &\quad - ((\partial_x\dot{x})(v_h - u), (\beta+\alpha\dot{x})\cdot\mathcal{X} - P_h((\beta+\alpha\dot{x})\cdot\mathcal{X})) \\ &\quad - ((\partial_x\dot{x})(v_h - P_h u), P_h((\beta+\alpha\dot{x})\cdot\mathcal{X})) \\ &\le C\{\|v_h - u\|_{-1} + h\|v_h - u\| + \|v_h - P_h u\|\}\|\mathcal{X}\|_1 \\ &\le C\{h\|u - P_h u\| + \|v_h - P_h u\|\}\|\mathcal{X}\|_1, \\ (E_1(S_h - \sigma), \mathcal{X}) &= -\left(S_h - \sigma, \frac{D\alpha}{Dt}\mathcal{X}\right) - (\alpha(\partial_x\dot{x} - \omega)(S_h - \sigma), \mathcal{X}) \\ &\quad - (S_h - \sigma, \alpha\omega\mathcal{X}) \\ &\le C\{\|S_h - \sigma\|_{-1} + h\|S_h - \sigma\|\}\|\mathcal{X}\|_1.\end{aligned}$$

Following the duality lemmas in [3] again and also using (6.15) we have

$$\begin{aligned}\left\|\frac{D}{Dt}v_h - P_h\frac{D}{Dt}u\right\| \le C\Big\{ & \|v_h - P_h u\| + h\left\|P_h\frac{D}{Dt}u - \frac{D}{Dt}u\right\| \\ & + h\left\|\frac{D}{Dt}\sigma - \Pi_h\frac{D}{Dt}\sigma\right\| + h\|S_h - \sigma\| + h^2\left\|P_h\partial_x\frac{D}{Dt}\sigma - \partial_x\frac{D}{Dt}\sigma\right\| \\ & + h\|u - P_h u\| + h^2\|\partial_x(S_h - \sigma)\|\Big\}.\end{aligned} \tag{6.16}$$

Note that $\|AD_t(u - v_h)\| \le C\|P_h\frac{D}{Dt}u - \frac{D}{Dt}v_h\| + C\|P_h u - v_h\|$, and $\|u_h - P_h u\|$ is dominated by $|\!|\!|(u - u_h, \sigma - \sigma_h)|\!|\!|_{D_t^*}$, the rest of the proof is similar to that of Theorem 11. □

7. Another Mixed Method. Consider the non-conservative form of the equation (2.1):

$$\begin{cases} \partial_t u - \nabla \cdot (a\nabla u) + b \cdot \nabla u + cu = f, & \text{on } Q, \\ u = 0, & \text{on } \partial\Omega \times (0,T), \\ u = u_0, & \text{for } t = 0, \end{cases} \tag{7.1}$$

Let $\sigma = a\nabla u$ and $\alpha = 1/a, \beta = b/a$. The mixed form becomes

$$\begin{cases} (\alpha\sigma, \mathcal{X}) + (u, div\, \mathcal{X}) = 0 & \forall \mathcal{X} \in H(div, \Omega), \\ \left(\dfrac{Du}{Dt} + div\, \sigma, r\right) + ((\beta - \dot{x}\alpha)\cdot\sigma, r) + (cu, r) = (f, r) & \forall r \in L^2(\Omega). \end{cases} \tag{7.2}$$

Note that with a little abuse of the notations, $a, b, c, u, \alpha, \beta, \sigma$ have been redefined. We will keep on using the relevant notations and results from previous sections unless specified.

The mixed method is to find $u_h : [0,T] \to V_h$ and $\sigma_h : [0,T] \to H_h$ such that

$$\begin{cases} (\alpha\sigma_h, \mathcal{X}) + (u_h, div\, \mathcal{X}) = 0 & \forall \mathcal{X} \in H_h, \\ \left(\dfrac{Du_h}{Dt} + div\, \sigma_h, r\right) + ((\beta - \dot{x}\alpha)\cdot\sigma_h, r) + (cu_h, r) = (f, r) & \forall r \in V_h. \end{cases} \tag{7.3}$$

We define the norm $|||(\cdot,\cdot)|||_c$ by

$$\begin{aligned} |||(\eta,\psi)|||_c^2 = \|\eta\|^2_{L^\infty(0,T:L^2(\Omega))} &+ \left\|A\frac{D\eta}{Dt}\right\|^2_{L^2(0,T:L^2(\Omega))} \\ &+ \|A(div\,\psi)\|^2_{L^2(0,T:L^2(\Omega))} + \|\psi\|^2_{L^2(0,T:L^2(\Omega))}. \end{aligned} \tag{7.4}$$

Let L_h be a linear operator $V_h(t) \to H_h(t)$ such that for any $v_h \in V_h(t)$,

$$(\alpha L_h(v_h), \mathcal{X}) + (v_h, div\, \mathcal{X}) = 0,\ \forall \mathcal{X} \in H_h.$$

We have the following theorem whose proof is similar to that of Theorem 9.

THEOREM 13. *Suppose Conditions 2 holds, and there exist constants c_1, c_2 such that*

$$\nabla_x \cdot \dot{x} \le c_1, \quad \text{and} \quad |\beta + \alpha\dot{x}| \le c_2,$$

for all $(x,t) \in Q$. Then there exists a constant $C > 0$ depending only on C_2, c_1, c_2, T, the bounds of coefficients a and c, and Ω such that, for any piecewise smooth function v_h with $v_h(\cdot,t) \in V_h(t)$,

$$|||(u - u_h, \sigma - \sigma_h)|||_c \le C|||(u - v_h, \sigma - L_h(v_h))|||_c.$$

Introduce another norm $|||(\cdot,\cdot)|||_{c^*}$ by

$$\begin{aligned} |||(\eta,\psi)|||_{c^*}^2 = \|P_h\eta\|^2_{L^\infty(0,T:L^2(\Omega))} + \left\|A\frac{D\eta}{Dt}\right\|^2_{L^2(0,T:L^2(\Omega))} + \|A(div\,\psi)\|^2_{L^2(0,T:L^2(\Omega))} \\ + \|A((\beta - \alpha\dot{x})\cdot\psi)\|^2_{L^2(0,T:L^2(\Omega))} + \|A(c\eta)\|^2_{L^2(0,T:L^2(\Omega))}. \end{aligned}$$

We have another theorem whose proof is similar to that of Theorem 10, also using Theorem 5.

THEOREM 14. *Suppose Condition 2 holds, and there exist constants c_1, c_2 such that*

$$\nabla_x \cdot \dot{x} \le c_1, \qquad \textit{and} \qquad |\beta + \alpha \dot{x}| \le c_2,$$

for all $(x,t) \in Q$. Then there exists a constant $C > 0$ depending only on C_2, c_1, c_2, T, the bounds of coefficients a and c, and Ω such that, for any piecewise smooth function v_h with $v_h(\cdot, t) \in V_h(t)$,

$$|\!|\!|(u - u_h, \sigma - \sigma_h)|\!|\!|_{c^*} \le C |\!|\!|(u - v_h, \sigma - L_h(v_h))|\!|\!|_{c^*}.$$

Parallel to what was done in Section 6, we derive an optimal convergence result for 1-d in the next theorem. In particular, the L^2-norm of $u_h - P_h u$ is super convergent.

Assume that a is sufficiently regular so that for any $\xi \in L^2(\Omega)$, the following equation

$$\begin{cases} -\partial_x(a\partial_x w) = g, & \text{in } \Omega, \\ \qquad\qquad w = 0. & \text{on } \partial\Omega, \end{cases} \tag{7.5}$$

has a unique solution satisfying $\|w\|_2 \le C\|g\|$. We have the following theorem.

THEOREM 15. *Suppose Condition 1 holds, and there exist constants c_1, c_2, c_3 such that $|\partial_x \dot{x}|, |\frac{D\alpha}{Dt}| \le c_1$; $|\beta + \alpha\dot{x}| \le c_2$; $|\partial_x c| \le c_3$ for all $(x,t) \in Q$. Then there exists a constant $C > 0$, depending only on C_1, c_1, c_2, c_3, T, the bounds on coefficients a and c, and Ω such that for any h sufficiently small,*

$$\begin{aligned} \|u_h - P_h u\| \le C \Big\{ & \|\underline{h}^{\min\{m+2,s+1\}}\sigma\|_{L^\infty[0,T;\underline{H}^{s+1}]} + \|h\underline{h}^{\min\{m+1,s\}}\sigma\|_{L^2[0,T;\underline{H}^{s+1}]} \\ & + \|h\underline{h}^{\min\{m+2,s\}}\frac{D\sigma}{Dt}\|_{L^2[0,T;\underline{H}^s]} + \|h^2\underline{h}^{\min\{m+1,s-1\}}\frac{D\sigma}{Dt}\|_{L^2[0,T;\underline{H}^s]} \\ & + \|h\underline{h}^{\min\{m+1,s\}}u\|_{L^2[0,T;\underline{H}^s]} \Big\}, \end{aligned}$$

and

$$\begin{aligned} \|u - u_h\| \le C \Big\{ & \|\underline{h}^{\min\{m+2,s\}}\sigma\|_{L^\infty[0,T;\underline{H}^s]} + \|h\underline{h}^{\min\{m+1,s-1\}}\sigma\|_{L^2[0,T;\underline{H}^s]} \\ & + \|h\underline{h}^{\min\{m+2,s-1\}}\frac{D\sigma}{Dt}\|_{L^2[0,T;\underline{H}^{s-1}]} + \|h^2\underline{h}_i^{\min\{m+1,s-2\}}\frac{D\sigma}{Dt}\|_{L^2[0,T;\underline{H}^{s-1}]} \\ & + \|h\underline{h}^{\min\{m+1,s\}}u\|_{L^2[0,T;\underline{H}^s]} \Big\}. \end{aligned}$$

Proof. The proof of the first estimate is an application of Theorem 14. Since $|\!|\!|(u - u_h, \sigma - \sigma_h)|\!|\!|_{c^*}$ dominates the term we want to bound, it suffices to show that $|\!|\!|(u - v_h, \sigma - L_h(v_h))|\!|\!|_{c^*}$ can be bounded by terms on the right- hand side of the first estimate. The second estimate follows from a triangle inequality.

Consider the following elliptic projection

$$\begin{cases} (\alpha(S_h - \sigma), \mathcal{X}) + (v_h - u, \partial_x \mathcal{X}) = 0, & \forall \mathcal{X} \in H_h, \\ (\partial_x(S_h - \sigma), r) = 0, & \forall r \in V_h. \end{cases} \tag{7.6}$$

Notice that $S_h = L_h(v_h)$.

Differentiating (7.6) with respect to time, using Lemma 7 and properties of $\frac{D}{Dt}$, we have

$$\begin{cases} (\alpha \dfrac{D}{Dt}(S_h - \sigma), \mathcal{X}) + \left(\dfrac{D}{Dt}(v_h - u), \partial_x \mathcal{X} \right) & \\ \qquad\qquad = (E_4(S_h - \sigma), \mathcal{X}), & \forall \mathcal{X} \in H_h, \\ \left(\partial_x \dfrac{D}{Dt}(S_h - \sigma), r \right) = 0, & \forall r \in V_h, \end{cases} \tag{7.7}$$

where $E_4 = -(\frac{D}{Dt}\alpha + \alpha \partial_x \dot{x})$. Using the duality lemma in [3] we have

$$\|v_h - P_h u\| \leq C\{h\|S_h - \sigma\| + h^2 \|\partial_x(S_h - \sigma)\|\}. \tag{7.8}$$

Also from the second equation of (7.6),

$$\|\partial_x(S_h - \Pi_h \sigma)\| = 0 \quad \text{and} \quad \|P_h \partial_x(S_h - \sigma)\| = 0,$$

so $\|\partial_x(S_h - \sigma)\| = \|P_h \partial_x \sigma - \partial_x \sigma\|$. From the first equation of (7.6),

$$\|S_h - \Pi_h \sigma\| \leq C\|\sigma - \Pi_h \sigma\|, \quad \text{so} \quad \|S_h - \sigma\| \leq C\|\sigma - \Pi_h \sigma\|.$$

Therefore

$$\|v_h - P_h u\| \leq C\{h\|\sigma - \Pi_h \sigma\| + h^2 \|P_h \partial_x \sigma - \partial_x \sigma\|\}.$$

Similarly for equation (7.7)

$$\begin{aligned} \left\| \frac{D}{Dt} v_h - P_h \frac{D}{Dt} u \right\| \leq C \Big\{ h \left\| \frac{D}{Dt} S_h - \frac{D}{Dt} \sigma \right\| & \\ + h^2 \left\| \partial_x (\frac{D}{Dt} S_h - \frac{D}{Dt} \sigma) \right\| + \|S_h - \sigma)\| \Big\}, \end{aligned} \tag{7.9}$$

and $\|P_h \partial_x (\frac{D}{Dt} S_h - \frac{D}{Dt} \sigma)\| = 0$, $\|\partial_x (\frac{D}{Dt} S_h - \Pi_h \frac{D}{Dt} \sigma)\| = 0$. So

$$\left\| \partial_x (\frac{D}{Dt} S_h - \frac{D}{Dt} \sigma) \right\| = \left\| P_h \partial_x \frac{D}{Dt} \sigma - \partial_x \frac{D}{Dt} \sigma \right\|.$$

Also from the first equation of (7.7)

$$\left\| \frac{D}{Dt} S_h - \Pi_h \frac{D}{Dt} \sigma \right\| \leq C \left\{ \|S_h - \sigma\| + \left\| \frac{D}{Dt} \sigma - \Pi_h \frac{D}{Dt} \sigma \right\| \right\}.$$

Therefore

$$\begin{aligned} \left\| \frac{D}{Dt} v_h - P_h \frac{D}{Dt} u \right\| \leq C \Big\{ \|S_h - \sigma\| + h \left\| \frac{D}{Dt} \sigma - \Pi_h \frac{D}{Dt} \sigma \right\| & \\ + h^2 \left\| \partial_x \frac{D}{Dt} \sigma - P_h \partial_x \frac{D}{Dt} \sigma \right\| \Big\}. \end{aligned} \tag{7.10}$$

In Theorem 15 choose v_h to be the solution of (7.6). Note that

$$\begin{aligned}
\left\|A\frac{D}{Dt}(u-v_h)\right\| &\le C\left\|P_h\frac{D}{Dt}u-\frac{D}{Dt}v_h\right\|,\\
\|A((\beta-\alpha\dot{x})\cdot(\sigma-S_h))\| &\le C\|\sigma-S_h\|\\
&\le C\|\sigma-\Pi_h\sigma\|,\\
\|A(c(u-v_h))\| &\le \|A((c-\bar{c})(u-v_h))\|+\|A(\bar{c}(u-v_h))\|\\
&\le C\|(c-\bar{c})(u-v_h)\|+\|A(\bar{c}P_h(u-v_h))\|\\
&\le Ch(\|u-P_hu\|+\|P_hu-v_h\|)+C\|P_hu-v_h\|,
\end{aligned}$$

where $\bar{c}|_{\Omega_i}\equiv(1/|\Omega_i|)\int_{\Omega_i}cdx,\forall i$, is a piece-wise constant function which commutes with P_h. The proof is completed using the approximation properties of the projections P_h and Π_h. □

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