MATH1502 A1B1, Test 1, June 7, 2007

NAME:

Be sure to show your work. Answers without explanations are not acceptable.

1. (10 points) Calculate the limit:

$$\lim_{x\to 0} \frac{\cos x - \cos 3x}{\sin(x^2)}.$$

$$= \lim_{x\to 0} \frac{-\sin x + 3\sin 3x}{2x\cos x^2} \quad (\text{by LH})$$

$$= \lim_{x\to 0} \frac{-\cos x + 9\cos 3x}{2\cos x^2 - 4x^2 \sin x^2} \quad (\text{by LH})$$

$$= \frac{-1+9}{2-0}$$

$$= 4$$

2. (5 points) Evaluate

$$\int_e^\infty \frac{1}{x(\ln x)^2} dx.$$

$$= \lim_{b\to\infty} \int_e^b \frac{1}{x(\ln x)^2} dx$$

$$= \lim_{b\to\infty} \int_e^b \frac{1}{(\ln x)^2} d(\ln x)$$

$$= \lim_{b\to\infty} -\frac{1}{\ln x}\Big|_e^b$$

$$= \lim_{b\to\infty} \left(1 - \frac{1}{\ln b}\right) = 1.$$

3. (10 points) Determine whether the following series converges (state the name of your test and show how it is applied): (a) $\sum \frac{2\cdot 4\cdot 6\cdots 2k}{(2k)!}$; (b) $\sum \frac{k}{k^{1.9}+1}$.

(a) Apply ratio test

$$\lim_{k\to\infty} \frac{\frac{2\cdot 4\cdots 2(k+1)}{(2(k+1))!}}{\frac{2\cdot 4\cdots 2k}{(2k)!}} = \lim_{k\to\infty} \frac{1}{2k+1} = 0 < 1$$

So the series converges by ratio test.

(b) Compare with $\sum \frac{1}{k^{0.9}}$

$$\lim_{k\to\infty} \frac{k/(k^{1.9}+1)}{1/k^{0.9}} = \lim_{k\to\infty} \frac{k^{1.9}}{k^{1.9}+1} = 1$$

Since $\sum \frac{1}{k^{0.9}}$ diverges, $\sum \frac{k}{k^{1.9}+1}$ diverges (by limit comparison test).

4. (10 points) Find the interval of convergence of $\sum(-1)^k(x-2)^k/2^k$.

Apply the root test.

$$\lim_{k\to\infty} \sqrt[k]{\left|\frac{(-1)^k(x-2)^k}{2^k}\right|} = \frac{|x-2|}{2}$$

Set $\frac{|x-2|}{2} < 1$, $0 < x < 4$

When $x=0$, the series is $\sum 1$ which diverges (by divergence test)

When $x=4$, the series is $\sum (-1)^k$ which diverges

So the interval of convergence is $(0,4)$.

5. (10 points) Use the Taylor series of $\sin x$ (in powers of x) to approximate $\int_0^1 \frac{\sin x}{x}\,dx$ within 0.001.

Taylor series $\sin x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$

nth remainder $R_n(x) = \frac{\sin^{(n+1)} c \; x^{n+1}}{(n+1)!}$, $c \in (0, x)$.

Note $\int_0^1 \frac{R_n(x)}{x}\,dx = \int_0^1 \frac{\sin^{(n+1)} c \cdot x^n}{(n+1)!}\,dx \leq \frac{1}{(n+1)!}$

The smallest n for $\frac{1}{(n+1)!} < 0.001$ is $n = 6$

So $\int_0^1 \frac{\sin x}{x}\,dx \approx \int_0^1 \sum_{k=0}^{3} \frac{(-1)^k x^{2k}}{(2k+1)!}\,dx$

$= \int_0^1 \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!}\right) dx$

$= 1 - \frac{1}{3 \cdot 3!} + \frac{1}{5 \cdot 5!} - \frac{1}{7 \cdot 7!}$

6. (5 points) Solve $y' - y = e^{2x}$ with $y(0) = 2$.

$H(x) = \int (-1)\,dx = -x$

$y = e^{-H(x)}\left(\int e^{H(x)} g(x)\,dx + c\right)$

$= e^{x}\left(\int e^{-x} \cdot e^{2x}\,dx + C\right)$

$= e^{2x} + C \cdot e^{x}$

Since $y(0) = 2$, $C = 1$.

Hence $y = e^{2x} + e^{x}$.