Positively Curved Cubic Plane Graphs Are Finite

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Abstract

Let G be an infinite plane graph such that G is locally finite and every face of G is bounded by a cycle. Then G is said to be *positively curved* if, for every vertex x of G, $1 - d(x)/2 + \Sigma_{x \in F} \frac{1}{|F|} > 0$, where the summation is taken over all facial cycles F containing x. Note that if G is positively curved then the maximum degree of G is at most 5. As a discrete analogue of a result in Riemannian geometry, Higuchi conjectured that if G is positively curved then G is finite. In this paper, we establish this conjecture for cubic graphs.

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1 Introduction

The graphs considered in this paper are simple, but may be finite or infinite. Let G be a graph. Then $V(G)$ and $E(G)$ denote the vertex set and edge set of G, respectively. We use $|G|$ to denote the number of vertices of G. For any $x \in V(G)$, $d_G(x)$ denotes the number of edges of G incident with x. (We use $d(x)$ if no confusion arises.) We say that G is *cubic* if $d(x) = 3$ for all $x \in V(G)$. We say that G is *locally finite* if $d(x)$ is finite for all $x \in V(G)$. If there is no danger of confusion we write $x \in G$ in stead of $x \in V(G)$. A *cycle* in G is a finite connected subgraph of G in which every vertex has degree 2. For subgraphs G and H of a graph, we use $G \cup H$ and $G \cap H$ to denote the union and intersection of G and H, respectively. For any graph G and any $X \subset V(G)$, $G - X$ denote the graph obtained from G by deleting X and all edges of G incident with X. If G is connected and $G - X$ is not connected, then X is called a *vertex cut* of G. If X is a vertex cut of G and $|X| = k$, then X is called a *k-cut*.

A *plane graph* is a graph drawn in the plane with no pair of edges crossing. Let G be a plane graph. Then the vertices and edges incident with a common face are called *cofacial*. We say that a face of G is *bounded by a cycle* if the edges of G incident with it induce a cycle in G, and such a cycle is called a *facial cycle* of G. Let C be a cycle in a plane graph. Then we can speak of two orientations on C: clockwise orientation and counter-clockwise orientation. Let u, v be distinct vertices of C. We use $C[u, v]$ to denote the colckwise path in C from u to v. We use $C(u, v)$ to denote the graph obtained from $C[u, v]$ by deleting u and v. We define $C[u, v)$ and $C(u, v]$ similarly.

In [2], the curvature of a plane graph is introduced as a discrete analogue of the sectional curvature of a Riemannian manifold, and a criterion is given for the hyperbolicity of a plane graph. For more detail, see [2] and the references in [2].

Let G be a plane graph (finite or infinite) such that (1) G is locally finite and (2) every face of G is bounded by a cycle. Then the *combinatorial curvature* of G is the function $\Phi_G : V(G) \to R$ such that for any $x \in V(G)$,

$$\Phi_G(x) = 1 - \frac{d(x)}{2} + \Sigma_{x \in F} \frac{1}{|F|},$$

where the summation is taken over all facial cycles of G containing x. See Figure 1 for an example. If $\Phi_G(x) > 0$ for all $x \in V(G)$, then we say that G is *positively curved*.

As pointed in [2], $\Phi_G(x)$ may be interpreted as the degree of difficulty for tiling the plane at x, and it is dual to another curvature introduced by Gromov [1]. Higuchi in [2] proves that under some minor requirements, if $\Phi_G(x) < 0$ for all $x \in V(G)$ then there exists a constant $\epsilon > 0$ such that $\Phi_G(x) < -\epsilon$ for all $x \in V(G)$. This is then used to derive a discrete analogue of a fact in Riemannian geometry concernning isoperimetric inequalities.

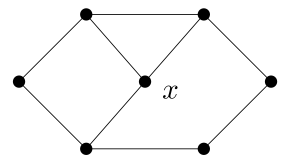

Figure 1: $\Phi_G(x) = 1 - 3/2 + (1/3 + 1/4 + 1/5) = 17/60$

The following conjecture is posed in [2] which is a discrete analogue of a result of Myers [3]: a complete Riemannian manifold with Ricci curvature bounded below by a positive number is compact and has finite fundamental group.

(1.1) Conjecture. Let G be a locally finite plane graph such that every face of G is bounded by a cycle. If $\Phi_G(x) > 0$ for all $x \in V(G)$ then G is a finite graph.

The plane graph in Figure 2 is obtained from two vertex disjoint cycles $u_1u_2\ldots u_nu_1$ and $v_1v_2\ldots v_nv_1$ by adding a perfect matching $\{u_iv_i : i = 1, 2, \ldots, n\}$. It is easy to verify that this graph is positively curved. Hence, there are arbitrarily large positively curved graphs.

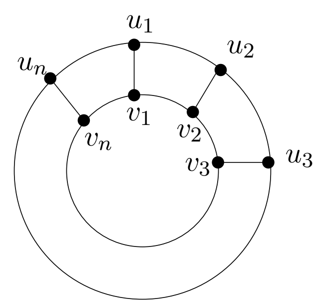

Figure 2: A positively curved graph.

Higuchi verified (1.1) for some special classes of graphs, and he noted that his method brings no insight to (1.1). Note that if $\Phi_G(x) > 0$ for all $x \in V(G)$ then $d(x) \leq 5$ for all $x \in V(G)$.

The main result of this paper is the following which establishes (1.1) for all cubic graphs. We believe that our method offers a possible approach to establish (1.1) completely.

(1.2) Theorem. *Let G be a cubic plane graph such that every face of G is bounded by a cycle. If $\Phi_G(x) > 0$ for all $x \in V(G)$ then G is a finite graph.*

The main idea of our proof is as follows. Assume that G is infinite. First, we prove the existence of an infinite sequence of disjoint cycles $(C_0, C_1, \ldots)$ which captures a lot of structural information of G. This is done without requiring G be cubic. We then assume that G is positively curved, and derive a contradiction by showing that for all sufficiently large n, $|C_n| > |C_{n+1}|$. This is done through case analysis.

This paper is organized as follows. In Section 2, we show first how to produce an infinite sequence of cycles mentioned above. We then use that sequence to derive a plane embedding of the same graph with the same curvature function, but is easier to deal with. In Section 3, we introduce necessary notation and derive further structural information about positively curved cubic plane graphs. In Sections 4 through 7, we prove (1.2).

2 Nice sequences

The main objective of this section is to derive some useful structural information about infnite plane graphs. To this end, we need the following convenient concept.

(2.1) Definition. Let H be a subgraph (finite or infinite) of a graph G (finite or infinite). An *H-bridge* of G is a subgraph of G which is induced by either (1) an edge $e \in E(K) - E(H)$ with both incident vertices on H, or (2) edges in a component D of $G - V(H)$ and edges from D to H. If B is an H-bridge of G, then the vertices in $V(H) \cap V(B)$ are *attachments* of B on H. Note that an H-bridge of G may be infinite.

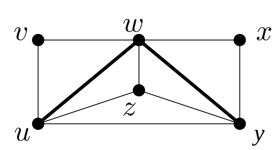

Figure 3: A path H and its bridges

In Figure 3, $H = uwy$ is a path. The H-bridges of G are the subgraphs induced by the following sets of edges: $\{uv, vw\}$, $\{wx, xy\}$, $\{zu, zw, zy\}$, and $\{uy\}$. Now we turn our attention to the description of a "nice" sequence of cycles.

Let G be an infinite plane graph such that G is locally finite and every face of G is bounded by a cycle. Let F be a facial cycle of G and let $R(F)$ denote the closure of the face of G bounded by F. (Thus, F is the boundary of $R(F)$.) For any cycle C in G, we define $R_F(C)$ as follows. By the Jordan curve theorem, C divides the plane into two closed regions whose intersection is C. We use $R_F(C)$ to denote the closed region of the plane bounded by C which contains $R(F)$. Hence, $R_F(F) = R(F)$. For any cycle C in G, we use $G_F(C)$ to denote the subgraph of G contained in $R_F(C)$.

(2.2) Definition. Let G be an infinite plane graph such that G is locally finite and every face of G is bounded by a cycle. Let F be a facial cycle of G. A sequence of disjoint cycles $(C_0, C_1, \ldots)$ in G is called a *nice sequence starting with* F if the following conditions hold:

(1) $C_0 = F$,

(2) for $i \geq 0$, $R_F(C_i) \subset R_F(C_{i+1})$ (and hence, $G_F(C_i) \subset G_F(C_{i+1})$),

(3) for $i \geq 0$, every $(G_F(C_i) \cup C_{i+1})$-bridge of $G_F(C_{i+1})$ has at most one attachment on C_{i+1}, and

(4) for $i \geq 0$, $G - V(G_F(C_i))$ is infinite.

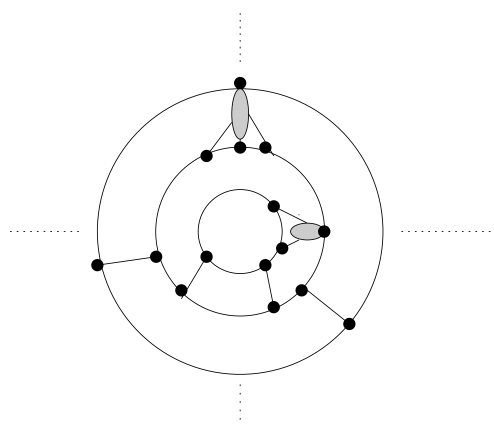

Figure 4: An example of a nice seqence

Figure 4 illustrates a nice sequence. The shaded regions represent subgraphs which are finite or infinite. Notice that a $(G_F(C_i) \cup C_{i+1})$-bridge B may be infinite, but the vertices and edges of B cofacial with a vertex of C_{i+1} form a finite subgraph B^* of G. Because every face of G is bounded by a cycle, B^* is the union of two finite paths.

We are now ready to describe and prove the main result of this section.

(2.3) Theorem. *Let G be a connected infinite plane graph such that G is locally finite and every face of G is bounded by a cycle, and let F be a facial cycle of G. Then G has a nice sequence starting with F.*

Proof. Let $C_0 = F$, let $G_F(C_0) = C_0$, and let $R_F(C_0)$ be the closure of the face bounded by F. Suppose that we have constructed $(C_0, \cdots, C_k)$ for some $k \geq 0$ such that

(1) $C_0 = F$,

(2) for $0 \le i \le k-1$, $R_F(C_i) \subset R_F(C_{i+1})$ and $G_F(C_i) \subset G_F(C_{i+1})$,

(3) for $0 \le i \le k-1$, every $(G_F(C_i) \cup C_{i+1})$-bridge of $G_F(C_{i+1})$ has at most one attachment on C_{i+1}, and

(4) for $0 \le i \le k$, $G - G_F(C_i)$ is infinite.

The remainder of this proof shows how to construct the next cycle for the desired sequence. Consider the graph $H := G - G_F(C_k)$. Note that H need not be connected.

First, we claim that H has only finitely many blocks. For, suppose H has infinitely many blocks. Because G is connected, every block of H has a neighbor on C_k. Since $|C_k|$ is finite, some vertex of C_k is adjacent to infinitely many blocks of H. This contradicts the assumption that G is a locally finite.

Therefore, since H is infinite, some block of H, say B, is infinite. Let C_{k+1} denote the subgraph of H consisting of vertices and edges of B cofacial with a vertex of C_k. Observe that C_{k+1} is finite because C_k is finite, G is locally finite, and every face of G is bounded by a cycle. Since B is 2-connected, C_{k+1} is a cycle.

Obviously, $R_F(C_k) \subset R_F(C_{k+1})$ and $G_F(C_k) \subset G_F(C_{k+1})$. Because B is a block of $G_F(C_k)$, every $(G_F(C_k) \cup C_{k+1})$-bridge has at most one attachment on C_{k+1}. Because B is infinite and C_{k+1} is finite, $B - V(C_{k+1})$ is infinite. Hence $G - V(G_F(C_{k+1}))$ is infinite. So the sequence $(C_0, \ldots, C_{k+1})$ statisies (1)-(4) above with $k+1$ replaceing k.

This process can be continued with $(C_0, \ldots, C_{k+1})$ replacing $(C_0, \ldots, C_k)$. Hence, the desired nice sequence $(C_0, C_1, \ldots)$ exists. $\square$

To facilitate later discussions, we will work with a "nice" embedding of a plane graph G which has the same combinatorial curvature as G.

(2.4) Theorem. *Suppose that G is an infinite locally finite plane graph with a facial cycle F, and let $(C_0, C_1, C_2, \cdots)$ be a nice sequence in G starting with F. Then G has an embedding G' in the plane such that*

(1) $G'_F(C_i)$ is contained in the closed disc bounded by C_i, and

(2) G and G' have the same combinatorial curvature.

Proof. Consider the graphs $H_i := G_F(C_{i+1}) - (G_F(C_i) - C_i)$ for all $i \ge 0$. Note that each H_i is a subgaph of G. Hence, H_i is a plane graph and both C_i and C_{i+1} are facial cycles of H_i. Therefore, H_i has an embedding H'_i in the plane such that

(a) for any cycle F of H_i, F is a facial cycle of H'_i if, and only if F is a facical cycle of H_i,

(b) the face of H'_i bounded by C_i is an open disc in the plane,

(c) the face of H'_i bounded by C_{i+1} is an unbounded region in the plane.

By assembling the embeddings H_i', for all $i \geq 0$, we obtain the desired embedding G' of G. $\square$

We say that G is *nicely embedded* with respect to a nice sequence $(C_0, C_1, \ldots)$ if, for each $i \geq 0$, $G_F(C_i)$ is contained in the closed disc bounded by C_i.

3 Notation and convention

Throughout this section, we assume that G is a cubic infinite plane graph such that every face of G is bounded by a cycle.

Let v be a vertex of G. Let F_1, F_2 and F_3 be the facial cycles of G containing v, and assume that $|F_1| \leq |F_2| \leq |F_3|$. We define $\ell(v) = (|F_1|, |F_2|, |F_3|)$. If $|F_1| \geq m_1$, $|F_2| \geq m_2$, and $|F_3| \geq m_3$, then we write $\ell(v) \geq (m_1, m_2, m_3)$.

For a vertex v of G with $\ell(v) = (m_1, m_2, m_3)$, $\Phi_G(v) > 0$ iff $1/m_1 + 1/m_2 + 1/m_3 > 1/2$. The following is a partial list of triples (m_1, m_2, m_3) such that $1/m_1 + 1/m_2 + 1/m_3 \leq 1/2$.

(3.1) Lemma. *Let G be a cubic infinite plane graph such that every face of G is bounded by a cycle and let $v \in V(G)$. Let*

$$\mathcal{T} = \{(3,10,15), (3,11,14), (3,12,12), (4,6,12), (4,7,10), (4,8,8), (5,5,10), (5,6,8), (5,7,7), (6,6,6)\}$$

. If $\ell(v) \in \mathcal{T}$, then $\Phi_G(v) \leq 0$.

Throughout the rest of the paper, we will derive contradictions to (3.1) by showing that in a positively curved cubic infinite plane graph there is a vertex v such that $\ell(v) \in \mathcal{T}$. For ease of presentation, we will not refer to (3.1) explicitely.

Let F be a facial cycle of G. By (2.3), G has a nice sequence $(C_0, C_1, \ldots)$ starting with F and satisfying (1)-(4) of (2.2). By (2.4), we may assume that G is nicely embedded with respect to $(C_0, C_2, \ldots)$.

A vertex v of G is called an *in-vertex* (respectively, *out-vertex*) if $v \in C_i$ for some $i \geq 0$ and v is incident with an edge contained in the annulus region between C_i and C_{i-1} (respectively, C_{i+1}). If v is an in-vertex on C_i, then we use $A(v), L(v), R(v)$ to denote the facial cycles of G containing v, where $A(v)$ is between C_i and C_{i+1} and $A(v), R(v), L(v)$ occur in this clockwise order around v. (We can think of $L(v)$ as to the left of v, and $R(v)$ as to the right of v.) If w is an out-vertex on C_i, then we use $B(w), L(w), R(w)$ to denote the facial cycles of G containing w, where $B(w)$ is between C_i and C_{i-1} and $B(w), L(w), R(w)$ occur around w in that clockwise order. (Again, we can think of $L(w)$ as to the left of w, and $R(w)$ as to the right of w.) See Figure 5.

By the choice of $(C_0, C_1, \ldots)$, we have the following two observations which will be used frequently (often without explicit reference).

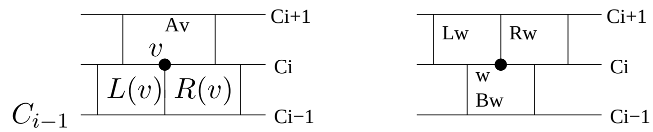

Figure 5: v is an in-vertex on C_i and w is an out-vertex on C_i

(3.2) Lemma. *Let G be a cubic infinite plane graph with a nice sequence $(C_0, C_1, \ldots)$ and assume that G is nicely embedded with respect to $(C_0, C_1, \ldots)$. For any in-vertex v on C_i, $|L(v)| \geq 4 \leq |R(v)|$. For any out-vertex w on C_i, $|B(w)| \geq 5$.*

(3.3) Lemma. *Let G be a cubic infinite plane graph with a nice sequence $(C_0, C_1, \ldots)$ and assume that G is nicely embedded with respect to $(C_0, C_1, \ldots)$. For any facial cycle F of length 4, $|F \cap C_i| \neq 1$.*

The next lemma allows us to get rid of certain 2-cuts of G in the proof of (1.2).

(3.4) Lemma. *If there is a cubic infinite plane graph with positive combinatorial curvature, then there is a cubic infnite plane graph G with positive combinatorial curvature such that*

(1) G has a nice sequence $(C_0, C_1, \ldots)$, and

(2) for any $k \geq 1$ and for any 2-cut T of G contained in $V(C_k)$, $\bigcup_{0 \leq i \leq k-1} C_i$ and $\bigcup_{i \geq k+1} C_i$ belong to different components of $G - T$.

Proof. Let G be a cubic infinite plane graph such that G is locally finite and every face of G is bounded by a cycle. Assume that G has positive combinatorial curvature. So for any $x \in V(G)$, $\Phi_G(x) > 0$. Suppose that there is some $k \geq 1$ and a 2-cut $T = \{u, v\}$ of G contained in $V(C_k)$ such that $\bigcup_{0 \leq i \leq k-1} C_i$ and $\bigcup_{i \geq k+1} C_i$ belong to the same component of $G - T$. Let B denote the T-bridge of G not containing $\bigcup_{i \neq k} C_i$. We may assume that T is chosen so that B is maximal. Then B contains at least two neighbors of u and at least two neighbors of v. So let H denote the plane graph obtained from G by replacing B with the edge uv. Then H is a cubic infinite plane graph. Moreover, for each vertex x of H, the length of any facical cycle of H containing x is not longer than the corresponding facial cycle of G. So H is also positively curved. Since for all $k \geq 1$, $|C_k|$ is finite, there are only finitely many 2-cuts contained in $V(C_k)$. So we can repeatedly perform the above operation to eliminate all 2-cuts cotained in $V(C_k)$ not satisfying (2). We can deal with $C_0, C_1, C_2, \ldots$ in that order, and we see that (3.4) holds. $\square$

The proof of (1.2) is divided into the following stages. Assume that G is a cubic infinite infinite plane graph. Then by the results in Section 2, G has a nice sequence

$(C_0, C_2, \ldots)$ and we can assume that G is nicely embedded with respect to that sequence. First, we show that for all sufficiently large i, there are at most three vertices of C_i between any two consecutive in-vertices on C_i. This is done in Section 4. We then use the result in Section 4 to show that for all sufficiently large i, there are at most two vertices of C_i between any two consecutive in-vertices on C_i, and this is done in Section 5. In Section 6, we further show that for all sufficiently large i, there are at most one vertex of C_i between any two consecutive in-vertices on C_i. Finally we complete the proof in Section 7 by showing that for all sufficiently large i, $|C_j| > |C_i|$ for all $j > i$.

4 Four vertices between consecutive in-vertices

For convenience, we assume throughout this section that G is a positively curved cubic infinite plane graph which is locally finite and whose faces are bounded by cycles. Moreover, G has a nice sequence $(C_0, C_1, \ldots)$ and G is nicely embedded with respect to $(C_0, C_1, \ldots)$.

The main result of this section is the following: if i is large enough, then there are at most three vertices of C_i between any two consecutive in-vertices on C_i. This is done through a series of lemmas. For the statement and proof of the first lemma, we refer to Figure 6.

(4.1) Lemma. *Let $i \geq 3$, and let a_1 and a_2 be consecutive in-vertices on C_i such that (i) $R(a_1) = L(a_2)$ and (ii) $|C_i(a_1, a_2)| \geq 4$. Let b_1, b_2 be the out-vertices on C_{i-1} such that $R(b_1) = L(b_2) = R(a_1)$. If $C_{i-1}(b_1, b_2) = \emptyset$, then $|L(b_1)| = |R(b_2)| = 3$.*

Proof. Suppose $C_{i-1}(b_1, b_2) = \emptyset$. Since G is cubic, $B(b_1) = B(b_2)$. So let c_1, c_2 be the out-vertices on C_{i-2} such that $R(c_1) = B(b_1) = L(c_2)$. By (ii), $|R(b_1)| = |L(b_2)| \geq 8$. Since $C_{i-1}(b_1, b_2) = \emptyset$, $|B(b_1)| = |B(b_2)| \geq 6$.

Therefore, $|L(b_1)| \leq 4$ and $|R(b_2)| \leq 4$; for otherwise, $\ell(b_1) \geq (5, 6, 8)$ or $\ell(b_2) \geq (5, 6, 8)$, a contradiction. Hence, we consider two cases.

Case 1. $|L(b_1)| = |R(b_2)| = 4$.

By (3.3), $|L(b_1) \cap C_{i-1}| \in \{2, 3\}$ and $|R(b_2) \cap C_{i-1}| \in \{2, 3\}$.

First, assume that $|L(b_1) \cap C_{i-1}| = 2 = |R(b_2) \cap C_{i-1}|$. Since G is cubic and $C_{i-1}(b_1, b_2) = \emptyset$, $|B(b_1)| \geq 8$. Hence $\ell(b_1) \geq (4, 8, 8)$, a contradiction.

Now assume that $|L(b_1) \cap C_{i-1}| = 2$ and $|R(b_2) \cap C_{i-1}| \neq 2$ or $|L(b_1) \cap C_{i-1}| \neq 2$ and $|R(b_2) \cap C_{i-1}| = 2$. By symmetry, assume the former. See Figure 6(a). Then $|R(b_1)| = |L(b_2)| \geq 9$, and since G is cubic and $C_{i-1}(b_1, b_2) = \emptyset$, $|B(b_1)| = |B(b_2)| \geq 7$. Thus, $\ell(b_1) = \ell(b_2) = (4, 7, 9)$, for otherwise, there would exist $i \in \{1, 2\}$ such that $\ell(b_i) \geq (5, 7, 9)$ or $(4, 8, 9)$ or $(4, 7, 10)$. Hence, $C_{i-2}(c_1, c_2) = \emptyset$ and c_2 is adjacent to C_{i-1}. So $|B(c_2)| \geq 6$ and $|R(c_2)| \geq 5$ (because $|R(b_2) \cap C_{i-1}| = 3$). If $|R(c_2)| = 5$

then $|B(c_2)| \geq 7$, and hence, $\ell(c_2) \geq (5,7,7)$, a contradiction. So $|R(c_2)| \geq 6$. Then $\ell(c_2) \geq (6,6,7)$, a contradiction.

So $|L(b_1) \cap C_{i-1}| = 3 = |R(b_2) \cap C_{i-1}|$. See Figure 6(b). Hence, $|R(b_1)| = |L(b_2)| \geq 10$. Since $|B(b_1)| = |B(b_2)| \geq 6$, $\ell(b_1) \geq (4,6,10)$ and $\ell(b_2) \geq (4,6,10)$. In fact, $|B(b_1)| = |B(b_2)| = 6$, as otherwise $\ell(b_1) \geq (4,7,10)$ or $\ell(b_2) \geq (4,7,10)$. So $C_{i-2}(c_1, c_2) = \emptyset$, $|L(c_1)| \geq 5 \leq |R(c_2)|$, and $|B(c_1)| = |B(c_2)| \geq 6$. Suppose $|L(c_1)| = |R(c_2)| = 5$. Then $|L(c_1) \cap C_{i-2}| = 2 = |R(c_2) \cap C_{i-2}|$. Since G is 2-connected and $C_{i-2}(c_1, c_2) = \emptyset$, $|B(c_1)| = |B(c_2)| \geq 8$. Hence $\ell(c_1) \geq (5,6,8)$, a contradiction. So $|L(c_1)| \geq 6$ or $|R(c_2)| \geq 6$. Then $\ell(c_1) \geq (6,6,6)$ or $\ell(c_2) \geq (6,6,6)$, a contradiction.

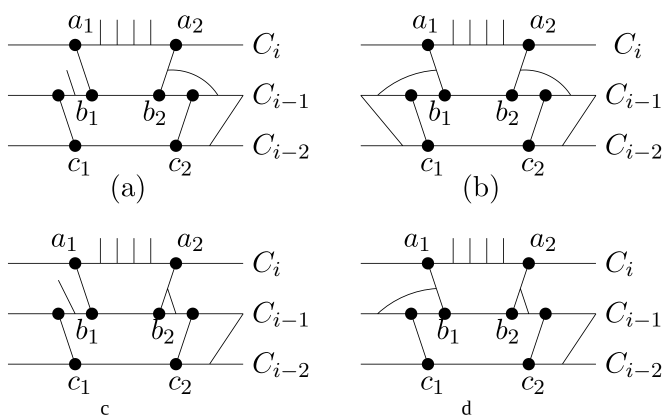

Figure 6: Proof of Lemma (4.1)

Case 2. $|L(b_1)| = 4$ and $|R(b_2)| = 3$, or $|L(b_1)| = 3$ and $|R(b_2)| = 4$.

By symmetry, we may assume the former. Since $|R(b_2)| = 3$ and $C_{i-1}(b_1, b_2) = \emptyset$, $|L(b_2)| = |R(b_1)| \geq 9$ and $|B(b_1)| = |B(b_2)| \geq 7$.

First, assume that $|L(b_1) \cap C_{i-1}| = 2$. See Figure 6(c). Then since G is cubic and $C_{i-1}(b_1, b_2) = \emptyset$, $|B(b_1)| \geq 8$. Therefore, $\ell(b_1) \geq (4,8,9)$, a contradiction.

So $|L(b_1) \cap C_{i-1}| = 3$. See Figure 6(d). Then $|B(b_1)| \geq 7$ and $|R(b_1)| \geq 10$. So $\ell(b_1) \geq (4,7,10)$, a contradiction. □

(4.2) Lemma. *Let $i \geq 5$, and let a_1 and a_2 be consecutive in-vertices on C_i such that (i) $R(a_1) = L(a_2)$ and (ii) $|C_i(a_1, a_2)| \geq 4$. Let b_1 and b_2 be the out-vertices on C_{i-1} such that $R(b_1) = L(b_2) = R(a_1)$. Then*

(1) $|C_{i-1}(b_1, b_2)| \neq 1$, and

(2) $|C_{i-1}(b_1, b_2)| \neq 2$ if $i \geq 7$.

Proof. (1) Suppose $|C_{i-1}(b_1,b_2)| = 1$. Let b be the only vertex in $C_{i-1}(b_1,b_2)$. Since G is 2-connected, b is an in-vertex and $B(b_1) = L(b)$. Let c_1, c, c_2 be the out-vertices on C_{i-2} such that $R(c_1) = L(c) = B(b_1)$ and $L(c_2) = B(b_2)$. See Figure 7.

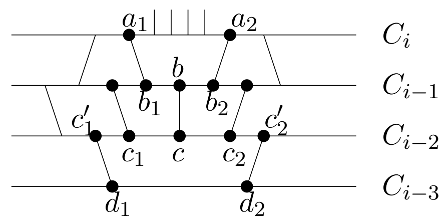

Figure 7: $C_{i-1}(b_1,b_2)| = 1$.

Note that $|R(b_1)| = |A(b)| = |L(b_2)| \geq 9$, $|L(b)| = |B(b_1)| \geq 5$, and $|R(b)| = |B(b_2)| \geq 5$. Hence $\ell(b) \geq (5,5,9)$. Therefore, $|L(b)| = |R(b)| = 5$ and $|A(b)| = 9$; otherwise, $\ell(b) \geq (5,6,9)$ or $(5,5,10)$, a contradiction.

Since $|L(b)| = |R(b)| = 5$, $C_{i-2}(c_1,c) = \emptyset = C_{i-2}(c,c_2)$, b is adjacent to c, and both c_1 and c_2 are adjacent to C_{i-1}. Therefore, $|B(c_1)| = |B(c)| = |B(c_2)| \geq 7$ and $|L(b_1)| \geq 5 \leq |R(b_2)|$. So $|L(b_1)| = 5 = |R(b_2)|$, for otherwise, $\ell(b_1) \geq (5,6,9)$ or $\ell(b_2) \geq (5,6,9)$, a contradiction.

Since G is 2-connected, let c_1', c_2' be in-vertices on C_{i-2} such that $R(c_1') = L(c_2')$, and let d_1, d_2 be the out-vertices on C_{i-3} such that $R(d_1) = L(d_2) = B(c_1)$.

We claim that $C_{i-2}(c_1', c_1) = \emptyset$ or $C_{i-2}(c_2, c_2') = \emptyset$. Otherwise, $|C_{i-2}(c_1', c_2')| \geq 5$. Since $i - 2 \geq 3$, it follows from (4.1) that $|L(d_1)| = 3 = |R(d_2)|$ or $C_{i-1}(d_1,d_2) \neq \emptyset$. So $|B(c)| \geq 10$, and hence $\ell(c) \geq (5,5,10)$, a contradiction.

So by symmetry, assume that $C_{i-2}(c_1', c_1) = \emptyset$. Since $|L(b_1)| = 5$ and a_1 is adjacent to b_1, we have $|A(c_1')| \geq 6$. So $|L(c_1)| = |A(c_1')| = 6$ and $|B(c_1)| = 7$, as otherwise, we would have $\ell(c_1) \geq (5,6,8)$ or $(5,7,7)$. Thus c_1' is adjacent to d_1, $C_{i-3}(d_1,d_2) = \emptyset$, and $|L(d_1)| \geq 5$. Moreover, if $|L(d_1)| = 5$ then $|B(d_1)| \geq 7$. So $\ell(d_1) \geq (5,7,7)$ or $(6,6,7)$, a contradiction.

(2) Now suppose $i \geq 7$ and $|C_{i-1}(b_1,b_2)| = 2$. Let b_3, b_4 denote the vertices in $C_{i-1}(b_1,b_2)$. By (3.4), both b_3 and b_4 are in-vertices. Without loss of generality, we may assume that $R(b_3) = L(b_4)$. See Figure 8.

Observe that $|R(b_1)| = |L(b_2)| = |A(b_3)| = |A(b_4)| \geq 10$, $|B(b_1)| = |L(b_3)| \geq 5 \leq |R(b_4)| = |B(b_2)|$, $|R(b_3)| = |L(b_4)| \geq 4$. Furthermore, $|R(b_3)| = |L(b_4)| = 4$ and $|L(b_3)| = |B(b_1)| \leq 6 \geq |R(b_4)| = |B(b_2)|$, for otherwise, there would exist $i \in \{3,4\}$ such that $\ell(b_i) \geq (5,5,10)$ or $(4,7,10)$, a contradiction.

So let $c_3, c_4 \in V(C_{i-3})$ such that $b_3c_3, b_4c_4 \in E(G)$. Because G is cubic and 2-connected, $C_{i-2}(c_4,c_3)$ has at least two in-vertices. We claim that $C_{i-2}(c_4,c_3)$ also

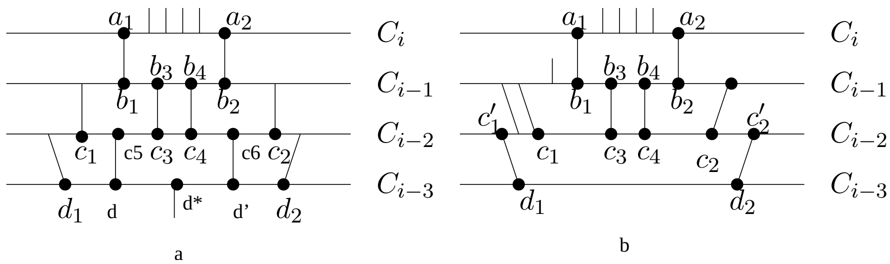

Figure 8: $i \geq 7$ and $|C_{i-1}(b_1, b_2)| = 2$.

contains at least two out-vertices. For otherwise, $C_{i-2}(c_4, c_3)$ contains at most one out-vertex. Because $|L(b_3)| \leq 6$, $C_{i-2}(c_4, c_3)$ contains exactly one out-vertex. So $|L(b_3)| = 6 = |R(b_4)|$, and $|L(b_1)| \geq 5$. Thus $\ell(b_1) \geq (5, 6, 10)$, a contradiction. So let c_1, c_2 be distinct out-vertices on C_{i-2} such that $R(c_1) = B(b_1)$ and $L(c_2) = B(b_2)$.

We claim that $C_{i-2}(c_1, c_3) = \emptyset = C_{i-2}(c_4, c_2)$. For otherwise, assume $C_{i-2}(c_1, c_3) \neq \emptyset$. See Figure 8(a). Because $|L(b_3)| \leq 6$, $C_{i-2}(c_1, c_3)$ consists of only one vertex, say c_5. Since G is 2-connected, c_5 is an in-vertex. Thus we see that $|A(c_5)| = 6$, $|L(c_5)| \geq 5$, and $|R(c_5)| \geq 6$. So $|L(c_5)| = 5$ and $|R(c_5)| \leq 7$, as otherwise $\ell(c_5) \geq (5, 6, 8)$ or $(6, 6, 6)$. Hence c_5 is adjacent to a vertex, say d, on C_{i-3}. Assume that $C_{i-2}(c_4, c_2) = \emptyset$, then $|R(c_5)| \geq 7$. Thus $|R(c_5)| = 7$, and hence $|R(c_5) \cap C_{i-3}| = 2$. Therefore, since $|L(c_5)| = 5$, $|B(d)| \geq 7$. So $\ell(d) \geq (5, 7, 7)$, a contradiction. Thus $C_{i-2}(c_4, c_2) \neq \emptyset$. Because $|R(b_4)| \leq 6$, $C_{i-2}(c_4, c_2)$ consists of only one vertex, say c_6. Since G is 2-connected, c_6 is an in-vertex. Note that $|R(c_6)| = 5$, for otherwise $\ell(c_6) \geq (6, 6, 6)$. Thus c_6 is adjacent to a vertex, say d', on C_{i-3}. If $C_{i-3}(d, d') = \emptyset$, then since $|L(c_5)| = |R(c_6)| = 5$, $|B(d)| \geq 8$ and we would have $\ell(d) \geq (5, 6, 8)$. So $C_{i-3}(d, d') \neq \emptyset$. Since $|R(c_5)| \leq 7$, $C_{i-3}(d, d')$ contains only one vertex, say d^*. Because $|L(c_5)| = 5 = |R(c_6)|$, $|L(d^*)| \geq 6 \leq |R(d^*)|$. So $\ell(d^*) \geq (6, 6, 7)$, a contradiction.

Let c_1', c_2' denote the in-vertices on C_{i-2} such that $R(c_1') = L(c_2') = B(c_1)$, and let d_1, d_2 be the out-vertices on C_{i-3} such that $L(d_1) = R(d_2) = B(c_1)$. See Figure 8(b). Since $i - 2 \geq 5$ and $|C_{i-2}(c_1' c_2')| \geq 4$, it follows from (1) that $|C_{i-3}(d_1, d_2)| \neq 1$. So by (4.1), $|L(d_1)| = |R(d_2)| = 3$ or $|C_{i-3}(d_1, d_2)| \geq 2$. Thus $|B(c_1)| = |B(c_2)| \geq 10$. Since $|R(c_1)| \geq 5 \leq |L(c_2)|$, $|L(c_1)| \leq 4 \geq |R(c_2)|$ (or else there would exist $i \in \{1, 2\}$ such that $\ell(c_i) \geq (5, 5, 10)$). Note $|R(b_1)| = |L(b_2)| \geq 10$ and $|B(b_1)| \geq 5 \leq |B(b_2)|$. So $|L(b_1)| \leq 4 \geq |R(b_2)|$ (otherwise there would exist some $i \in \{1, 2\}$ such that $\ell(b_i) \geq (5, 5, 10)$).

We claim that $|L(b_1)| = 4$ and $|L(b_1) \cap C_{i-1}| = 2$ or $|R(b_2)| = 4$ and $|R(b_2) \cap C_{i-1}| = 2$. For otherwise, $|R(b_1)| = |L(b_2)| \geq 12$. Then $|L(b_3)| = |B(b_1)| = 5$, or else $\ell(b_3) \geq (4, 6, 12)$. Hence $|L(b_1)| = 4$ and $|L(b_1) \cap C_{i-1}| = 3$, and so, $|L(c_1)| \geq 5$, a contradiction.

Without loss of generality assume that $|L(b_1)| = 4$ and $|L(b_1) \cap C_{i-1}| = 2$. See Figure

8(b). Then $|B(b_1)| \geq 6$. In fact, $|B(b_1)| = 6$; otherwise, $\ell(b_1) \geq (4,7,10)$, a contradiction.

So $|L(c_1)| = 4$ (otherwise $\ell(c_1) \geq (5,6,8)$). Hence $|L(c_1) \cap C_{i-2}| = 2$ and $|B(c_1)| \geq 11$. In fact, $|B(c_1)| = 11$ as otherwise $\ell(c_1) \geq (4,6,12)$. So c_1' is adjacent to $L(c_1)$ and $C_{i-2}(c_2, c_2') = \emptyset$. Thus $|A(c_1')| \geq 5$, $|A(c_2')| = |R(c_2)| = 4$, and $|A(c_2') \cap C_{i-2}| = 3$.

If $C_{i-3}(d_1, d_2) \neq \emptyset$, then since $i - 2 \geq 5$ and by (1), $|C_{i-3}(d_1, d_2)| \geq 2$. Thus $|B(c_1)| = 11$ implies that c_2' is adjacent to d_2. Since $|A(c_2')| = 4$, $|R(d_2)| \geq 5$. Hence $\ell(d_2) \geq (5,5,11)$, a contradiction. Therefore, $C_{i-3}(d_1, d_2) = \emptyset$. Then by (4.1), $|L(d_1)| = 3$, and hence $|L(c_1')| \geq 5$. But now $\ell(c_1') \geq (5,5,11)$, a contradiction. $\square$

(4.3) Lemma. *Let $i \geq 7$, let a_1 and a_2 be consecutive in-vertices on C_i such that (i) $R(a_1) = L(a_2)$ and (ii) $|C_i(a_1, a_2)| \geq 4$. Let b_1 and b_2 be the out-vertices on C_{i-1} such that $R(b_1) = L(b_2) = R(a_1)$. Assume that $C_{i-1}(b_1, b_2) = \emptyset$, and let c_1, c_2 denote the out-vertices on C_{i-2} such that $R(c_1) = L(c_2) = B(b_1) = B(b_2)$. Then $C_{i-2}(c_1, c_2) \neq \emptyset$.*

Proof. Suppose for a contradiction that $C_{i-2}(c_1, c_2) = \emptyset$. Since G is 2-connected and cubic, let b_1', b_2' denote the consecutive in-vertices on C_{i-1} such that $R(b_1') = L(b_2') = B(b_1) = B(b_2)$; let c_1', c_2' be the consecutive in-vertices on C_{i-2} such that $R(c_1') = L(c_2') = B(c_1) = B(c_2)$; and let d_1, d_2 denote the out-vertices on C_{i-3} such that $R(d_1) = L(d_2) = B(c_1)$. See Figure 9.

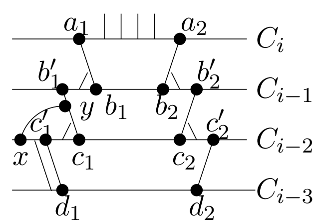

Figure 9: Proof of Lemma (4.3)

Since $C_{i-1}(b_1, b_2) = \emptyset$ and $i \geq 7$, it follows from (4.1) that $|L(b_1)| = |R(b_2)| = 3$. Therefore, since G is cubic, there are at least four consecutive out-vertices in $C_{i-1}(b_1', b_2')$. Since $C_{i-2}(c_1, c_2) = \emptyset$ and since $i-1 \geq 6$, it follows from (4.1) that $|L(c_1)| = |R(c_2)| = 3$. See Figure 9. Hence $|R(b_1')| = |B(b_1)| = |B(b_2)| \geq 10$.

We claim that $|B(b_1)| = |B(b_2)| \geq 12$. For otherwise, b_1' is adjacent to both $L(b_1)$ and $L(c_1)$, or b_2' is adjacent to both $R(b_2)$ and $R(c_2)$. By symmetry, we may assume the former. Then since G is cubic, $|L(b_1')| \geq 5 \leq |A(b_1')|$. Since $|R(b_1')| \geq 10$, $\ell(b_1') \geq (5,5,10)$, a contradiction.

Then $|B(c_1)| = |B(c_2)| < 12$; for otherwise, $\ell(c_1) \geq (3,12,12)$, a contradiction.

Since $i - 2 \geq 5$, it follows from (4.1) that $|L(d_1)| = |R(d_2)| = 3$ or $|C_{i-3}(d_1, d_2)| \neq 0$. So by (4.2) that $|L(d_1)| = |R(d_2)| = 3$ or $|C_{i-3}(d_1, d_2)| \geq 2$. Thus $B(c_1)| = |B(c_2)| \geq$

10. Since $|B(d_1)| \geq 5 \leq |B(d_2)|$, $|L(d_1)| \leq 4 \geq |R(d_2)|$ (or else, $\ell(d_1) \geq (5,5,10)$ or $\ell(d_2) \geq (5,5,10)$). Since $|B(c_1)| = |B(c_2)| < 12$, either c_1' is adjacent to $L(c_1)$, or c_2' is adjacent to $R(c_2)$. By symmetry, assume the former. Then $|L(c_1')| = |L(d_1)| = 4$, and so, $|A(c_1')| \geq 6$. Since $|R(c_1')| \geq 10$, $|A(c_1')| = 6$ (or else $\ell(c_1') \geq (4,7,10)$, a contradiction). So G has an edge xy such that $x \in V(C_{i-2})$, y is strictly between C_{i-1} and C_{i-2}, and y is adjacent to $L(c_1)$. See Figure 9. Since G is cubic, we can check that $\ell(y) \geq (4,6,12)$, a contradiction. $\square$

(4.4) Lemma. *Let $i \geq 8$, let a_1 and a_2 be consecutive in-vertices on C_i such that (i) $R(a_1) = L(a_2)$ and (ii) $|C_i(a_1,a_2)| \geq 4$. Let b_1 and b_2 be out-vertices on C_{i-1} such that $R(b_1) = L(b_2) = R(a_1)$. Let b_3, b_4 be the vertices on $C_{i-1}(b_1,b_2)$ such that $C_{i-1}(b_1,b_3) = \emptyset = C_{i-1}(b_4,b_2)$. Assume $|C_{i-1}(b_1,b_2)| \geq 3$. Then*

(1) both b_3 and b_4 are out-vertices on C_{i-1},

(2) $|L(b_1)| = |R(b_2)| = 3$ and $|R(b_3)| = |L(b_4)| = 3$, and

(3) both $R(b_3)$ and $L(b_4)$ use three consecutive vertices on C_{i-1}.

Proof. Since $|C_{i-1}(b_1,b_2)| \geq 3$, $|R(b_1)| = |L(b_2)| \geq 10$.

(1) Suppose b_3 is an in-vertex on C_{i-1}. Then $|A(b_3)| = |R(b_1)| \geq 11$, $|L(b_3)| \geq 5$, and $|R(b_3)| \geq 4$. Hence, $|R(b_3)| = 4$; otherwise, $\ell(b_3) \geq (5,5,11)$, a contradiction. So let b, c, c_3 be the vertices of $R(b_3)$ such that $b \in C_{i-1}$, $c, c_3 \in C_{i-2}$, c is adjacent to b, and c_3 is adjacent to b_3. See Figure 10.

Since $|B(b_1)| \geq 5$, $|L(b_1)| \leq 4$, for otherwise, $\ell(b_1) \geq (5,5,11)$. In fact $|L(b_1)| = 4$, for otherwise $|A(b_3)| = |R(b_1)| \geq 12$ and $|L(b_3)| = |B(b_1)| \geq 6$, and so, $\ell(b_3) \geq (4,6,12)$, a contradiction. By (3.3), we see that $|L(b_1) \cap C_{i-1}| \in \{2,3\}$. Hence we distinguish two cases.

First, assume that $|L(b_1) \cap C_{i-1}| = 2$. See Figure 10(a). Then $|B(b_1)| \geq 6$. Hence $|R(b_1)| = 11$ and $|B(b_1)| = 6$, for otherwise, $\ell(b_1) \geq (4,6,12)$ or $(4,7,11)$. Then $|C_{i-1}(b_1,b_2)| = 3$, and b_4 is also an in-vertex. Hence, $|R(b_4)| \geq 5$, $|L(b_4)| \geq 4$, and $|A(b_4)| = 11$. So $|L(b_4)| = 4$, as otherwise, $\ell(b_4) \geq (5,5,11)$. Thus b_4 is adjacent to a vertex c_4 on C_{i-2}. Now $|B(b_2)| \geq 5$. Since $|L(b_2)| = 11$, a_2 is adjacent to b_2, and hence $|R(b_2)| \geq 4$. Moreover, $|R(b_2)| = 4$ and $|B(b_2)| = 6$, for otherwise, $\ell(b_2) \geq (5,5,11)$ or $(4,7,11)$. Let c_1, c_2 be the out-vertices on C_{i-2} such that $R(c_1) = B(b_1)$ and $L(c_2) = B(b_2)$. Then $c_1 \neq c_2$ (since $i \geq 7$ and G is 2-connected). See Figure 10(a). Since $|B(b_1)| = |B(b_2)| = 6$, $C_{i-2}(c_1,c_3) = \emptyset = C_{i-2}(c_4,c_2)$. Since $i-2 \geq 5$, by (4.1) and (4.2), $|B(c_1)| = |B(c_2)| \geq 11$. So $|L(c_1)| \leq 4$, as otherwise $\ell(c_1) \geq (5,6,10)$. Since $|R(c_1)| = 6$, we have $|L(c_1)| = 4$ and $|L(c_1) \cap C_{i-2}| = 2$. Therefore, $|B(c_1)| \geq 12$, and so, $\ell(c_1) \geq (4,6,12)$, a contradiction.

So $|L(b_1) \cap C_{i-1}| = 3$. See Figure 10(b) and Figure 10(c). Then $|R(b_1)| \geq 12$ (since $|C_{i-1}(b_1,b_2)| \geq 3$, and b_3 and b are in-vertices on C_{i-1}). So $|B(b_1)| = 5$, for otherwise,

$\ell(b_3) \geq (4,6,12)$. Let c_1 denote the out-vertex on C_{i-2} such that $R(c_1) = B(b_1)$. Then $C_{i-2}(c_1, c_3) = \emptyset$, and so, $|B(c_1)| = |B(c)| \geq 7$. Note that $|L(c_1)| \in \{5,6\}$, or else $\ell(c_1) \geq (5,7,7)$. Let d_1, d_2 be the out-vertices on C_{i-3} such that $R(d_1) = L(d_2) = B(c_1)$. Assume $|L(c_1)| = 6$. See Figure 10(b). Then $|B(c_1)| = |R(d_1)| = 7$, for otherwise, $\ell(c_1) \geq (5,6,8)$. Thus $C_{i-3}(d_1, d_2) = \emptyset$, and so, $|B(d_1)| \geq 6$. So $|L(d_1)| = 5$ as otherwise, $\ell(d_1) \geq (6,6,7)$. Since $|L(d_1)| = 5$, $d_2 \notin L(d_1)$. Therefore, $|B(d_1)| \geq 7$, and $\ell(d_1) \geq (5,7,7)$, a contradiction. Now assume $|L(c_1)| = 5$. See Figure 10(c). Then $c \notin L(c_1)$. Let c'_1, c'_2 denote the in-vertices on C_{i-2} such that $R(c'_1) = L(c'_2) = B(c_1)$. Then $|C_{i-2}(c'_1, c'_2)| \geq 4$. Since $i - 2 \geq 5$, by (4.1) and (4.2), we have $|B(c_1)| \geq 10$. Thus $\ell(c_1) \geq (5,5,10)$, a contradiction.

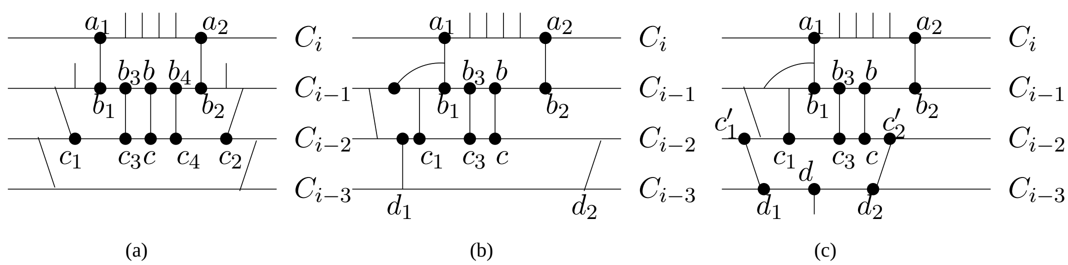

Figure 10: Proof of Lemma (4.4)

(2) By symmetry, we only prove (2) for b_3 and b_1. By (1), b_3 is an out-vertex, and so, $|L(b_3)| = |R(b_1)| \geq 10$ and $|B(b_1)| = |B(b_3)| \geq 6$. So $|L(b_1)| \leq 4$; otherwise, $\ell(b_1) \geq (5,6,10)$, a contradiction.

First, assume that $|L(b_1)| = 4$. Then $|L(b_1) \cap C_{i-1}| \in \{2,3\}$. If $|L(b_1) \cap C_{i-1}| = 2$, then $|B(b_1)| \geq 7$, and so, $\ell(b_1) \geq (4,7,10)$, a contradiction. So $|L(b_1) \cap C_{i-1}| = 3$. Then $|R(b_1)| \geq 11$. Further, $|R(b_1)| = 11$ and $|B(b_1)| = 6$; otherwise, $\ell(b_1) \geq (4,6,12)$ or $\ell(b_1) \geq (4,7,11)$, a contradiction. Because $|R(b_1)| = 11$, b_3 is adjacent to b_4 and a_2 is adjacent to b_2. See Figure 11(a). So $|B(b_2)| \geq 6$ and $|R(b_2)| \geq 4$, and if $|R(b_2)| = 4$ then $|B(b_2)| \geq 7$. Thus $\ell(b_2) \geq (4,7,11)$ or $(5,6,11)$, a contradiction.

Thus $|L(b_1)| = 3$. So $|B(b_1)| = |B(b_3)| \geq 7$ and $|L(b_3)| = |R(b_1)| \geq 11$. Therefore, $|R(b_3)| = 3$, as otherwise $\ell(b_3) \geq (4,7,11)$.

(3) By symmetry, we will only show that $R(b_3)$ uses three consecutive vertices on C_{i-1}. Suppose on the contrary that $R(b_3)$ has a vertex, say b, not on C_{i-1}. See Figure 11(b). Let b' denote the vertex in $R(b_3) - \{b, b_3\}$. Note that $C_{i-1}(b_3, b') = \emptyset$ and $|R(b_1)| \geq 13$ (by (2)). Let b'_1, b^* denote the in-vertices on C_{i-1} such that $R(b'_1) = L(b^*) = B(b_1)$. Note that $|C_{i-1}(b'_1, b^*)| \geq 4$. Then $b^* \in C_{i-1}(b', b_2)$, for otherwise, $|B(b_1)| \geq 12$ (by (4.1)) and $\ell(b_1) \geq (3,12,13)$. Since b_4 is an out-vertex and $|L(b_4)| = 3$, b is not adjacent to b_4, and so, $|R(b_1)| = |L(b_2)| \geq 14$.

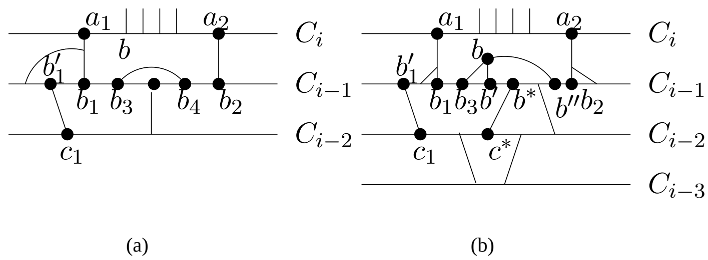

Figure 11: Proof of Lemma (4.4)

Let c_1, c^* denote the in-vertices on C_{i-2} such that $R(c_1) = L(c^*) = B(b_1)$.

We claim that $|L(c_1)| = |R(c^*)| = 3$. Since $|C_{i-1}(b_1', b^*)| \geq 4$ and $i - 1 \geq 7$, it follows from (4.1) that $|L(c_1)| = |R(c^*)| = 3$ or $|C_{i-2}(c_1, c^*)| \neq 0$. If $|C_{i-2}(c_1, c^*)| \neq 0$, then it follows from (4.2) that $|C_{i-2}(c_1, c^*)| \geq 3$. Hence $|L(c_1)| = |R(c^*)| = 3$.

Thus $|B(b_1)| \geq 10$ and $|L(b_1')| \geq 5$. Then $|A(b_1')| \leq 4$, for otherwise, $\ell(b_1') \geq (5, 5, 10)$. Hence b_1' can not be adjacent to $L(b_1)$. So $|B(b_1)| \geq 11$. Therefore $\ell(b_1) \geq (3, 11, 14)$, a contradiction. $\square$

Let us state the claim in the proof of (3) of (4.4) as a lemma.

(4.5) Lemma. *Let $i \geq 8$, let a_1 and a_2 be consecutive in-vertices on C_i such that (i) $R(a_1) = L(a_2)$ and (ii) $|C_i(a_1, a_2)| \geq 4$. Let b_1 and b_2 be the out-vertices on C_{i-1} such that $R(b_1) = L(b_2) = R(a_1)$. Then $|L(b_1)| = |R(b_2)| = 3$.*

(4.6) Lemma. *Let $i \geq 8$, let a_1 and a_2 be consecutive in-vertices on C_i such that (i) $R(a_1) = L(a_2)$ and (ii) $|C_i(a_1, a_2)| \geq 4$. Let b_1 and b_2 be out-vertices on C_{i-1} such that $R(b_1) = L(b_2) = R(a_1)$. Suppose $|C_{i-1}(b_1, b_2)| \geq 3$. Let b_3, b_4 be vertices on $C_{i-1}(b_1, b_2)$ such that $C_{i-1}(b_1, b_3) = \emptyset = C_{i-1}(b_4, b_2)$. Let b, b^* be the vertices in $C_{i-1}(b_3, b_4)$ such that $C_{i-1}(b_3, b) = \emptyset = C_{i-1}(b^*, b_4)$, and let b', b'' be the neighbors of b, b^*, respectively, such that $b', b'' \notin C_{i-1}$. Then*

(1) $b', b'' \notin C_{i-2}$ and

(2) b' is contained in a facial triangle of G which also contains two consecutive vertices on C_{i-2}, and b'' is contained in a facial triangle of G which also contains two consecutive vertices on C_{i-2}.

Proof. By (4.4), $R(b_3) = A(b)$ and $L(b_4) = A(b^*)$ are facial triangles, and b and b^* are in-vertices on C_{i-1}. By symmetry, we only need to show the conclusions for b'.

(1) Suppose $b' \in C_{i-2}$. See Figure 12(a). Then b' is an out-vertex on C_{i-2}, and so, $|B(b')| \geq 5$. By (2) of (4.4), $|R(b')| \geq 5$ and $|L(b')| = |B(b_1)| \geq 7$. If $|B(b')| = 5$ then

$|L(b')| \geq 8$ and $|R(b')| \geq 6$, and we would have $\ell(b') \geq (5,6,8)$. So $|B(b')| \geq 6$. Then $|R(b')| = 5$ or else $\ell(b') \geq (6,6,7)$. So $R(b') \neq B(b_1)$. Therefore, if $|B(b')| = 6$ then $|L(b')| \geq 8$. So $\ell(b') \geq (5,6,8)$ or $(5,7,7)$, a contradiction.

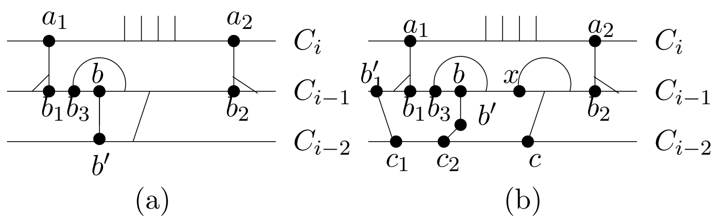

Figure 12: (1) of Lemma (4.6)

(2) First we show that b' is contained in a facial triangle. Suppose on the contrary that b' is not contained in any facial triangle. Since $b' \notin C_{i-2}$, $|L(b)| \geq 8$. Hence $|R(b)| \leq 7$ as otherwise $\ell(b') \geq (4,8,8)$. Thus $b \neq b^*$, for otherwise, $|R(b)| \geq 8$. See Figure 12(b). Hence $|R(b_1)| \geq 14$.

Let x denote the vertex on $C_{i-1}(b, b^*) - A(b)$ such that x is adjacent to $A(b)$. Then x is an out-vertex; otherwise $|L(x)| \geq 6$ and $|R(x)| \geq 4$, and so, $\ell(x) \geq (4,6,14)$, a contradiction. Thus $|R(b)| = 7$. This implies that $R(x)$ is a triangle (otherwise $\ell(x) \geq (4,7,14)$) and $R(x)$ uses three consecutive vertices of C_{i-1}. Now let c denote the out-vertex on C_{i-2} such that $L(c) = R(b)$. Then $|L(c)| = 7$. Hence $|R(c)| \geq 5$ and $|B(c)| \geq 6$. Furthermore, if $|R(c)| = 5$ then $|B(c)| \geq 7$. So $\ell(c) \geq (5,7,7)$ or $(6,6,7)$, a contradiction.

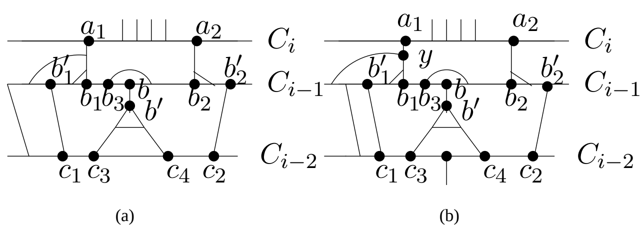

Figure 13: (2) of Lemma (4.6)

Next we show that the facial triangle containing b' also contains two consecutive vertices on C_{i-2}. Note that $|R(b_1)| = |L(b_2)| \geq 12$ and $|B(b_1)| \geq 8 \leq |B(b_2)|$ (because $b', b'' \notin C_{i-2}$ by (1)). Also note that $|B(b_1)| \leq 11 \geq |B(b_2)|$, as otherwise, we would have $\ell(b_1) \geq (3,12,12)$ or $\ell(b_2) \geq (3,12,12)$.

Let b'_1, b'_2 denote the in-vertices on C_{i-1} such that $R(b'_1) = B(b_1)$ and $L(b'_2) = B(b_2)$. Let c_1, c_2, c_3, c_4 be the out-vertices on C_{i-2} such that $R(c_1) = L(c_3) = B(b_1)$ and $L(c_2) =$

$R(c_4) = B(b_2)$. See Figure 13.

For convenience, let P denote the clockwise subpath of $B(b_1)$ from b' to c_3. We wish to show that $|P| = 2$, and therefore, since G is cubic, the facial triangle of G containing b' also contains two consecutive vertices on C_{i-2}.

We claim that $|P| \leq 3$. Suppose this is false, then $|P| \geq 4$. Thus $|B(b_1)| \geq 10$. Note that $|B(b_1)| \leq 11$, as otherwise, $\ell(b_1) \geq (3, 12, 12)$. First, assume that $C_{i-2}(c_1, c_3) \neq \emptyset$. Then $|B(b_1)| = 11$, and $C_{i-2}(c_1, c_3)$ consists of only one vertex, say c. It is easy to see that $\ell(c) \geq (5, 5, 11)$, a contradiction. So $C_{i-2}(c_1, c_3) = \emptyset$. Thus $|B(c_1)| \geq 6$. So $|L(c_1)| \leq 4$; otherwise, $\ell(c_1) \geq (5, 6, 10)$, a contradiction. If $|L(c_1)| = 4$ and $|L(c_1) \cap C_{i-2}| = 2$, then $|B(c_1)| \geq 7$, and we would have $\ell(c_1) \geq (4, 7, 10)$. So $|L(c_1)| = 3$ or $|L(c_1)| = 4$ and $|L(c_1) \cap C_{i-2}| = 3$. Then c_1 is not adjacent to b'_1. So $|B(b_1)| = 11$, and hence, b'_1 is adjacent to $L(b_1)$. Therefore $|A(b'_1)| \geq 5 \leq |L(b'_1)|$, and so, $\ell(b'_1) \geq (5, 5, 11)$, a contradiction.

Now assume that $|P| = 3$. Since b' is contained in a facial triangle and $|P| = 3$, $|R(c_3)| \geq 4$. See Figure 13. Note that $9 \leq |B(b_1)| \leq 11$. Also note that $c_1 \neq c_4$, and so, $c_3 \notin L(c_1)$.

It is easy to see that b'_1 is adjacent to $L(b_1)$. Otherwise, $|C_{i-1}(b'_1, b)| \geq 4$. By (4.5), $|R(c_3)| = 3$, a contradiction. Thus $|A(b'_1)| \geq 5$.

Assume $|A(b'_1)| = 5$. See Figure 13(a). Then $|L(b'_1)| = 5$ and $|R(b'_1)| = |B(b_1)| = 9$, for otherwise, $\ell(b'_1) \geq (5, 6, 9)$ or $(5, 5, 10)$. Hence $C_{i-2}(c_1, c_3) = \emptyset$ and $|B(c_1)| \geq 7$. Therefore, $\ell(c_1) \geq (5, 7, 9)$, a contradiction.

So $|A(b'_1)| \geq 6$. See Figure 13(b). Then $|L(b'_1)| = 4$, or else $\ell(b'_1) \geq (5, 6, 9)$. Thus b'_1 is adjacent to c_1 and $|L(b'_1) \cap C_{i-2}| = 2$. Assume $C_{i-2}(c_1, c_3) = \emptyset$. Then $|B(c_1)| \geq 7$. In fact, $|B(c_1)| = 7$, for otherwise, $\ell(c_1) \geq (4, 8, 9)$. So $|R(c_3)| \geq 5$, and $\ell(c_3) \geq (5, 7, 9)$, a contradiction. Thus $C_{i-2}(c_1, c_3) \neq \emptyset$, and so, $|B(b_1)| \geq 10$. This implies that $|A(b'_1)| = 6$, or else $\ell(b'_1) \geq (4, 7, 10)$. Thus $A(b'_1)$ has an edge xy such that $x \in C_{i-1}$, $y \in R(a_1) \cap A(b'_1)$, and $x, y \notin L(b_1)$. Now it is easy to see that $\ell(y) \geq (4, 6, 12)$, a contradiction. □

(4.7) Lemma. *Let $i \geq 10$, let a_1 and a_2 be consecutive in-vertices on C_i such that (i) $R(a_1) = L(a_2)$ and (ii) $|C_i(a_1, a_2)| \geq 4$. Let b_1 and b_2 be the out-vertices on C_{i-1} such that $R(b_1) = L(b_2) = R(a_1)$. Suppose $|C_{i-1}(b_1, b_2)| \geq 3$. Then $|C_{i-1}(b_1, b_2)| = 3$.*

Proof. Let b_3, b_4 be vertices on $C_{i-1}(b_1, b_2)$ such that $C_{i-1}(b_1, b_3) = \emptyset = C_{i-1}(b_4, b_2)$. Let b, b^* be the vertices in $C_{i-1}(b_3, b_4)$ such that $C_{i-1}(b_3, b) = \emptyset = C_{i-1}(b^*, b_4)$.

By (4.4), $|A(b)| = |A(b^*)| = 3 = |L(b_1)| = |R(b_2)|$, and $A(b)$ and $A(b^*)$ each consist of three consecutive vertices on C_{i-1}. So if $b = b^*$ then $|C_{i-1}(b_1, b_2)| = 3$. Hence we may assume that $b \neq b^*$.

Let b', b'' be the neighbors of b, b^*, respectively, such that $b', b'' \notin C_{i-1}$. By (4.6), $b', b'' \notin C_{i-2}$, there is a facial triangle containing b' and two consecutive vertices on C_{i-2} and there is a facial triangle containing b'' and two consecutive vertices on C_{i-2}.

Since G is cubic, $A(b) \cap A(b^*) = \emptyset$. So $|R(b_1)| = |L(b_2)| \geq 14$. Also $|B(b_1)| \geq 8 \leq |B(b_2)|$ (since $b', b'' \notin C_{i-2}$), and $|B(b_1)| \leq 10 \geq |B(b_2)|$ (otherwise, $\ell(b_1) \geq (3, 11, 14)$ or $\ell(b_2) \geq (3, 11, 14)$, a contradiction).

Let b'_1, b'_2 be the in-vertex of C_{i-1} such that $R(b'_1) = B(b_1)$ and $L(b'_2) = B(b_2)$, and let c_1, c_2, c_3, c_4 be the out-vertices of C_{i-2} such that $R(c_1) = L(c_3) = B(b_1)$ and $R(c_4) = L(c_2) = B(b_2)$. See Figure 14(a). Then by (4.6), c_3 is adjacent to b', c_4 is adjacent to b'', and $|R(c_3)| = |L(c_4)| = 3$. Since $|B(b_1)| \leq 10 \geq |B(b_2)|$, $|C_{i-2}(c_1, c_3)| \leq 2 \geq |C_{i-2}(c_4, c_2)|$.

Case 1. b'_1 is adjacent to $L(b_1)$ or b'_2 is adjacent to $R(b_2)$.

By symmetry, assume that b'_1 is adjacent to $L(b_1)$. Thus $|A(b'_1)| \geq 5$. See Figure 14(b).

We claim that $|B(b_1)| \leq 9$. Suppose otherwise, then $|B(b_1)| = 10$. So $|R(b_1)| = 14$, as otherwise $\ell(b_1) \geq (3, 10, 15)$. Then a_1 is adjacent to $L(b_1)$, and so, $|A(b'_1)| \geq 6$. Then $|L(b'_1)| = 4$ as otherwise $\ell(b'_1) \geq (5, 6, 10)$. But then $|A(b'_1)| \geq 7$, and so $\ell(b'_1) \geq (4, 7, 10)$, a contradiction.

Since $|B(b_1)| \leq 9$, $|C_{i-2}(c_1, c_3)| \leq 1$. Suppose $|C_{i-2}(c_1, c_3)| = 1$. Let c denote the only vertex in $C_{i-2}(c_1, c_3)$. Then c is an in-vertex, $|A(c)| = |B(b_1)| = 9$ and $|L(c)| \geq 5$. Since $|R(c_3)| = 3$, $|R(c)| \geq 6$. Thus $\ell(c) \geq (5, 6, 9)$, a contradiction. Therefore $|C_{i-2}(c_1, c_3)| = 0$, and hence $|B(c_1)| \geq 7$. So $|L(c_1)| \leq 4$, as otherwise, $\ell(c_1) \geq (5, 7, 8)$.

If $|L(c_1)| = 4$ and $|L(c_1) \cap C_{i-2}| = 2$, then $|B(c_1)| \geq 8$ and we would have $\ell(c_1) \geq (4, 8, 8)$. So $|L(c_1)| = 3$ or $|L(c_1)| = 4$ and $|L(c_1) \cap C_{i-2}| = 3$. Then $|R(b'_1)| = |R(c_1)| = 9$ and $|L(b'_1)| \geq 5$. Further, if $|L(b'_1)| = 5$, then $|A(b'_1)| \geq 6$. Thus $\ell(b'_1) \geq (5, 6, 9)$, a contradiction.

Case 2. b'_1 is not adjacent to $L(b_1)$ and b'_2 is not adjacent to $R(b_2)$.

Then $|C_{i-1}(b'_1, b)| \geq 4 \leq |C_{i-1}(b^*, b'_2)|$. Thus since $i - 1 \geq 9$ and by (4.5), $|L(c_1)| = |R(c_2)| = 3$. Hence $|B(b_1)| \geq 10 \leq |B(b_2)|$. In fact, $|B(b_1)| = 10 = |B(b_2)|$ and $|R(b_1)| = 14 = |L(b_2)|$; for otherwise $\ell(b_1) \geq (3, 11, 14)$ or $(3, 10, 15)$. Thus there are exactly six vertices in $C_{i-1}(b_1, b_2)$, and $C_{i-2}(c_1, c_3) = \emptyset = C_{i-2}(c_4, c_2)$.

Let c'_1 and c' denote the in-vertices on C_{i-2} such that $R(c'_1) = L(c') = B(c_1)$. We claim that $c' \in C_{i-2}(c_3, c_4)$. Otherwise, $|C_{i-2}(c, c')| \geq 8$. See Figure 14(c). Since $i-2 \geq 8$ and by (4.5), $R(c'_1) \geq 14$. Thus $R(c'_1) = 14 = |B(c_1)|$, or else $\ell(c_1) \geq (3, 10, 15)$. Then c'_1 is adjacent to $L(c_1)$ and $|A(c'_1)| \geq 6$. Therefore $\ell(c'_1) \geq (4, 6, 14)$, a contradiction.

If c' is adjacent to $R(c_3)$ or $L(c_4)$, then it is easy to check that $\ell(c') \geq (4, 8, 8)$, a contradiction.

So c' is not adjacent to $R(c_3)$ and c' is not adjacent to $L(c_4)$. Let x be the vertex in $C_{i-2}(c_3, c_4) - V(R(c_3) \cup L(c_4))$ such that x is adjacent to $R(c_3)$. See Figure 14(d). Then x is an out-vertex on C_{i-2} and $|L(x)| = |R(b)| \geq 10$. Further, $|L(x)| = |R(b)| = 10$, as otherwise, $\ell(b) \geq (3, 11, 14)$. Also $|R(x)| = |L(y)| = 3$, or else $\ell(x) \geq (4, 10, 10)$. Hence $|C_{i-2}(c_3, c_4)| = 5 = |C_{i-2}(c'_1, c')|$. Let y be the vertex in $C_{i-2}(c', c_4)$ and let c'_2

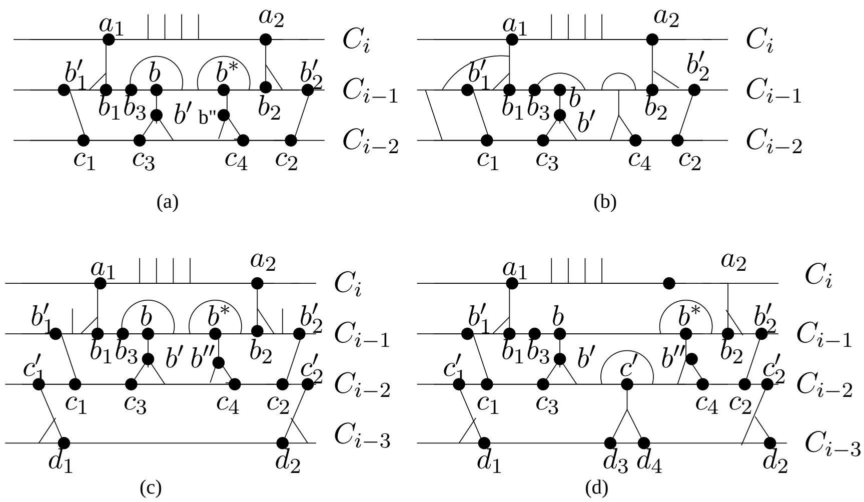

Figure 14: Proof of Lemma (4.7)

be the in-vertex on C_{i-2} such that $L(c'_2) = B(c_2)$. Let d_1, d_2, d_3, d_4 be out-vertices on C_{i-3} such that $R(d_1) = L(d_3) = B(c_1) = B(c_3)$ and $R(d_4) = L(d_2) = B(c_2) = B(c_4)$. Since $C_{i-2}(c_1, c_3) = \emptyset$ and $|C_{i-1}(b'_1, b)| \geq 4$, it follows from (4.3) that $C_{i-3}(d_1, d_3) \neq \emptyset$. Similarly, we can show that $C_{i-3}(d_4, d_2) \neq \emptyset$. Since $i - 2 \geq 8$ and by (4.5), $|L(d_1)| = |R(d_3)| = 3 = |L(d_4)| = |R(d_2)|$. So $|L(c')| \geq 12 \leq |R(c')|$. Thus $\ell(c') \geq (3, 12, 12)$, a contradiction. $\square$

(4.8) Lemma. *Let $i \geq 11$, let a_1 and a_2 be consecutive in-vertices on C_i such that (i) $R(a_1) = L(a_2)$ and (ii) $|C_i(a_1, a_2)| \geq 4$. Let b_1, b_2 denote the out-vertices on C_{i-1} such that $R(b_1) = L(b_2) = R(a_1)$. Then $|C_{i-1}(b_1, b_2)| = \emptyset$.*

Proof. Suppose on the contrary that $|C_{i-1}(b_1, b_2)| \neq 0$. Since $i \geq 11$ and by (4.2), $|C_{i-1}(b_1, b_2)| \geq 3$. Therefore, by (4.7), $|C_{i-1}(b_1, b_2)| = 3$. Let b_3, b, b_4 be the vertices on $C_{i-1}(b_1, b_2)$ in this clockwise order from b_1 to b_2. See Figure 15. By (4.4), $|L(b_1)| = |R(b_2)| = 3$ and $|A(b)| = |R(b_3)| = |L(b_4)| = 3$. Let b' be the neighbor of b not on C_{i-1}. By (4.6), $b' \notin C_{i-2}$ and there is a facial triangle containing b' and two consecutive vertices on C_{i-2}. Let b'_1, b'_2 denote the in-vertices on C_{i-1} such that $R(b'_1) = B(b_1)$ and $L(b'_2) = B(b_2)$. Let c_1, c_2, c_3, c_4 denote the out-vertices on C_{i-2} such that $R(c_1) = L(c_3) = B(b_1)$ and $R(c_2) = L(c_4) = B(b_2)$.

Note that $|R(b_1)| = |L(b_2)| \geq 12$. So $|B(b_1)| \leq 11 \geq |B(b_2)|$, for otherwise we would have $\ell(b_1) \geq (3, 12, 12)$ or $\ell(b_2) \geq (3, 12, 12)$. Also note that $|B(b_1)| \geq 8 \leq |B(b_2)|$.

Case 1. b'_1 is adjacent to $L(b_1)$, or b'_2 is adjacent to $R(b_2)$.

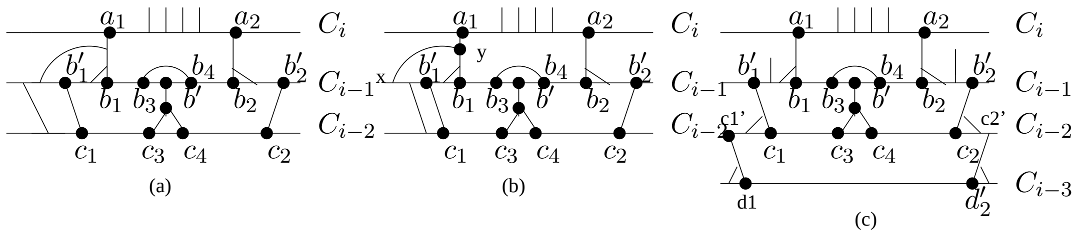

Figure 15: Proof of Lemma (4.8)

By symmetry we may assume that b'_1 is adjacent to $L(b_1)$. Then $|A(b'_1)| \geq 5$. First, assume $|A(b'_1)| = 5$. See Figure 15(a). Then $|A(b'_1) \cap C_{i-1}| = 3$ and $|L(b'_1)| \geq 5$. In fact $|L(b'_1)| = 5$ and $|L(b'_1) \cap C_{i-2}| = 2$, for otherwise $\ell(b'_1) \geq (5, 6, 8)$. So b'_1 is adjacent to c_1 and $|B(c_1)| \geq 6$. Hence $\ell(c_1) \geq (5, 6, 8)$, a contradiction. So $|A(b'_1)| \geq 6$. Then $|L(b'_1)| \leq 4$, as otherwise, we would have $\ell(b'_1) \geq (5, 6, 8)$. In fact $|L(b'_1)| = 4$ (since G is cubic), b'_1 is adjacent to c_1, and $|L(b'_1) \cap C_{i-2}| = 2$. See Figure 15(b). Note that $c_4 \notin L(b'_1)$ (because G is cubic and $|L(b'_1)| = 4$). Then $C_{i-2}(c_1, c_3) \neq \emptyset$, otherwise, $|B(c_1)| \geq 8$, and we would have $\ell(c_1) \geq (4, 8, 8)$. Suppose $C_{i-2}(c_1, c_3)$ consists of only one vertex, say c. Then $|L(c)| \geq 6$, $|R(c)| \geq 6$, and $|A(c)| \geq 9$. So $\ell(c) \geq (6, 6, 9)$, a contradiction. Hence, $|C_{i-2}(c_1, c_3)| \geq 2$. Then $|B(b_1)| \geq 10$. So $|A(b'_1)| = 6$, as otherwise, $\ell(b'_1) \geq (4, 7, 10)$. Thus $A(b'_1)$ has an edge xy such that $x \in C_{i-1}$, $y \in R(a_1) \cap A(b'_1)$, and $x, y \notin L(b_1)$. Now it is easy to see that $\ell(y) \geq (4, 6, 12)$, a contradiction.

Case 2. b'_1 is not adjacent to $L(b_1)$, and b'_2 is not adjacent to $R(b_2)$.

Then $|C_{i-1}(b'_1, b)| \geq 4$ and $|C_{i-1}(b, b'_2)| \geq 4$. Since $i - 1 \geq 10$ and by (4.5), $|L(c_1)| = |R(c_2)| = 3$. Thus $|B(b_1)| \geq 10 \leq |B(b_2)|$, $|B(c_1)| \geq 6 \leq |B(c_2)|$ and $L(c_1) \cap R(c_2) = \emptyset$.

We claim that $C_{i-2}(c_1, c_3) = \emptyset = C_{i-2}(c_4, c_2)$. For otherwise, we may assume by symmetry that $C_{i-2}(c_1, c_3) \neq \emptyset$. Then $|B(b_1)| \geq 11$. Thus $|B(b_1)| = 11$ and $C_{i-2}(c_1, c_3)$ consists of only one vertex, say c. Now $\ell(c) \geq (6, 6, 11)$, a contradiction.

Let c'_1, c'_2 denote the in-vertices on C_{i-2} such that $R(c'_1) = L(c'_2) = B(c_1)$, and let d_1, d_2 denote the out-vertices on C_{i-3} such that $R(d_1) = L(d_2) = B(c_1)$. See Figure 15(c).

Since $i-2 \geq 9$ and $|C_{i-2}(c'_1, c'_2)| \geq 4$, it follows from (4.5) that $|L(d_1)| = |R(d_2)| = 3$. Thus $|B(c_1)| \geq 12$. Since $|C_{i-1}(b'_1, b)| \geq 4$ and $C_{i-2}(c_1, c_3) = \emptyset$, it follows from (4.3) that $|C_{i-3}(d_1, d_2)| \neq 0$. By (4.2), $|C_{i-3}(d_1, d_2)| \geq 3$. Therefore, $|B(c_1)| \geq 14$. In fact $|B(c_1)| = 14$, as otherwise, $\ell(c_1) \geq (3, 10, 15)$. So c'_1 is adjacent to $L(c_1)$. Then $|A(c'_1)| \geq 5$. Since $|L(d_1)| = 3$, $\ell(c'_1) \geq (5, 5, 14)$, a contradiction. $\square$

We are now ready to prove our main lemma in this section.

(4.9) Lemma. *Let $i \geq 12$, let a_1 and a_2 be consecutive in-vertices on C_i such that $R(a_1) = L(a_2)$. Then $|C_i(a_1, a_2)| \leq 3$.*

Proof. Suppose for a contradiction that $|C_i(a_1, a_2)| \geq 4$. Then by (4.5), $|C_{i-1}(b_1, b_2)| = 0$. So by (4.1), $|L(b_1)| = |R(b_2)| = 3$. Let b_1', b_2' be the in-vertices on C_{i-1} such that $R(b_1') = L(b_2') = B(b_1)$, and let c_1, c_2 be the out-vertices on C_{i-2} such that $R(c_1) = L(c_2) = B(b_1)$. See Figure 16.

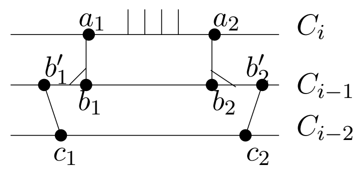

Figure 16: Proof of Lemma (4.9)

Then $|C_{i-1}(b_1', b_2')| \geq 4$. Since $i - 1 \geq 11$ and by (4.3), $|C_{i-2}(c_1, c_2)| \neq 0$. Since $i - 1 \geq 11$, we have a contradiction to (4.8). $\square$

5 Three vertices between consecutive in-vertices

In this section we show that for sufficiently large i, there are at most two vertices of C_i between any two consecutive in-vertices of C_i. As in Section 4, this is done through a series of lemmas.

(5.1) Lemma. *Let $i \geq 15$, let a_1 and a_2 be consecutive in-vertices on C_i such that (i) $R(a_1) = L(a_2)$ and (ii) $|C_i(a_1, a_2)| = 3$. Let b_1, b_2 denote the out-vertices on C_{i-1} such that $R(b_1) = L(b_2) = R(a_1)$. Then $|C_{i-1}(b_1, b_2)| \geq 2$.*

Proof. Suppose on the contrary that $|C_{i-1}(b_1, b_2)| \leq 1$. Let b_1', b_2' denote the in-vertices on C_{i-1} such that $R(b_1') = B(b_1)$ and $B(b_2) = L(b_2')$. Let c_1, c_2 be the out-vertices on C_{i-2} such that $R(c_1) = R(b_1')$ and $L(c_2) = L(b_2')$. See Figure 17.

Case 1. $|C_{i-1}(b_1, b_2)| = 1$.

Let b be the only vertex in $C_{i-1}(b_1, b_2)$. See Figure 17(a). Since G is 2-connected, b is an in-vertex. Note that $|A(b)| \geq 8$ and $|L(b)| \geq 5 \leq |R(b)|$. In fact, $|L(b)| = |R(b)| = 5$; for otherwise, $\ell(b) \geq (5, 6, 8)$. So $C_{i-1}(b_1', b_1) = \emptyset = C_{i-1}(b_2, b_2')$. Hence $c_1 \neq c_2$, $b_1' \neq b_2'$, $|B(c_1)| \geq 7$, and $|C_{i-2}(c_1, c_2)| = 1$.

Moreover, a_1 is adjacent to b_1, or a_2 is adjacent to b_2. Otherwise, $|R(b_1)| \geq 10$, and $\ell(b) \geq (5, 5, 10)$. By symmetry assume that a_1 is adjacent to b_1. Then $|L(b_1)| \geq 5$.

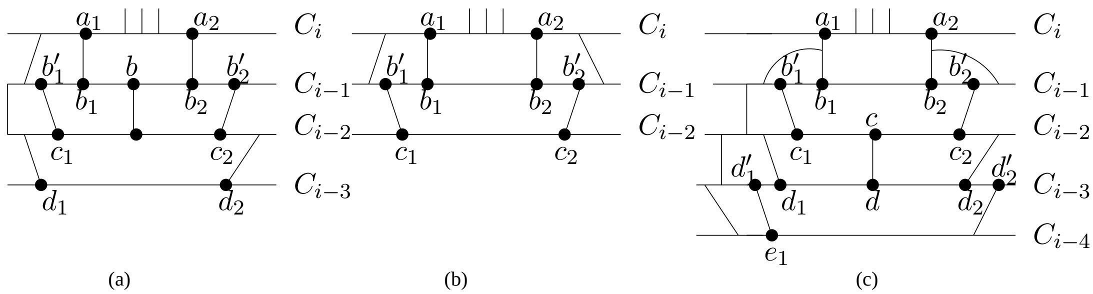

Figure 17: Proof of Lemma (5.1)

Further, $|L(b_1)| = 5$, for otherwise $\ell(b_1) \geq (5,6,8)$. So $|L(b_1) \cap C_{i-1}| = 3$. This implies that $|L(c_1)| \geq 5$.

We claim that $|L(c_1)| = 6$ and $|L(c_1) \cap C_{i-2}| = 3$. If $|L(c_1) \cap C_{i-2}| = 2$, then there are four consecutive out-vertices on C_{i-2}, contradicting (4.9). So $|L(c_1) \cap C_{i-2}| \geq 3$. So $|L(c_1)| = 6$, or else $\ell(c_1) \geq (5,7,7)$. Since $|L(b_1) \cap C_{i-1}| = 3$ and $|L(c_1) \cap C_{i-2}| \geq 3$, we have $|L(c_1) \cap C_{i-2}| = 3$.

So $|B(c_1)| = 7$, for otherwise $\ell(c_1) \geq (5,6,8)$. Now let d_1, d_2 denote the out-vertices on C_{i-3} such that $R(d_1) = L(d_2) = B(c_1)$. Since $|B(c_1)| = 7 = |R(d_1)|$, $C_{i-3}(d_1,d_2) = \emptyset$. So $|B(d_1)| \geq 6$ and $|L(d_1)| \geq 5$. Since $|L(c_1) \cap C_{i-2}| = 3$, $|L(d_1)| = 5$ implies $|B(d_1)| \geq 7$. So $\ell(d_1) \geq (5,7,7)$ or $(6,6,7)$, a contradiction.

Case 2. $|C_{i-1}(b_1,b_2)| = 0$.

By (4.9), $C_{i-1}(b_1',b_1) = \emptyset$ or $C_{i-1}(b_2,b_2') = \emptyset$. By symmetry, we may assume $C_{i-1}(b_1',b_1) = \emptyset$. So $|L(b_1)| \geq 4$, $|R(b_1)| \geq 7$, and $|B(b_1)| \geq 6$. See Figure 17(b).

Claim 1. $|L(b_1)| = 4$ and $|L(b_1) \cap C_{i-1}| = 3$.

Suppose Claim 1 is false. Then $|L(b_1)| \geq 5$. In fact $|L(b_1)| = 5$ as otherwise $\ell(b_1) \geq (6,6,7)$. Moreover, a_2 is adjacent to b_2 and a_1 is adjacent to b_1, for otherwise $\ell(b_1) \geq (5,6,8)$. So $|L(b_1) \cap C_{i-1}| \geq 3$. See Figure 17(b).

Also $|B(b_1)| = 6$, or else $\ell(b_1) \geq (5,7,7)$. So $C_{i-2}(c_1,c_2) = \emptyset$, c_1 is adjacent to b_1', c_2 is adjacent to b_2', and $C_{i-1}(b_2,b_2') = \emptyset$. Thus $|R(c_1)| = 6$ and $|B(c_1)| \geq 6$.

Hence $|L(c_1)| = 5$, as otherwise $\ell(c_1) \geq (6,6,6)$. Therefore $|L(c_1) \cap C_{i-2}| = 2$. Since $C_{i-1}(b_2,b_2') = \emptyset$ and because a_2 is adjacent to b_2, $|R(b_2)| \geq 5$. So by a symmetric argument as above, we have $|R(c_2)| \geq 5$ and $|R(c_2) \cap C_{i-2}| = 2$. Then C_{i-2} has four distinct consecutive out-vertices, contradicting (4.9).

So $|L(b_1)| = 4$. Therefore, since $|C_{i-1}(b_1',b_1)| = 0$, $|L(b_1) \cap C_{i-1}| = 3$.

Claim 2. $|R(b_2)| = 4$ and $|R(b_2) \cap C_{i-1}| = 3$.

Since $|L(b_1)| = 4$ and $|L(b_1) \cap C_{i-1}| = 3$, $|R(b_1)| \geq 8$ and $|L(c_1)| \geq 5$. See Figure 17(b) (but with $|L(b_1)| = 4$ and $|L(b_1) \cap C_{i-1}| = 3$).

Assume $C_{i-1}(b_2, b_2') \neq \emptyset$. Then $|B(b_1)| \geq 7$. In fact, $|B(b_1)| = 7$, for otherwise $\ell(b_1) \geq (4,8,8)$. So $C_{i-2}(c_1, c_2) = \emptyset$ and c_1 is adjacent to b_1'. Therefore, $|B(c_1)| \geq 6$, $|L(c_1)| \geq 5$, and $|R(c_1)| = |B(b_1)| = 7$. Moreover, if $|L(c_1)| = 5$ then $|B(c_1)| \geq 7$. Hence $\ell(c_1) \geq (5,7,7)$ or $(6,6,7)$, a contradiction.

Thus $C_{i-1}(b_2, b_2') = \emptyset$. Therefore $|R(b_2)| \geq 4$. In fact $|R(b_2)| = 4$, otherwise $\ell(b_2) \geq (5,6,8)$. So $|R(b_2) \cap C_{i-1}| = 3$.

By Claims 1 and 2, $|R(b_1)| = |L(b_2)| \geq 9$, $|L(c_1)| \geq 5$, and $|R(c_2)| \geq 5$. See Figure 17(c).

Claim 3. $|C_{i-2}(c_1, c_2)| = 1$.

Suppose $|C_{i-2}(c_1, c_2)| = 0$. Then $|B(c_2)| = |B(c_1)| \geq 6$. Hence $|R(c_2)| = |L(c_1)| = 5$, for otherwise $\ell(c_1) \geq (6,6,6)$ or $\ell(c_2) \geq (6,6,6)$. So $|L(c_1) \cap C_{i-2}| = 2 = |R(c_2) \cap C_{i-2}|$. Then there are four consecutive out-vertices on C_{i-2}, contradicting (4.9). So $C_{i-2}(c_1, c_2) \neq \emptyset$. In fact $|C_{i-2}(c_1, c_2)| = 1$, as otherwise $|B(b_1)| \geq 8$ and $\ell(b_1) \geq (4,8,9)$.

By Claim 3, let c denote the only vertex in $C_{i-2}(c_1, c_2)$. Then $|L(c)| \geq 5 \leq |R(c)|$. By the symmetry between $R(c)$ and $L(c)$, we may assume that $|L(c)| = 5$, otherwise, $\ell(c) \geq (6,6,7)$. Then $|L(c_1)| \geq 6$. In fact $|L(c_1)| = 6$, or else $\ell(c_1) \geq (5,7,7)$. Similarly, $|R(c)| \in \{5,6\}$, for otherwise $\ell(c) \geq (5,7,7)$. Let d, d_1, d_2 denote the out-vertices on C_{i-3} such that d is adjacent to c, $R(d_1) = B(c_1)$, and $L(d_2) = B(c_2)$. Because $|L(c)| = 5$, $C_{i-3}(d_1, d) = \emptyset$.

We claim that $C_{i-3}(d, d_2) = \emptyset$. For otherwise, $|B(c_2)| = |R(c)| = 6$. So $|R(c_2)| \geq 6$, and $\ell(c_2) \geq (6,6,7)$, a contradiction.

Hence d_1, d, d_2 are consecutive out-vertices on C_{i-3}. Let d_1', d_2' denote the in-vertices on C_{i-3} such that $R(d_1') = B(d_1) = B(d_2) = L(d_2')$. Let e_1 be the out-vertex on C_{i-4} such that $R(e_1) = B(d_1)$. Then, since $i - 3 \geq 12$ and by (4.9), $C_{i-3}(d_1, d_2) = \emptyset$. So $|B(d_1)| = |R(d_1')| \geq 7$ and $|A(d_1')| \geq 6$. Hence $|L(d_1')| = 5$, as otherwise $\ell(d_1') \geq (6,6,7)$. Therefore d_1' is adjacent to e_1, $|L(e_1)| \geq 5$, $|B(e_1)| \geq 6$. Moreover, if $|L(e_1)| = 5$ then either $|B(e_1)| \geq 7$ or $|R(e_1)| \geq 8$. So $\ell(e_1) \geq (5,7,7)$ or $(5,6,8)$, a contradiction. $\square$

(5.2) Lemma. *Let $i \geq 15$, let a_1 and a_2 be consecutive in-vertices on C_i such that (i) $R(a_1) = L(a_2)$ and (ii) $|C_i(a_1, a_2)| = 3$. Let b_1, b_2 denote the out-vertices on C_{i-1} such that $R(b_1) = L(b_2) = R(a_1)$. Suppose $|C_i(a_1, a_2)| = 3$. Then $|C_{i-1}(b_1, b_2)| \geq 3$.*

Proof. Suppose on the contrary $|C_{i-1}(b_1, b_2)| \leq 2$. By (5.1), $C_{i-1}(b_1, b_2)$ has exactly two vertices, say b_3 and b_4. See Figure 18. Without loss of generality, assume that $b_3 \in C_{i-1}(b_1, b_4)$. Since G is 2-connected and by (3.4), both b_3 and b_4 are in-vertices. So $|R(b_1)| = |L(b_2)| \geq 9$. Note that $|L(b_3)| = |B(b_1)| \geq 5 \leq |B(b_2)| = |R(b_4)|$. So $|R(b_3)| = |L(b_4)| \geq 5$, for otherwise $\ell(b_3) \geq (5,6,9)$.

Case 1. $|R(b_3)| = |L(b_4)| = 5$.

Then $|B(b_1)| = |B(b_2)| = 5$, as otherwise $\ell(b_1) \geq (5,6,9)$ or $\ell(b_2) \geq (5,6,9)$. Therefore $|R(b_3) \cap C_{i-2}| = 3$. Let c be the only vertex in $R(b_3) \cap C_{i-2}$. Then c is an in-vertex on C_{i-2} and $|L(c)| \geq 6 \leq |R(c)|$. See Figure 18(a). Note that c has a neighbor, say d, on C_{i-3}; for otherwise, $\ell(c) \geq (5,7,7)$. Note that $|L(d)| \geq 6 \leq |R(d)|$ and $|B(d)| \geq 5$. Further, if $|B(d)| = 5$ then $|L(d)| \geq 7 \leq |R(d)|$. So $\ell(d) \geq (5,7,7)$ or $(6,6,6)$, a contradiction.

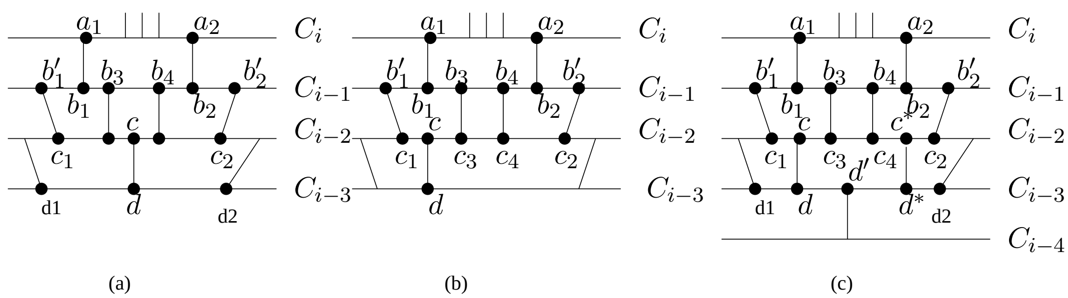

Figure 18: Proof of Lemma (5.2)

Let $c_3, c_4 \in C_{i-2}$ be the neighbors of b_3, b_4, respectively. Let c_1, c_2 be the out-vertices on C_{i-2} such that $R(c_1) = B(b_1)$ and $L(c_2) = B(b_2)$. See Figure 18(b).

We claim that $|C_{i-2}(c_1, c_3)| \leq 1 \geq |C_{i-2}(c_4, c_2)|$. For otherwise, we may assume by symmetry that $|C_{i-2}(c_1, c_3)| \geq 2$. Then $|L(b_3)| \geq 7$. In fact, $|L(b_3)| = 7$, or else $\ell(b_3) \geq (4,8,9)$. Therefore $|L(b_1)| \geq 5$, and $\ell(b_1) \geq (5,7,9)$, a contradiction.

We further claim that $C_{i-2}(c_1, c_3) \cup C_{i-2}(c_4, c_2)$ has an in-vertex on C_{i-2}. For otherwise, $C_{i-2}(c_1, c_3) \cup C_{i-2}(c_4, c_2) = \emptyset$. Since G is cubic and 2-connected, $c_1 \neq c_2$ and $|C_{i-2}(c_2, c_1)| \neq \emptyset$. Therefore, C_{i-2} has four consecutive out-vertices, contradicting (4.9).

By symmetry we may assume that $C_{i-2}(c_1, c_3)$ has an in-vertex. So let c denote the only vertex in $C_{i-2}(c_1, c_3)$. See Figure 18(b). Then $|A(c)| \geq 6$ and $|L(c)| \geq 5$. Hence c is adjacent to some vertex d on C_{i-3}, for otherwise $\ell(c) \geq (6,6,6)$.

Suppose $C_{i-2}(c_4, c_2) = \emptyset$. Then $|R(c)| \geq 7$. So $|L(c)| = 5$ and $|R(c)| = 7$; otherwise, we would have $\ell(c) \geq (5,6,8)$ or $(6,6,7)$. Then $|L(d)| = 5$, $|R(d)| = 7$, and $|B(d)| \geq 7$. This implies $\ell(d) \geq (5,7,7)$, a contradiction.

So $|C_{i-2}(c_4, c_2)| = 1$. Let c^* denote the only vertex in $C_{i-2}(c_4, c_2)$. See Figure 18(c). Then c^* is adjacent to some vertex d^* in C_{i-3}, for otherwise $\ell(c^*) \geq (6,6,6)$. Note that $|R(c)| = |L(c^*)| \geq 6$. Then $|L(c)| = 5 = |R(c^*)|$, otherwise $\ell(c) \geq (6,6,6)$ or $\ell(c^*) \geq (6,6,6)$. So $|R(c)| = |L(c^*)| \leq 7$, or else $\ell(c) \geq (5,6,8)$. Since $i - 3 \geq 12$ and by (4.9), $C_{i-3}(d, d^*) \neq \emptyset$. So $|C_{i-3}(d, d^*)| = 1$ (because $|R(c)| \leq 7$). Let d' denote the only vertex in $C_{i-3}(d, d^*)$. Then $\ell(d') \geq (6,6,7)$, a contradiction. □

(5.3) Lemma. *Let $i \geq 17$, let a_1 and a_2 be consecutive in-vertices on C_i such that (i) $R(a_1) = L(a_2)$ and (ii) $|C_i(a_1, a_2)| = 3$. Let b_1, b_2 denote the out-vertices on C_{i-1} such that $R(b_1) = L(b_2) = R(a_1)$, and let b_3, b_4 be the vertice in $C_{i-1}(b_1, b_2)$ such that $C_{i-1}(b_1, b_3) = \emptyset = C_{i-1}(b_4, b_2)$. Then both b_3 and b_4 are in-vertices.*

Proof. Suppose this lemma is false. By symmetry assume that b_3 is an out-vertex. Then $|B(b_1)| = |B(b_3)| \geq 6$. By (5.2), $|C_{i-1}(b_1, b_2)| \geq 3$, and so, $|R(b_1)| = |L(b_2)| \geq 9$. Hence $|L(b_1)| \leq 4$, as otherwise, $\ell(b_1) \geq (5, 6, 9)$. Let b_1', b be the in-vertices on C_{i-1} such that $R(b_1') = L(b) = B(b_1)$, and let c_1, c be the out-vertices on C_{i-2} such that $R(c_1) = L(c) = B(b_1)$. See Figure 19. Note that $b \in C_{i-1}(b_3, b_4)$ as otherwise G has a 2-cut contained in $V(C_{i-1})$, contradicting (3.4). Since $|L(b_1)| \leq 4$, we have two cases to consider.

Case 1. $|L(b_1)| = 3$.

Then $|R(b_1)| \geq 10$. Since $i - 1 \geq 16$ and $|C_{i-1}(b_1', b)| \geq 4$, it follows from (5.2) that $|C_{i-2}(c_1, c_3)| \geq 3$, and so, $|B(b_1)| \geq 9$. Hence $|R(b_3)| = 3$, as otherwise $\ell(b_3) \geq (4, 9, 10)$. By (4.9), b_1' is adjacent to $L(b_1)$ and $C_{i-1}(b_3, b) = \emptyset$. As a consequence, $|A(b_1')| \geq 5$ and $R(b_3) \subset C_{i-1}$. See Figure 19(a).

We claim that b is not adjacent to c. For otherwise, $|L(c)| \geq 9$ and $|R(c)| \geq 5 \leq |B(c)|$. Further, if $|B(c)| = 5$ then $|R(c)| \geq 6$. Thus $\ell(c) \geq (5, 6, 9)$, a contradiction.

Hence, $|L(c)| = |R(b_1')| \geq 10$. Then $|L(b_1')| = 4$, or else $\ell(b_1') \geq (5, 5, 10)$. Therefore $|A(b_1')| \geq 6$. In fact, $|A(b_1')| = 6$ as otherwise, $\ell(b_1') \geq (4, 7, 10)$. So $R(a_1) \cap A(b_1') \neq \emptyset$. Let y be the vertex in $R(a_1) \cap A(b_1')$ such that the clockwise path in $R(a_1)$ from y to a_1 is shortest. So $|R(a_1)| \geq 11$. Let C denote the facial cycle of G containing y such that $C \neq R(a_1)$ and $C \neq A(b_1')$. Note that $|C| \geq 4$. If $|C| \geq 5$, then $\ell(y) \geq (5, 6, 11)$. If $|C| = 4$, then $|R(a_1)| \geq 12$ and $\ell(y) \geq (4, 6, 12)$.

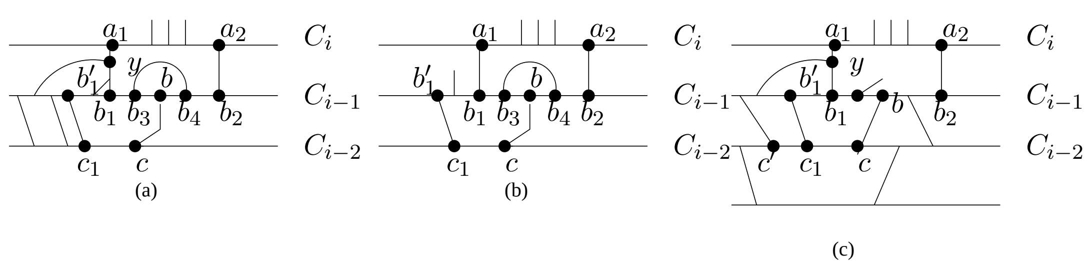

Figure 19: Proof of Lemma (5.3)

Case 2. $|L(b_1)| = 4$.

First, assume that $|L(b_1) \cap C_{i-1}| = 2$. See Figure 19(b). Then $|C_{i-1}(b_1', b)| \geq 3$. Since $i - 1 \geq 16$ and by (5.2), $|B(b_1)| \geq 9$. Thus $\ell(b_1) \geq (4, 9, 9)$, a contradiction.

So $|L(b_1) \cap C_{i-1}| = 3$. See Figure 19(c). Then $|R(b_1)| \geq 10$. Thus $|B(b_1)| = 6$, as otherwise $\ell(b_1) \geq (4, 7, 10)$. So b is adjacent to c, b'_1 is adjacent to c_1, $|C_{i-1}(b'_1, b)| = 2$, and $C_{i-2}(c_1, c) = \emptyset$. Now $|B(c_1)| \geq 6$. Because $|L(b_1) \cap C_{i-1}| = 3$ and $|L(b_1)| = 4$, $|L(c_1)| \geq 5$. In fact, $|L(c_1)| = 5$ and $|L(c_1) \cap C_{i-2}| = 2$, as otherwise $\ell(c_1) \geq (6, 6, 6)$. Let c' be the vertex on C_{i-2} such that $R(c') = L(c_1)$. Then c', c_1 and c are three consecutive out-vertices on C_{i-2}. Since $i - 2 \geq 15$ and by (5.1), $|B(c_1)| \geq 9$. Thus $\ell(c_1) \geq (5, 6, 9)$, a contradiction. $\square$

(5.4) Lemma. *Let $i \geq 17$, let a_1 and a_2 be consecutive in-vertices on C_i such that $R(a_1) = L(a_2)$. Then $|C_i(a_1, a_2)| \leq 2$.*

Proof. Suppose this lemma is false. Then by (4.9), $|C_i(a_1, a_2)| = 3$. Let b_1, b_2 denote the out-vertices on C_{i-1} such that $R(b_1) = L(b_2) = R(a_1)$. By (5.2), $|C_{i-1}(b_1, b_2)| \geq 3$. Let b_3, b_4 be vertices in $C_{i-1}(b_1, b_2)$ such that $C_{i-1}(b_1, b_3) = \emptyset = C_{i-1}(b_4, b_2)$. By (5.3), both b_3 and b_4 are in-vertices. See Figure 20. So $|R(b_1)| = |L(b_2)| \geq 10$ and $|B(b_1)| = |L(b_3)| \geq 5 \leq |R(b_4)| = |B(b_2)|$.

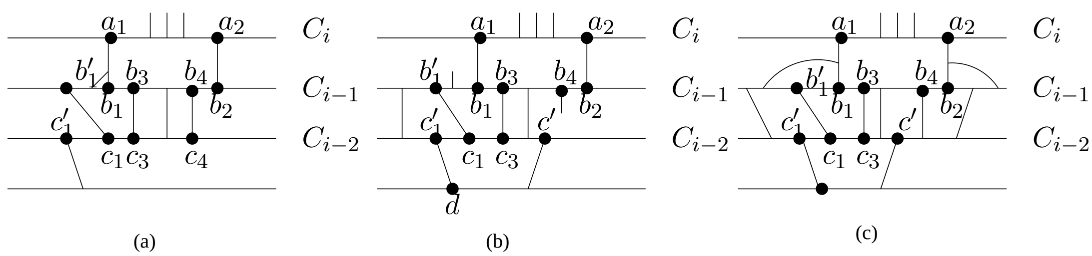

Figure 20: Proof of Lemma (5.4)

Then $|R(b_3)| = 4 = |L(b_4)|$, for otherwise $\ell(b_3) \geq (5, 5, 10)$ or $\ell(b_4) \geq (5, 5, 10)$. Also $|L(b_1)| \leq 4 \geq |R(b_2)|$, otherwise, $\ell(b_1) \geq (5, 5, 10)$ or $\ell(b_2) \geq (5, 5, 10)$.

Let b'_1 be the in-vertex on C_{i-1} such that $R(b'_1) = B(b_1)$. Let c_1, c_3, c_4 be out-vertice on C_{i-2} such that $R(c_1) = L(c_3) = B(b_1)$ and $R(c_4) = B(b_2)$. Since $|R(b_3)| = |L(b_4)| = 4$, c_3 is adjacent to b_3 and c_4 is adjacent to b_4. Let c'_1, c^* be the in-vertices on C_{i-2} such that $R(c'_1) = L(c^*) = B(c_1)$.

We claim that $|L(b_1)| = 4 = |R(b_2)|$. For otherwise, by symmetry we may assume that $|L(b_1)| = 3$. See Figure 20(a). Then $|R(b_1)| \geq 11$ and $|B(b_1)| \geq 6$. Indeed, $|R(b_1)| = 11$ and $|B(b_1)| = 6$, as otherwise $\ell(b_3) \geq (4, 7, 11)$ or $(4, 6, 12)$. So $|C_{i-1}(b_1, b_2)| = 3$. Since $|R(b_3)| = |L(b_4)| = 4$, $C_{i-2}[c_3, c_4]$ consists of three consecutive out-vertices on C_{i-2}. Since $i - 2 \geq 15$ and by (4.9), c_1 and c_3 cannot be adjacent. Thus $|B(b_1)| \geq 7$, a contradiction.

We further claim that $|L(b_1) \cap C_{i-1}| = 3 = |R(b_2) \cap C_{i-1}|$. For otherwise, by symmetry we may assume $|L(b_1) \cap C_{i-1}| = 2$. See Figure 20(b). Then $|B(b_1)| \geq 6$. Indeed

$|B(b_1)| = 6$, otherwise $\ell(b_1) \geq (4, 7, 10)$. Thus b_3 is adjacent to c_3, b_1' is adjacent to c_1, and $C_{i-2}(c_1, c_3) = \emptyset$. Thus $|C_{i-2}(c_1', c^*)| \geq 3$ (because $|R(b_3)| = 4$). By (4.9), $C_{i-2}(c_1', c_1) = \emptyset$, and so $|L(c_1)| \geq 5$. By (5.2), $|B(c_1)| \geq 9$. Since $|R(c_1)| = |B(b_1)| = 6$, $\ell(c_1) \geq (5, 6, 9)$, a contradiction.

So $|R(b_1)| = |L(b_2)| \geq 12$. See Figure 20(c). Then $|L(b_3)| = 5$, otherwise $\ell(b_3) \geq (4, 6, 12)$. Thus b_1' is adjacent to c_1, and $C_{i-2}(c_1, c_3) = \emptyset$. Again, $|C_{i-2}(c_1', c^*)| \geq 3$ (because $|R(b_3)| = 4$). By (4.9), $C_{i-2}(c_1', c_1) = \emptyset$, and so $|L(c_1)| \geq 6$. By (5.2), $|B(c_1)| \geq 9$. Therefore $\ell(c_1) \geq (5, 6, 9)$, a contradiction. $\square$

6 Two vertices between consecutive in-vertices

In this section, we show that for sufficiently large i, there is at most one vertex between any two consecutive in-vertices on C_i. Again this is done through a series of lemmas.

(6.1) Lemma. *Let $i \geq 20$, let a_1 and a_2 be consecutive in-vertices on C_i such that (i) $R(a_1) = L(a_2)$ and (ii) $|C_i(a_1, a_2)| = 2$. Let b_1, b_2 denote the out-vertices on C_{i-1} such that $R(b_1) = L(b_2) = R(a_1)$. Then $|C_{i-1}(b_1, b_2)| \neq 0$.*

Proof. Suppose $|C_{i-1}(b_1, b_2)| = 0$. Then $|B(b_1)| = |B(b_2)| \geq 6$. Let b_1', b_2' be the in-vertices on C_{i-1} such that $R(b_1') = B(b_1) = L(b_2')$, and let c_1, c_2 denote the out-vertices on C_{i-2} such that $R(c_1) = L(c_2) = B(b_1)$. Because $i-1 \geq 19$ and by (5.4), $|C_{i-1}(b_1', b_2')| = 2$.

Case 1. $|C_{i-2}(c_1, c_2)| = 0$.

Let c_1', c_2' denote the in-vertices on C_{i-2} such that $R(c_1') = B(c_1) = L(c_2')$. See Figure 21(a). Since $i - 2 \geq 18$ and by (5.4), $|C_{i-2}(c_1', c_2')| = 2$. Thus $C_{i-2}(c_1', c_1) = \emptyset = C_{i-2}(c_2, c_2')$. Note that $|B(c_2)| \geq 6$. Then $|L(c_1)| = |R(c_2)| = 5$, otherwise $\ell(c_1) \geq (6, 6, 6)$ or $\ell(c_2) \geq (6, 6, 6)$. This forces $|L(b_1)| \geq 5 \leq |R(b_2)|$. Further $|L(b_1)| = 5 = |R(b_2)|$, otherwise $\ell(b_1) \geq (6, 6, 6)$ or $\ell(b_2) \geq (6, 6, 6)$. Therefore $|R(b_1)| = |L(b_2)| \geq 8$, and so $\ell(b_1) \geq (5, 6, 8)$, a contradiction.

Case 2. $|C_{i-2}(c_1, c_2)| = 1$.

Let c be the only vertex in $C_{i-2}(c_1, c_2)$. See Figure 21(b). Then $|A(c)| \geq 7$ and $|L(c)| \geq 5 \leq |R(c)|$. If $|L(c) \cap C_{i-3}| = 2 = |R(c) \cap C_{i-3}|$, then C_{i-3} has three consecutive out-vertices, contradicting (5.4) (because $i - 3 \geq 17$). So by symmetry assume that $|R(c) \cap C_{i-3}| \geq 3$. Hence $|R(c)| \geq 6$. Indeed $|R(c)| = 6$ and $|L(c)| = 5$, otherwise $\ell(c) \geq (5, 7, 7)$ or $(6, 6, 7)$. So $|R(c) \cap C_{i-2}| = 3$, and hence, $|R(c_2)| \geq 5$. In fact, $|R(c_2)| = 5$, for otherwise $\ell(c_2) \geq (6, 6, 7)$. Therefore $|R(b_2)| \geq 5$. Further, if $|R(b_2)| = 5$ then $|L(b_2)| \geq 7$. So $\ell(b_2) \geq (5, 7, 7)$ or $(6, 6, 7)$, a contradicton.

Case 3. $|C_{i-2}(c_1, c_2)| \geq 2$.

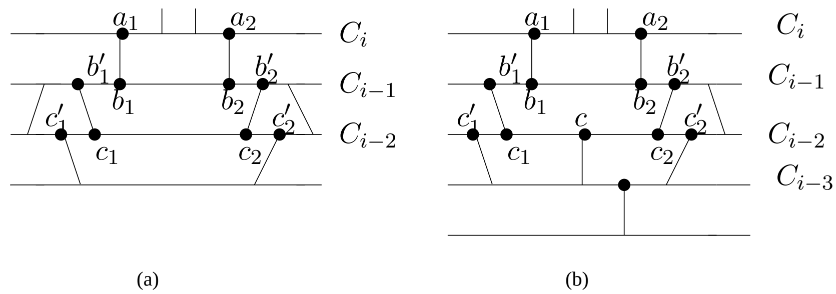

Figure 21: Proof of Lemma (6.1)

Then $|B(b_1)| = |B(b_2)| \geq 8$. Hence $|L(b_1)| = |R(b_2)| = 4$; otherwise $\ell(b_1) \geq (5, 6, 8)$ or $\ell(b_2) \geq (5, 6, 8)$, a contradiction. Since $i-1 \geq 19$ and by (5.4), $|L(b_1) \cap C_{i-1}|$=3=$|R(b_2) \cap C_{i-1}|$. Thus $\ell(b_1) \geq (4, 8, 8)$, a contradiction. $\square$

(6.2) Lemma. *Let $i \geq 21$, let a_1 and a_2 be consecutive in-vertices on C_i such that (i) $R(a_1) = L(a_2)$ and (ii) $|C_i(a_1, a_2)| = 2$. Let b_1, b_2 denote the out-vertices on C_{i-1} such that $R(b_1) = L(b_2) = R(a_1)$. Then $|C_{i-1}(b_1, b_2)| \neq 1$.*

Proof. Suppose $|C_{i-1}(b_1, b_2)| = 1$. Then $|R(b_1)| = |L(b_2)| \geq 7$. Let b be the only vertex in $C_{i-1}(b_1, b_2)$. Since G is 2-connected, b is an in-vertex. See Figure 22. Note that $|L(b)| = |B(b_1)| \geq 5 \leq |B(b_2)| = |R(b)|$.

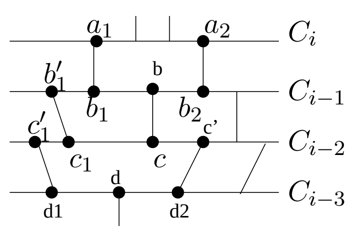

Figure 22: Proof of Lemma (6.2)

Now $|L(b)| = 5$ or $|R(b)| = 5$, as otherwise $\ell(b) \geq (6, 6, 7)$. By symmetry assume that $|L(b)| = 5$. Then b is adjacent to a vertex on C_{i-3}, say c. Let c_1 denote the out-vertex on C_{i-2} such that $R(c_1) = B(b_1)$, and let b_1' be the in-vertex on C_{i-1} such that $R(b_1') = B(b_1)$. Since $|L(b)| = 5$, $C_{i-2}(c_1, c) = \emptyset$, b_1' is adjacent to c_1, and $C_{i-1}(b_1', b_1) = \emptyset$.

Let c_1', c' denote the in-vertices on C_{i-2} such that $R(c_1') = L(c') = B(c_1)$. Since $i - 2 \geq 19$ and by (5.4), $|C_{i-2}(c_1', c')| = 2$ and $C_{i-2}(c, c') = \emptyset$. Thus $|A(c')| \geq 6$. Indeed,

$|A(c')| = 6$, as otherwise $\ell(b) \geq (5,7,7)$. Now $|R(c')| = 5$ and $|L(c')| \leq 7$, for otherwise $\ell(c') \geq (5,6,8)$ or $(6,6,7)$. Let d_1, d_2 denote the out-vertices on C_{i-3} such that $R(d_1) = L(d_2) = B(c_1)$. Since $|R(c')| = 5$, c' is adjacent to d_2 and $|R(c') \cap C_{i-3}| = 3$. Further we have $|B(d_2)| \geq 6$. Since $i-3 \geq 18$ and by (5.4), $C_{i-3}(d_1, d_2) \neq \emptyset$. Since $|L(c')| \leq 7$, we may assume d is the only vertex in $C_{i-3}(d_1, d_2)$. Then $|R(d)| = |B(d_2)| \geq 6$, and $|L(d)| = |B(d_1)| \geq 5$. In fact, $|L(d)| = 5$ and $|R(d)| = 6$, for otherwise $\ell(d) \geq (6,6,7)$ or $(5,7,7)$. Thus $|L(d) \cap C_{i-4}| = 2 = |R(d) \cap C_{i-4}|$. Hence C_{i-4} has three consecutive out-vertices. Since $i-4 \geq 17$, this is a contradiction to (5.4). □

(6.3) Lemma. *Let $i \geq 24$, let a_1 and a_2 be consecutive in-vertices on C_i such that (i) $R(a_1) = L(a_2)$ and (ii) $|C_i(a_1, a_2)| = 2$. Let b_1, b_2 denote the out-vertices on C_{i-1} such that $R(b_1) = L(b_2) = R(a_1)$. Suppose $|C_{i-1}(b_1, b_2)| \geq 2$. Let b_3, b_4 be the vertices in $C_{i-1}(b_1, b_2)$ such that $C_{i-1}(b_1, b_3) = \emptyset = C_{i-1}(b_4, b_2)$. Then both b_3 and b_4 are out-vertices.*

Proof. Suppose this is false. By symmetry, assume that b_3 is an in-vertex. Since $|C_{i-1}(b_1, b_2)| \geq 2$, $|A(b_3)| = |R(b_1)| = |L(b_2)| \geq 8$. Hence b_3 is adajcent to C_{i-2}; otherwise, $|L(b_3)| \geq 6$ and $|R(b_3)| \geq 5$, and so, $\ell(b_3) \geq (5,6,8)$, a contradiction.

Let c denote the neighbor of b_3 on C_{i-2}. We consider two cases.

Case 1. $|R(b_3)| = 4$.

Let c_2 be the out-vertex on C_{i-2} such that $L(c_2) = R(c)$. See Figure 23(a). Since $|R(b_3)| = 4$, $C_{i-2}(c, c_2) = \emptyset$. Let c_1', c_2' denote the in-vertices on C_{i-2} such that $R(c_1') = B(c) = L(c_2')$, and let d_1, d_2 be the out-vertices on C_{i-3} such that $R(d_1) = B(c) = L(d_2)$. Since $i-2 \geq 22$ and by (6.1) and (6.2), $|C_{i-3}(d_1, d_2)| \geq 3$. Hence $|B(c)| = |B(c_2)| \geq 8$. By (5.4), $C_{i-2}(c_1', c) = \emptyset = C_{i-2}(c_2, c_2')$. So $|B(b_1)| = |A(c_1')| \geq 6$ and $|R(c_1')| = |B(c)| \geq 8$. Hence $|L(c_1')| = 4$, as otherwise $\ell(c_1') \geq (5,6,8)$. Therefore c_1' is adjacent to d_1. Note $|L(d_1)| = |L(c_1')| = 4$ and $|R(d_1)| = |R(c_1')| \geq 8$. Since $i-3 \geq 21$ and since $L(d_1) \cap C_{i-3}$ consists of two consecutive out-vertices on C_{i-3}, it follows from (6.1) and (6.2) that $|B(d_1)| \geq 8$. Thus $\ell(d_1) \geq (4,8,8)$, a contradiction.

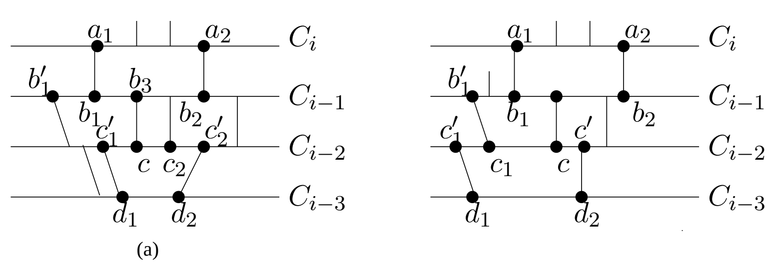

Figure 23: Proof of Lemma (6.3)

Case 2. $|R(b_3)| \geq 5$.

In this case, $|L(b_3)| \geq 5 \leq |R(b_3)|$. So $|L(b_3)| = |R(b_3)| = 5$, for otherwise, $\ell(b_3) \geq (5,6,8)$. Also $|R(b_1)| \leq 9$, or else $\ell(b_3) \geq (5,5,10)$.

Let b_1' be the in-vertex on C_{i-1} such that $R(b_1') = B(b_1)$, and let c_1, c be the out-vertices on C_{i-2} such that $R(c_1) = L(c) = B(b_1)$. See Figure 23(b). Since $|L(b_3)| = 5$, $C_{i-2}(c_1, c) = \emptyset$, b_3 is adjacent to c, and b_1' is adjacent to c_1.

Let c_1', c' be the in-vertices on C_{i-2} such that $R(c_1') = L(c') = B(c_1)$. Since $i-2 \geq 22$ and by (5.4), $C_{i-2}(c_1', c_1) = \emptyset = C_{i-2}(c, c')$. Since $i-3 \geq 21$ and by (6.1) and (6.2), $|C_{i-3}(d_1, d_2)| \geq 2$. Hence $|B(c_1)| = |B(c)| = |L(c')| \geq 8$.

Let d_1, d_2 denote the out-vertices on C_{i-3} such that $R(d_1) = L(d_2) = B(c_1)$. Since $|R(b_3)| = 5$, b_3 is adjacent to c, and $C_{i-2}(c, c') = \emptyset$, we have $|R(c) \cap C_{i-2}| = 3$. Thus c' is adjacent to d_2, for otherwise, $\ell(c') \geq (5,6,8)$. So $|R(d_2)| \geq 5$. Further, if $|R(d_2)| = 5$, then $|B(d_2)| \geq 6$. Therefore, $\ell(d_2) \geq (5,6,8)$, a contradiction. □

(6.4) Lemma. *Let $i \geq 24$, let a_1 and a_2 be consecutive in-vertices on C_i such that $R(a_1) = L(a_2)$. Then $|C_i(a_1, a_2)| \leq 1$.*

Proof. Suppose $|C_i(a_1, a_2)| \geq 2$. Then by (5.4), $|C_i(a_1, a_2)| = 2$. Let b_1, b_2 denote the out-vertices on C_{i-1} such that $R(b_1) = L(b_2) = R(a_1)$. See Figure 24. By (6.1) and (6.2), $|C_{i-1}(b_1, b_2)| \geq 2$ and $|R(b_1)| = |L(b_2)| \geq 8$. Let b_3, b_4 be the vertices in $C_{i-1}(b_1, b_2)$ such that $C_{i-1}(b_1, b_3) = \emptyset = C_{i-1}(b_4, b_2)$. By (6.3), both b_3 and b_4 are out-vertices. Let c_1, c be the out-vertices on C_{i-2} such that $R(c_1) = B(b_1) = L(c)$. Since $i - 1 \geq 23$, it follows from (6.1) and (6.2) that $|C_{i-2}(c_1, c)| = 0$. Hence $|B(b_1)| = |B(b_3)| \geq 8$.

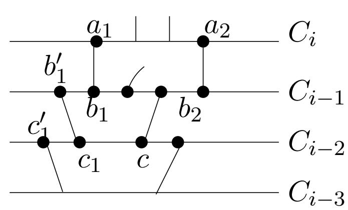

Figure 24: Proof of Lemma (6.4)

By (5.4), $C_{i-1}(b_1', b_1) = \emptyset$. Thus $|L(b_1)| \geq 4$. Therefore $\ell(b_1) \geq (4,8,8)$, a contradiction. □

7 The proof of the main result

In this section, we complete the proof of (1.2). Let G be a positively curved cubic plane graph such that G is locally finite and every face of G is boudned by a cycle.

Assume that G is infinite. Then G has a nice sequence $(C_0, C_1, \ldots)$. By (2.4), we may assume that G is nicely embedded with respect to $(C_0, C_1, \ldots)$.

(7.1) Lemma. *For $i \geq 26$, $|C_{i-1}| > |C_i|$.*

Proof. Let $a_1, a_2, \ldots, a_n$ denote the in-vertices on C_i and occur on C_i in that clockwise order. For each $j \in \{1, \ldots, n\}$, let b_j, b'_j be the out-vertices on C_{i-1} such that $R(b_j) = R(a_j)$ and $L(b'_j) = L(a_j)$. See Figure 25. For convenience, let $b_{n+1} = b_1$, $b'_{n+1} = b'_1$, and $a_{n+1} = a_1$.

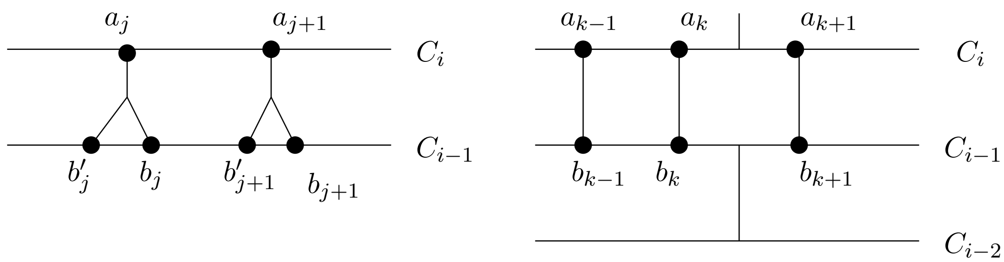

Figure 25: Proof of Lemma (7.1)

To prove the lemma, it suffices to show that for each $j \in \{1, \ldots, n\}$, $|C_{i-1}(b_j, b'_{j+1})| \geq |C_i(a_j, a_{j+1})|$, and there is some k such that $|C_{i-1}(b_k, b'_{k+1})| > |C_i(a_k, a_{k+1})|$.

By (6.4), $|C_i(a_j, a_{j+1})| \leq 1$.

If $|C_i(a_j, a_{j+1})| = 0$, then clearly $|C_{i-1}(b_j, b'_{j+1})| \geq |C_i(a_j, a_{j+1})|$. Now assume that $|C_i(a_j, a_{j+1})| = 1$. Since $i - 1 \geq 24$, it follows from (6.4) that $|C_{i-1}(b_j, b'_{j+1})| \geq 1$. So $|C_{i-1}(b_j, b'_{j+1})| \geq |C_i(a_j, a_{j+1})|$.

Hence we may assume that $b_j = b'_j$ for all $j \in \{1, \ldots, n\}$, for otherwise, we have $|C_{i-1}| > |C_i|$. Because G is connected and $(C_0, C_1, \ldots)$ is an infinite sequence, there is some $k \in \{1, \ldots, n\}$ such that $|C_i(a_k, a_{k+1})| = 1$ for some k. Now $|B(b_k)| \geq 5$.

If $|B(b_k)| = 5$, then $|B(b_k) \cap C_{i-2}| = 2$, contradicting (6.4) (because $i - 2 \geq 24$). So $|B(b_k)| \geq 6$. Hence $|L(b_k)| \leq 5$, or else $\ell(b_k) \geq (6, 6, 6)$. In fact, $|L(b_k) \cap C_{i-1}| \geq 3$ by (6.4) (since $i - 1 \geq 25$). So $|C_{i-1}(b_{k-1}, b'_k)| = |C_{i-1}(b_{k-1}, b_k)| > |C_i(a_{k-1}, a_k)|$. Therefore, $|C_{i-1}| > |C_i|$. □

It is now easy to see that (1.2) holds because of the contradiction caused by (7.1) and the infinite sequence $(C_0, C_1, \ldots)$.

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