Long Cycles in 3-Connected Graphs in Orientable Surfaces*

Laura Sheppardson

Xingxing Yu [†]

School of Mathematics

Georgia Institute of Technology

Atlanta, Georgia 30332

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Abstract

In this paper we apply a cutting theorem of Thomassen to show that there is a function $f : \mathbf{N} \rightarrow \mathbf{N}$ such that if G is a 3-connected graph which can be embedded in the orientable surface of genus g with face-width at least $f(g)$, then G contains a cycle of length at least $c(n^{\log_3 2})$, where c is a constant not dependent on g .

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1 Introduction

Whitney [11] proved in 1931 that every 4-connected triangulation of the plane contains a Hamilton cycle. In 1956, Tutte [9] showed that in fact all 4-connected planar graphs are Hamiltonian. Thomas and Yu [7] have extended this to show that 4-connected projective planar graphs are also Hamiltonian. Archdeacon, Hartsfield, and Little [1] showed that this result does not extend to graphs embedded in other surfaces. They proved that for each n , there is a triangulation of an orientable surface which is n -connected and in which every spanning tree contains a vertex of degree at least n . (Note that a Hamilton path is a spanning tree with maximum degree two.) However, in any 5-connected triangulation of surfaces with large face-width (see definition below), Yu [12] proved that there is a Hamiltonian path. (This is still open for non-triangulations.)

We also lose Hamiltonicity in planar graphs if we relax the connectivity condition. Moon and Moser [6] showed that there exist 3-connected planar triangulations on n vertices whose longest cycles are of length $O(n^{\log_3 2})$. They conjectured, however, that any 3-connected planar graph on n vertices must contain a cycle of length at least $O(n^{\log_3 2})$. Chen and Yu [3] recently proved this conjecture, not only for planar graphs, but also for graphs embeddable in the projective plane, or the torus, or the Klein Bottle.

This result can be extended to graphs on other surfaces under certain conditions. The *face-width* (or *representativity*) of a graph G embedded in a surface Σ is defined to be the number $\min\{|\Gamma \cap V(G)| : \Gamma \text{ is a homotopically nontrivial closed curve in } \Sigma, \text{ and } \Gamma \cap G \subset V(G)\}$. We will show that if a graph on n vertices is embeddable in some fixed orientable surface with sufficiently large face-width, then it contains a cycle of length at least $cn^{\log_3 2}$, where $c = \frac{1}{2}(\frac{1}{4})^{\log_3 2}$.

The techniques used here are similar to those in [3], combined with a cutting technique of Thomassen in [8]. In this paper, we first review the definitions to be used in graph cutting and the necessary lemmas. Our main theorem is stated

following these definitions. We then define circuit graphs and annulus graphs, special classes of 2-connected planar graphs. We show the existence of “heavy” paths in vertex-weighted circuit or annulus graphs. Finally, the previous results are combined to prove the main theorem. The theorem addresses “heavy” cycles in vertex-weighted graphs, which easily extends to long cycles by applying a uniform weighting.

Graph Cutting If H is a graph embedded in an orientable surface Σ , and C is a cycle in H , we talk about cutting H and Σ along C . To cut along C , where $C = x_1x_2 \dots x_kx_1$ is an orientation preserving cycle, we add cycle $C' = x'_1x'_2 \dots x'_kx'_1$ immediately to the left of C . Where edge vx_i is incident with C on its left, we remove vx_i and replace it with edge vx'_i . We then remove the open region of Σ between C and C' , and attach a disc to each of C and C' . This yields a new graph H' embedded in a new surface Σ' , or new graphs H'_1 and H'_2 embedded in surfaces Σ'_1 and Σ'_2 , respectively.

Let H be a graph embedded in some surface Σ , and let C_0 and C_1 be two disjoint cycles in H which are homotopic as curves in Σ . Let G be the 2-connected subgraph of H contained in the annulus region of Σ between C_0 and C_1 , including C_0 and C_1 . Note that G can be viewed as a plane graph with C_0 and C_1 as its facial cycles and C_1 bounding its infinite face. Hence, we call G a *cylinder* with *outer cycle* C_1 and *inner cycle* C_0 . The *cylinder-width* of G is the largest integer q such that G contains q pairwise disjoint cycles $R_0, \dots, R_{q-1}$, all homotopic to C_0 and C_1 in Σ .

Suppose that H is embedded in the orientable surface Σ of genus g , and that G is an induced subgraph of H which is a cylinder with inner cycle C_0 and outer cycle C_1 . Cutting along both C_0 and C_1 produces graphs H' and G on some set of surfaces. Here we say that the graph H' is produced from H by *cutting and deleting* the cylinder G . Note that if C_0 and C_1 are homotopically non-trivial, then H' is embedded in a surface Σ' of genus at most $g - 1$. We use

S_g to identify the orientable surface of genus g , which has Euler genus $2g$.

The following result is proven in [8] for triangulations, and is given in [2] as a simple extension to all graphs.

Lemma 1 *For any natural numbers g, r there exists a natural number $f(g, r)$ such that any 2-connected graph H in S_g having face-width at least $f(g, r)$ contains g pairwise disjoint cylinders $Q_1, \dots, Q_g$ of cylinder-width at least r whose cutting and deletion results in a connected plane graph.*

Let C_i be the outer cycle of Q_i and let D_i be the inner cycle of Q_i . Let $Q'_i \subset Q_i$ be a cylinder of H with outer cycle C'_i and inner cycle D'_i such that C'_i and D'_i are homotopic to C_i and D_i (as curves in S_g). We observe that, after cutting and deletion of $Q'_1, \dots, Q'_g$ from H , the resulting graph is also a connected plane graph. To see this, let P_i denote the cylinder of G with outer cycle C_i and inner cycle C'_i , and let R_i denote the cylinder of G with outer cycle D'_i and inner cycle D_i . Assume that C'_i and D'_i are chosen such that $P_i \cap R_i = \emptyset$. Let G be the resulting graph after cutting and deletion of $Q_1, \dots, Q_g$ from H , and let G' be the resulting graph after cutting and deletion of $Q'_1, \dots, Q'_g$ from H . Clearly, $G' = G \cup (\bigcup_{i=1}^g (P_i \cup R_i))$. Hence, G' is also a connected plane graph.

Note that $f(g, r)$ is $O(r4^g)$ and $f(g, r) \geq 4^g$. Using Lemma 1, Böhme, Mohar, and Thomassen [2] proved that there is a function $c(g)$ such that every 3-connected graph G with n vertices embeddable in S_g contains a cycle of length at least $c(g)n^{0.4}$. With the result of Chen and Yu [3], the exponent 0.4 can be improved to $\log_3 2 \approx 0.63$. No face-width requirement is placed on the graph G , but $c(g)$ depends on the genus of the surface. In this paper we show that by imposing a lower bound on the face-width of G , $c(g)$ can be replaced by a constant which is not dependent on g . To do this, we prove a stronger result for weighted graphs.

We will work with graphs which have non-negative vertex weighting. Where

there is no danger of confusion, we use $v \in G$ to mean $v \in V(G)$. Let $\mathbb{R}^+$ denote the set of non-negative real numbers. Let G be a graph, and $w : V(G) \rightarrow \mathbb{R}^+$. If H is a subgraph of G , we define $w(H) = \sum_{v \in H} w(v)$. Define $w(\emptyset) = 0$.

We now have the definitions required to precisely state the main result of this paper.

Theorem 1 *Let G be a 3-connected graph embedded in S_g and having face-width $\geq f(g, 6)$, and let $w : V(G) \rightarrow \mathbb{R}^+$. Then G contains a cycle R such that $\sum_{v \in R} w(v)^{\log_3 2} \geq \frac{1}{2} \left[\frac{1}{4} w(G) \right]^{\log_3 2}$.*

2 Circuit Graphs - Definitions and Background

Let us begin with some notation.

Let S be a subgraph of a graph G . Denote by $G - S$ the subgraph of G induced by $V(G) \setminus V(S)$. Let H be a component of $G - S$, and let B be the subgraph induced by $E(H) \cup \{xy \in E(G) : x \in V(S), y \in V(H)\}$. Then B is called an *S-bridge* of G . We also call a subgraph B an *S-bridge* of G if B is induced by a single edge $xy \in E(G) - E(S)$ where $x, y \in V(S)$. In both cases, the vertices in $V(B) \cap V(S)$ are called the *attachments* of B .

Let $x, y \in V(G)$ for a graph G , and let P be a path in G with $x, y \in V(P)$. By xPy we mean the subpath of P between x and y , inclusive. Where C is a cycle in a plane graph G and $x, y \in V(C)$, we use xCy to indicate the subpath of C from x to y , inclusive, which follows the clockwise orientation of C . Where x and y are the endpoints of a path R in G , we call R an *x - y path*. Let $Y \subset V(G), x \notin Y$. If R is an *x - y path* with $R \cap Y = \{y\}$, we say that R is an *x - Y path*. A set of *x - Y paths* is called an *x - Y fan* if each pair of the paths have only x in common. If the number of paths in an *x - Y fan* is n , we call it an *x - Y n-fan*. Given a 2-connected plane graph G , we call the cycle boundary of the infinite face of G the *outer cycle* of G .

A *circuit graph* is a pair (G, C) , where G is a 2-connected plane graph and C is a facial cycle of G , such that, for every 2-cut S of G , every component of $G - S$ contains a vertex of C . Given a circuit graph (G, C) and distinct vertices $x, y \in V(C)$, we say that (G, xCy) is a *strong circuit graph* if, for every 2-cut S of G , $S \cap (yCx - \{x, y\}) \neq \emptyset$.

An *annulus graph* is a triple (G, C_1, C_2) , where G is a 2-connected plane graph, and C_1 and C_2 are facial cycles of G , such that, for any 2-cut S of G , every component of $G - S$ contains a vertex of $C_1 \cup C_2$.

Where G is a connected graph which is not 2-connected, we describe the structure of G using Tutte's *block graph* construction [10]. A *block* of G is a maximal 2-connected subgraph. For this purpose, we define any subgraph induced by a single edge to be 2-connected. We construct a graph $Blk(G)$ such that $V(Blk(G)) = \{\text{cutvertices } u_i \text{ of } G\} \cup \{\text{blocks } B_j \text{ of } G\}$, and u_i is adjacent to B_j in $Blk(G)$ if $u_i \in B_j$ in G . We call a block B an *extremal* block of G if B contains exactly one cutvertex. If the graph $Blk(G)$ is a simple path, we say that G is a *chain* of blocks. Note that for a connected graph G this is equivalent to the condition that each block of G contains at most two cutvertices, and each cutvertex of G is contained in at most two blocks. Suppose that G is a chain of blocks, and that each block of G is induced by a single edge or is a circuit graph (B, C) with all cutvertices of G in B lying on the cycle C . Then G is called a *chain of circuit graphs*.

Let G be a chain of circuit graphs such that, for each block B of G , one of the following holds:

- (i) $B = G$; either B is induced by an edge xy , or there is a facial cycle C of B and there are distinct $x, y \in V(C)$ such that (B, xCy) is a strong circuit graph.
- (ii) $B \neq G$ is an extremal block containing cutvertex y of G ; either B is induced by an edge xy , or there is a facial cycle C of G containing y and

there is some $x \in V(C) - \{y\}$ such that one of (B, xCy) or (B, yCx) is a strong circuit graph.

- (iii) $B \neq G$ is a non-extremal block containing distinct cutvertices x, y of G ; either $V(B) = \{x, y\}$ or there is some facial cycle C in B such that $\{x, y\} \in V(C)$ and one of (B, xCy) or (B, yCx) is a strong circuit graph.

We call G a *chain of strong circuit graphs*.

We will repeatedly make use of the following fact.

Lemma 2 *Let $n_i \in \mathbb{R}^+, i = 1, \dots, m$, and let $0 < r \leq 1$. Then $\sum_{i=1}^m n_i^r \geq (\sum_{i=1}^m n_i)^r$.*

Lemma 2 is obvious if $r = 1$. So assume $0 < r < 1$. Define $f(x) = x^r + (1 - x)^r$. One may show that, with $x \in [0, 1]$, $f(x)$ has a unique critical point at $x = 1/2$. Since $f(1/2) > 1$, $f(x) \geq 1$ for all $x \in [0, 1]$. Now let $x = \frac{n_1}{n_1 + n_2}$ to find that $n_1^r + n_2^r \geq (n_1 + n_2)^r$, and apply induction on m .

Lemma 3 *Let (G, xCy) be a strong circuit graph, and let $w : V(G) \rightarrow \mathbb{R}^+$. Then G contains an $x - y$ path P such that $\sum_{v \in P-y} w(v)^{\log_3 2} \geq w(G - y)^{\log_3 2}$.*

Lemma 3 is given in [3]. The vertex y is excluded in the above inequality for technical reasons, (counting becomes straightforward when piecing together paths), which will be adopted in this paper. As a warm-up, we prove a similar result for the larger class of circuit graphs.

Lemma 4 *Let (G, C) be a circuit graph, let $x, y \in V(C)$ be distinct, and let $w : V(G) \rightarrow \mathbb{R}^+$. Then G contains an $x - y$ path P such that $\sum_{v \in P-y} w(v)^{\log_3 2} \geq [\frac{1}{2}w(G - y)]^{\log_3 2}$.*

Proof We assume without loss of generality that G is embedded in the plane with outer cycle C . If either of (G, xCy) or (G, yCx) is a strong circuit graph, we simply apply Lemma 3. Let us assume then that neither is a strong circuit

graph. Hence G has a 2-cut contained in $V(xCy)$ and a 2-cut contained in $V(yCx)$. Let $(S_k)_k$ be the collection of 2-cuts of G with $S_k \subseteq V(xCy)$, and let $(T_l)_l$ be the collection of 2-cuts of G with $T_l \subseteq V(yCx)$. Let B_k be the union of all S_k -bridges of G which do not contain yCx . Define $H = (\bigcup_k B_k) - \{x, y\}$. Similarly define A_l as the union of T_l -bridges of G which do not contain xCy , and $L = (\bigcup_l A_l) - \{x, y\}$. Since (G, C) is a circuit graph, every component of $G - S_k$ (respectively $G - T_l$) contains a vertex of C . Hence $H \cap L = \emptyset$.

By symmetry, assume that $w(H) \leq w(L)$. A 2-cut $S_k \subseteq V(xCy)$ is called *maximal* if there is no other 2-cut $S_j \subseteq V(xCy)$ for which $B_k \subset B_j$, $B_k \neq B_j$. For each maximal 2-cut S_k of G , remove the subgraph $B_k - S_k$ from G and replace it with a single edge e_k between the vertices of S_k so that the resulting graph is a plane graph. Call the resulting graph G' , and its outer cycle C' .

Now $(G', xC'y)$ is a strong circuit graph. To see this, note that any 2-cut S' in G' would also be a 2-cut in the original graph G . Hence $S' \subset C'$. Note also that since we replaced the bridges of all maximal 2-cuts contained in $V(xCy)$, $S' \not\subseteq xC'y$. Hence $S' \cap (yC'x - \{x, y\}) \neq \emptyset$, and so, $(G', xC'y)$ is a strong circuit graph.

Since $w(H) \leq w(L)$ and $H \cap L = \emptyset$, we have $w(G' - y) \geq \frac{1}{2}w(G - y)$. Applying Lemma 3 to $(G', xC'y)$, we find an $x - y$ path P' in G' such that

$$\sum_{v \in P' - y} w(v)^{\log_3 2} \geq [w(G' - y)]^{\log_3 2} \geq [\frac{1}{2}w(G - y)]^{\log_3 2}.$$

We then expand P' to an $x - y$ path P in G as follows: if $e_k \in E(P')$ and $S_k = \{u, v\}$, simply replace e_k with any $u - v$ path in B_k . Now

$$\sum_{v \in P - y} w(v)^{\log_3 2} \geq [\frac{1}{2}w(G - y)]^{\log_3 2}. \quad \square$$

Let G be a chain of circuit graphs, and let B be a block of G , where (B, C) is a circuit graph or B is induced by a single edge. Call $x \in V(B)$ an *outer vertex* of B if x is not a cutvertex of G , and either $x \in V(C)$ or x has degree 1 in B .

Lemma 5 *Let G be a chain of circuit graphs, and let $w : V(G) \rightarrow \mathbb{R}^+$. If G has only one block, let x, y be distinct outer vertices of G . If G has at least two blocks, let $x \in V(G)$ be an outer vertex of one extremal block of G , and let $y \in V(G)$ be an outer vertex of the other extremal block of G . Then G contains an $x - y$ path P such that $\sum_{v \in P-y} w(v)^{\log_3 2} \geq [\frac{1}{2}w(G - y)]^{\log_3 2}$*

Proof Let the blocks of G be labeled by $B_1, \dots, B_k$, let the cutvertices be labeled $x_1, \dots, x_{k-1}$, and let $x_0 = x$ and $x_k = y$, so that $x_{i-1}, x_i \in B_i$ for $i = 1, \dots, k$. To construct the desired long path in G , we apply Lemma 4 to each of B_i , finding an $x_{i-1} - x_i$ path P_i such that $\sum_{v \in P_i - x_i} w(v)^{\log_3 2} \geq [\frac{1}{2}w(B_i - x_i)]^{\log_3 2}$. Let $P = \bigcup_{i=1}^k P_i$. By applying Lemma 2, we have

$$\begin{aligned} \sum_{v \in P-y} w(v)^{\log_3 2} &= \sum_i \sum_{v \in P_i - x_i} w(v)^{\log_3 2} \geq \sum_i [\frac{1}{2}w(B_i - x_i)]^{\log_3 2} \\ &\geq [\frac{1}{2}w(G - y)]^{\log_3 2}. \quad \square \end{aligned}$$

Let us make a few simple observations about the structure of circuit and annulus graphs. These are stated in a fairly weak form, but they will be helpful to simplify the extension of our results to annulus graphs. Where G is a plane graph and C is any cycle in G , we define $\text{int}(C)$ to be the subgraph of G contained in the closed disc in the plane which is bounded by C .

Lemma 6 *Let G be a 2-connected plane graph with outer cycle C and another facial cycle D . Let F be any cycle in G .*

- (i) *If (G, C) is a circuit graph, then $(\text{int}(F), F)$ is a circuit graph.*
- (ii) *If (G, C, D) is an annulus graph and $D \subseteq \text{int}(F)$, then $(\text{int}(F), F, D)$ is an annulus graph.*
- (iii) *If (G, C, D) is an annulus graph and $D \not\subseteq \text{int}(F)$, then $(\text{int}(F), F)$ is a circuit graph.*

Proof If $\text{int}(F)$ is 3-connected then it is clearly a circuit graph, so we assume that $\text{int}(F)$ is not 3-connected. Let S be an arbitrary 2-cut in $\text{int}(F)$, and let T be any component of $\text{int}(F) - S$.

(i) Assume (G, C) is a circuit graph. Suppose that $T \cap F = \emptyset$, and hence $T \cap C = \emptyset$. Since G is planar, we see that there can be no edges from T to $G - \text{int}(F)$. So S is a 2-cut in G , and T is a component of $G - S$ with $T \cap C = \emptyset$. But this is a contradiction to the hypothesis that (G, C) is a circuit graph. Hence $T \cap F \neq \emptyset$. Since S and T are arbitrary, $(\text{int}(F), F)$ is a circuit graph.

(ii) Suppose (G, C, D) is an annulus graph and $D \subseteq \text{int}(F)$. If $T \cap (F \cup D) = \emptyset$, then S is a 2-cut of G and T is a component of $G - S$, and $T \cap (C \cup D) = \emptyset$, contradicting the assumption that (G, C, D) is an annulus graph. Hence $T \cap (F \cup D) \neq \emptyset$. Since S and T are arbitrary, $(\text{int}(F), F, D)$ is an annulus graph.

(iii) Now suppose (G, C, D) is an annulus graph and $D \not\subseteq \text{int}(F)$. If $T \cap F = \emptyset$ then S is a 2-cut of G and T is a component of $G - S$, and $T \cap (C \cup D) = \emptyset$, contradicting the assumption that (G, C, D) is an annulus graph. Since S and T are arbitrarily chosen, $(\text{int}(F), F)$ must be a circuit graph. $\square$

3 Long Paths in Annulus Graphs

Lemma 7 *Let (A, C, D) be an annulus graph. Let $w : V(A) \rightarrow \mathbb{R}^+$, and let $x, y \in V(C \cup D)$ be distinct. Then there is an $x - y$ path P in A such that*

$$\sum_{v \in P - y} w(v)^{\log_3 2} \geq \left[\frac{1}{4} w(A - y) \right]^{\log_3 2}.$$

Proof Throughout the proof, we assume without loss of generality that A is embedded in the plane with outer cycle C .

Case 1 $|C \cap D| \geq 3$.

We will define two subgraphs of A , each of which is a circuit graph or chain of circuit graphs. We begin by defining $U = V(C \cap D)$. Label the vertices of U by $u_1, \dots, u_m$ in clockwise order along C , and let $u_{m+1} = u_1$. Let M_i denote the union of $\{u_i, u_{i+1}\}$ -bridges of A not containing $u_{i+1}Cu_i \cup u_{i+1}Du_i$.

We claim that there are M_k and M_l such that either (1) $x \in M_k - u_{k+1}$, $y \in M_l - u_l$, and $u_k \neq u_{l+1}$, or (2) $x \in M_k - u_k$, $y \in M_l - u_{l+1}$, and $u_l \neq u_{k+1}$. Assume without loss of generality that $x \in M_k - u_{k+1}$ and $y \in M_l - u_l$. If $u_k \neq u_{l+1}$, then (1) holds. So let $u_k = u_{l+1}$. Then $u_{k+1} \neq u_l$ because $m \geq 3$. If $x \neq u_k$ and $y \neq u_k$, then we have $x \in M_k - u_k$, $y \in M_l - u_{l+1}$, and $u_l \neq u_{k+1}$, and so, (2) holds. So either $x = u_k$ or $y = u_k$. If $x = u_k$, then $y \neq u_k$, and so, $x \in M_l - u_l$, $y \in M_l - u_{l+1}$, and $u_{l+1} \neq u_l$, that is, (2) holds with $M_k = M_l$. So $y = u_k$ and $x \neq u_k$. Then $x \in M_k - u_k$, $y \in M_k - u_{k+1}$, and $u_k \neq u_{k+1}$, and so, (2) holds with $M_l = M_k$.

By symmetry, assume (1) holds (see Figure 1). Let $B_1 = \cup_{i=k}^l M_i$, and let $B_2 = A - (B_1 - \{u_k, u_{l+1}\})$. Note that each M_i is either a subgraph of A induced by a single edge or (by (iii) of Lemma 6) a circuit graph with its outer cycle as the special cycle. Hence, B_1 and B_2 are chains of circuit graphs. Note that when $k < l$, x and y lie in distinct extremal blocks of B_1 . Also note that either $w(B_1 - y) \geq \frac{1}{2}w(A - y)$ or $w(B_2 - y) \geq \frac{1}{2}w(A - y)$; otherwise, we have the contradiction that $w(B_1 - y) + w(B_2 - y) < w(A - y)$.

(1A) Suppose $w(B_1 - y) \geq \frac{1}{2}w(A - y)$. We apply Lemma 5 to find an $x - y$ path P in B_1 with

$$\begin{aligned} \sum_{v \in P - y} w(v)^{\log_3 2} &\geq \left[\frac{1}{2}w(B_1 - y)\right]^{\log_3 2} \\ &\geq \left[\frac{1}{4}w(A - y)\right]^{\log_3 2}. \end{aligned}$$

Figure 1: Case 1, $|C \cap D| \geq 3$

(1B) Now suppose $w(B_1 - y) < \frac{1}{2}w(A - y)$, so $w(B_2 - y) \geq \frac{1}{2}w(A - y)$. By Lemma 5 there is an $u_k - u_{l+1}$ path P_2 in B_2 with $\sum_{v \in P_2 - u_{l+1}} w(v)^{\log_3 2} \geq [\frac{1}{2}w(B_2 - u_{l+1})]^{\log_3 2}$. Note also that B_1 contains two disjoint paths P_x, P_y from x, y to u_k, u_{l+1} , respectively, which are internally disjoint from B_2 .

We define an $x - y$ path by $P = P_x \cup P_2 \cup P_y$. Then, with an application of Lemma 2, we find

$$\begin{aligned}
 \sum_{v \in P - y} w(v)^{\log_3 2} &= \sum_{v \in P_x - u_k} w(v)^{\log_3 2} + \sum_{v \in P_2 - u_{l+1}} w(v)^{\log_3 2} + \sum_{v \in P_y - y} w(v)^{\log_3 2} \\
 &\geq \sum_{v \in P_2 - u_{l+1}} w(v)^{\log_3 2} + w(u_{l+1})^{\log_3 2} \\
 &\geq [\frac{1}{2}w(B_2 - u_{l+1})]^{\log_3 2} + w(u_{l+1})^{\log_3 2} \\
 &\geq [\frac{1}{2}w(B_2)]^{\log_3 2} \\
 &\geq [\frac{1}{4}w(A - y)]^{\log_3 2}.
 \end{aligned}$$

Case 2 $|C \cap D| \leq 2$.

First we define subgraphs B_1 and B_2 . Their definitions depend upon the size of $C \cap D$ and are illustrated in Figure 2.

Figure 2: Case 2, $|C \cap D| \leq 2$

When $|C \cap D| = 2$: Let $U = \{u, v\} = C \cap D$. If $uCv = uDv$, then let $B_1 = uCv$, which is induced by $uv \in E(G)$. Otherwise, let $B_1 = \text{int}(F_1)$ where $F_1 = uCv \cup uDv$. If $vCu = vDu$, let $B_2 = vCu$, which is induced by $uv \in E(G)$. Otherwise, let $B_2 = \text{int}(F_2)$ where $F_2 = vCu \cup vDu$.

When $|C \cap D| = 0$: Since A is 2-connected, there must be disjoint paths Q and R from $V(C)$ to $V(D)$. We may assume that each path intersects C and D only at its endpoints. Let $q_c \in V(C)$ and $q_d \in V(D)$ be the endpoints of Q , and $r_c \in V(C)$ and $r_d \in V(D)$ the endpoints of R . Let $U = \{q_c, q_d, r_c, r_d\}$. Define $B_1 = \text{int}(F_1)$ where $F_1 = q_cCr_c \cup Q \cup q_dDr_d \cup R$. Define $B_2 = \text{int}(F_2)$ where $F_2 = r_cCq_c \cup Q \cup r_dDq_d \cup R$.

When $|C \cap D| = 1$: Let $\{u\} = C \cap D$. Since A is 2-connected, $A - u$ is connected. We may find a path Q from $V(C)$ to $V(D)$ in $A - u$, such that Q intersects C and D only at its endpoints. Let $\{q_c\} = Q \cap C$ and $\{q_d\} = Q \cap D$. Let $U = \{q_c, q_d, u\}$. Define $B_1 = \text{int}(F_1)$ where $F_1 = uCq_c \cup Q \cup uDq_d$. Define $B_2 = \text{int}(F_2)$ where $F_2 = q_cCu \cup Q \cup q_dDu$.

We proceed in the same manner for all three of the above situations.

By the above definitions, $D \not\subseteq B_1$ and $D \not\subseteq B_2$. From (iii) of Lemma 6 both (B_1, F_1) and (B_2, F_2) are circuit graphs, except when B_1 or B_2 is induced by a

single edge.

By symmetry, assume that $w(B_1 - y) \geq w(B_2 - y)$. Then $w(B_1 - y) \geq \frac{1}{2}w(A - y)$. In this case, B_1 cannot be induced by a single edge, and so, (B_1, F_1) is a circuit graph.

(2A) First we consider the case $x, y \in B_1$. Then by Lemma 4, there is an $x - y$ path P in B_1 with

$$\sum_{v \in P-y} w(v)^{\log_3 2} \geq \left[\frac{1}{2}w(B_1 - y)\right]^{\log_3 2} \geq \left[\frac{1}{4}w(A - y)\right]^{\log_3 2}.$$

(2B) Now consider the case $x, y \notin B_1$, noting that B_2 may not be induced by a single edge in this case. We recall that $x, y \in V(C \cup D)$. Hence B_2 contains disjoint paths P_x, P_y from x, y to vertices $u_x, u_y \in U$, respectively, with $(P_x \cup P_y) \cap U = (P_x \cup P_y) \cap B_1 = \{u_x, u_y\}$.

By Lemma 4 there is a $u_x - u_y$ path P_1 in B_1 with

$$\sum_{v \in P_1 - u_y} w(v)^{\log_3 2} \geq \left[\frac{1}{2}w(B_1 - u_y)\right]^{\log_3 2}.$$

Now we define an $x - y$ path by $P = P_x \cup P_1 \cup P_y$. Then, with an application of Lemma 2, we have

$$\begin{aligned} \sum_{v \in P-y} w(v)^{\log_3 2} &= \sum_{v \in P_x - u_x} w(v)^{\log_3 2} + \sum_{v \in P_1 - u_y} w(v)^{\log_3 2} + \sum_{v \in P_y - y} w(v)^{\log_3 2} \\ &\geq \sum_{v \in P_1 - u_y} w(v)^{\log_3 2} + w(u_y)^{\log_3 2} \\ &\geq \left[\frac{1}{2}w(B_1 - u_y)\right]^{\log_3 2} + w(u_y)^{\log_3 2} \\ &\geq \left[\frac{1}{2}w(B_1)\right]^{\log_3 2} \\ &\geq \left[\frac{1}{4}w(A - y)\right]^{\log_3 2}. \end{aligned}$$

(2C) Finally, suppose $x \notin B_1, y \in B_1$. The case $x \in B_1, y \notin B_1$ is symmetric. Since $x \in V(C \cup D)$, B_2 contains a path P_2 from x to some vertex $u_x \in U$, such that $P_2 \cap U = P_2 \cap B_1 = \{u_x\}$.

By Lemma 4, there is a $u_x - y$ path P_1 in B_1 with $\sum_{v \in P_1 - y} w(v)^{\log_3 2} \geq [\frac{1}{2}w(B_1 - y)]^{\log_3 2}$.

Now we define an $x - y$ path by $P = P_1 \cup P_2$. Then

$$\begin{aligned} \sum_{v \in P - y} w(v)^{\log_3 2} &\geq \sum_{v \in P_1 - y} w(v)^{\log_3 2} \\ &\geq [\frac{1}{2}w(B_1 - y)]^{\log_3 2} \\ &\geq [\frac{1}{4}w(A - y)]^{\log_3 2}. \quad \square \end{aligned}$$

4 Proof of Theorem 1

Given a 3-connected graph G embedded in S_g with sufficiently large face-width, we apply Lemma 1 to create a connected plane graph G' . We do not use the graph G' directly. Instead, we use observations about the cutting cylinders used to create G' from G , to construct from G a 3-connected plane graph H . With the results of [3] we find that H has a heavy cycle. A modification of this cycle is then used to produce a heavy cycle in the original graph G .

Where $S, T \subset V(G)$ for a graph G , we define $[S, T] = \{xy \in E(G) : x \in S, y \in T\}$.

(1) Selecting the Cutting Cylinders Let G be a 3-connected graph embedded in S_g with face-width at least $f(g, 6)$. Then by Lemma 1, G contains g pairwise disjoint cylinders $Q_1, \dots, Q_g$ of cylinder-width at least six whose cutting and deletion results in a connected plane graph G' . Let C_i^0 and C_i^5 be the cycles in $Q_i, i = 1, \dots, g$ along which we cut, with C_i^0 the inner cycle of Q_i and C_i^5 its outer cycle. Note that each of these cutting cycles is homotopically

nontrivial in S_g (because otherwise G' is not connected). By the observation following Lemma 1, we may choose $Q_1, \dots, Q_g$ such that any cylinder of G contained in Q_i with outer cycle and inner cycle homotopic to C_i^0 and C_i^5 have cylinder-width < 6 .

In the following discussion, where C is a cycle in G we will also use C to refer to the corresponding cycle in G' or in Q_i . Let us view each of the deleted cylinders, Q_i , as a plane graph with C_i^5 as its outer cycle and C_i^0 as a facial cycle. Since Q_i has cylinder-width at least six, there exist disjoint cycles $C_i^1, C_i^2, C_i^3, C_i^4$ in Q_i , also disjoint from C_i^0 and C_i^5 , with $C_i^0 \subseteq \text{int}(C_i^1) \subset \dots \subseteq \text{int}(C_i^4) \subseteq \text{int}(C_i^5)$. We choose $C_i^1, C_i^2, C_i^3, C_i^4$ such that the following graphs are minimal: $\text{int}(C_i^1)$, $\text{int}(C_i^3) - (\text{int}(C_i^2) - C_i^2)$, and $\text{int}(C_i^5) - (\text{int}(C_i^4) - C_i^4)$.

Claim 1 There are no vertices between C_i^0 and C_i^1 , between C_i^2 and C_i^3 , and between C_i^4 and C_i^5 . More precisely, $V(\text{int}(C_i^1) \setminus \text{int}(C_i^0)) = V(C_i^1)$, $V(\text{int}(C_i^3) \setminus \text{int}(C_i^2)) = V(C_i^3)$, and $V(\text{int}(C_i^5) \setminus \text{int}(C_i^4)) = V(C_i^5)$.

Proof Suppose there is some vertex $v \in \text{int}(C_i^1) \setminus \text{int}(C_i^0)$ with $v \notin V(C_i^1)$. Since G is 3-connected, there is a $v - V(C_i^0 \cup C_i^1)$ 3-fan in $\text{int}(C_i^1)$. That is, there are three distinct paths from v to the set $V(C_i^0 \cup C_i^1)$ which are disjoint except for x and which intersect $V(C_i^0 \cup C_i^1)$ only at their endpoints. Two of these paths, say P and Q , must end on the same cycle. Let p, q be the endpoints of P, Q , respectively, other than v .

Case 1 . $p, q \in C_i^1$.

Then either $C_i^0 \subset \text{int}(P \cup Q \cup pC_i^1q)$ or $C_i^0 \subset \text{int}(P \cup Q \cup qC_i^1p)$. Assume by symmetry that $C_i^0 \subset \text{int}(P \cup Q \cup pC_i^1q)$. Since $E(qC_i^1p)$ is outside $\text{int}(P \cup Q \cup pC_i^1q)$, $\text{int}(C_i^1)$ is properly contained in $\text{int}(P \cup Q \cup pC_i^1q)$. Note that $P \cup Q \cup pC_i^1q$ is disjoint from $C_i^0, C_i^2, C_i^3, C_i^4, C_i^5$. Hence, $P \cup Q \cup pC_i^1q$ contradicts the choice of C_i^1 .

Case 2 . $p, q \in C_i^0$.

Then either $C_i^0 \subset \text{int}(P \cup Q \cup pC_i^0q)$ or $C_i^0 \subset \text{int}(P \cup Q \cup qC_i^0p)$. By symmetry, assume that $C_i^0 \subset \text{int}(P \cup Q \cup pC_i^0q)$, and let $C'_i = P \cup Q \cup pC_i^0q$. Let Q'_i be the cylinder of G with outer cycle C_i^5 and inner cycle C'_i . Clearly, C'_i is homotopic to C_i^0 in S_g , and Q'_i has cylinder width 6, contradicting the choice of Q_i .

Hence, we proved that $V(\text{int}(C_i^1) \setminus \text{int}(C_i^0)) = V(C_i^1)$. By a similar argument, we can prove $V(\text{int}(C_i^5) \setminus \text{int}(C_i^4)) = V(C_i^5)$. By an argument similar to Case 1, we can show that $V(\text{int}(C_i^3) \setminus \text{int}(C_i^2)) = V(C_i^3)$.

Figure 3: Cutting cylinder Q_i

Let $A_i = \text{int}(C_i^2) - C_i^0$. Then (A_i, C_i^2, C_i^1) can be viewed as an annulus graph. (That is, it has a natural planar embedding with outer cycle C_i^2 which is an annulus graph). Let $A'_i = \text{int}(C_i^4) - \text{int}(C_i^2)$, with (A'_i, C_i^4, C_i^3) also seen as an annulus graph. (See Figure 3.)

(2) Constructing the New Graph Recall that G' is the connected plane graph obtained from H by cutting and deleting $Q_1, \dots, Q_g$.

Claim 2 G' is 2-connected, and for any 2-cut S of G' , (1) $S \subset V(C_i^0)$ or $S \subset V(C_i^5)$ for some $1 \leq i \leq g$, and (2) $G' - S$ has exactly two components, each containing a vertex of C_i^0 if $S \subset V(C_i^0)$ or C_i^1 if $S \subset V(C_i^1)$.

Proof Let $S \subset V(G')$ be a cutset in G' with $|S| \leq 2$. Let $T_1, \dots, T_k$ be the components of $G' - S$. For each $j \in \{1, \dots, k\}$, there is a simple closed curve

γ_j in the plane such that $\gamma_j \cap G' = S$, and T_j is contained in one open region of the plane bounded by γ_j while $\bigcup_{i \neq j} T_i$ is contained in the other open region bounded by γ_j . Let Γ_j denote the simple closed curve in S_g corresponding to γ_j . Then $\Gamma_j \cap G = S$. Since G has face-width greater than three, Γ_j must be homotopically trivial in S_g . Since G is 3-connected, we see that $T_j \cap D_j \neq \emptyset$ for some cycle $D_j \in \{C_i^0, C_i^5 : 1 \leq i \leq q\}$. But since D_j is homotopically non-trivial, no D_j is contained in the open region in the plane bounded by γ_j and containing T_j . Hence $S \subset D_j$ for all $j \in \{1, \dots, k\}$. This forces $k = 2$ and $|S| = 2$. This proves the claim.

Now we turn to the construction of the graph H . Beginning from G , remove all $[C_i^2, C_i^3]$, $i = 1, \dots, g$. Contract each subgraph A_i to the vertex a_i and contract each A'_i to the vertex a'_i , deleting multiple edges, and letting $\tilde{w}(a_i) = w(A_i)$, $\tilde{w}(a'_i) = w(A'_i)$, for $i = 1, \dots, g$. Let $\tilde{w}(x) = w(x)$ for all $x \in V(G - (A_i \cup A'_i))$. Call the resulting graph H . Then $\tilde{w}(H) = w(G)$. Note that $G - \bigcup_{i=1}^g (A_i \cup A'_i) = G'$. By Claim 2 and since the face-width of G is at least three, we see that H is a 3-connected plane graph.

(3) Constructing a Long Cycle Let C be a facial cycle of H , and let $e = xy \in E(C)$. Since H is 3-connected, (H, xCy) is a strong circuit graph. Applying Lemma 3 we find that there is some $x - y$ path P in H with

$$\sum_{v \in P-y} \tilde{w}(v)^{\log_3 2} \geq \tilde{w}(H-y)^{\log_3 2}.$$

Note that we may choose P so that $V(P) \neq \{x, y\}$. Now let T be the cycle in H obtained from P by adding the edge e . By Lemma 2, we find that

$$\sum_{v \in T} \tilde{w}(v)^{\log_3 2} \geq \tilde{w}(H-y)^{\log_3 2} + \tilde{w}(y)^{\log_3 2} \geq \tilde{w}(H)^{\log_3 2}.$$

However, the cycle T may pass through contracted vertices a_i or a'_i .

Note that any cycle T in H which passes through the contracted vertex a_i can be easily extended to a cycle in G which passes through the annulus

subgraph A_i . To reach a_i in H , T must pass through two distinct vertices u_i, v_i on the cycle surrounding a_i , which corresponds to the cutting cycle C_i^0 in G . We will call vertex a_i *good* in cycle T if there exist edges $u_i x_i, v_i y_i \in E(G)$ with x_i, y_i distinct vertices on the cycle C_i^1 . If no such edges exist, we call a_i *bad* in T . Similar definitions are used for a'_i , with respect to C_i^4 and C_i^5 . (Note that if $a_i \notin V(T)$, then a_i is considered neither good nor bad in T .)

The notation of the following discussion will be simplified by a re-indexing of the vertices a'_i . We let $a_{i+g} = a'_i$ for $i = 1, \dots, g$, so that $\{a_i\}_{i=1}^{2g} = \{a_i\}_{i=1}^g \cup \{a'_i\}_{i=1}^g$. Similarly let $A_{i+g} = A'_i$ for $i = 1, \dots, g$. Define $W = T - \{a_i\}_{i=1}^{2g}$. Let $I = \{i \in \mathbb{Z} : a_i \text{ is good in } T\}$, $J = \{j \in \mathbb{Z} : a_j \text{ is bad in } T\}$.

We construct a cycle R in G based on the cycle T in H as follows. For each $i \in I$, we may apply Lemma 7 to find an $x_i - y_i$ path R_i in A_i with $\sum_{v \in R_i - y_i} w(v)^{\log_3 2} \geq [\frac{1}{4}w(A_i - y_i)]^{\log_3 2}$. For each $j \in J$, let $x_j \in A_j$ with $u_j x_j, v_j x_j \in E(G)$. Define the cycle R in G as $W \cup (\bigcup_{i \in I} u_i x_i R_i y_i v_i) \cup (\bigcup_{j \in J} u_j x_j v_j)$.

Note that from our choice of the original cycle T in H , we have

$$\begin{aligned} \sum_{v \in W} \tilde{w}(v)^{\log_3 2} + \sum_{i \in I} \tilde{w}(a_i)^{\log_3 2} + \sum_{j \in J} \tilde{w}(a_j)^{\log_3 2} \\ = \sum_{v \in T} \tilde{w}(v)^{\log_3 2} \\ \geq \tilde{w}(H)^{\log_3 2} \\ = w(G)^{\log_3 2}. \end{aligned}$$

So considering the weight of the new cycle R , with an application of Lemma 2, we have

$$\begin{aligned}
\sum_{v \in R} w(v)^{\log_3 2} &= \sum_{v \in W} w(v)^{\log_3 2} + \sum_{i \in I} \sum_{v \in R_i - y_i} w(v)^{\log_3 2} + \sum_{i \in I} w(y_i)^{\log_3 2} + \sum_{j \in J} w(x_j)^{\log_3 2} \\
&\geq \sum_{v \in W} w(v)^{\log_3 2} + \sum_{i \in I} \left[\frac{1}{4} w(A_i - y_i) \right]^{\log_3 2} + \sum_{i \in I} \left[\frac{1}{4} w(y_i) \right]^{\log_3 2} \\
&\geq \left[\frac{1}{4} \right]^{\log_3 2} \left[\sum_{v \in W} \tilde{w}(v)^{\log_3 2} + \sum_{i \in I} \tilde{w}(a_i)^{\log_3 2} \right].
\end{aligned}$$

If we have $\sum_{v \in W} \tilde{w}(v)^{\log_3 2} + \sum_{i \in I} \tilde{w}(a_i)^{\log_3 2} \geq \frac{1}{2} \sum_{v \in T} \tilde{w}(v)^{\log_3 2}$ then we find

$$\sum_{v \in R} w(v)^{\log_3 2} \geq \frac{1}{2} \left[\frac{1}{4} w(G) \right]^{\log_3 2}.$$

We may assume then that

$$\sum_{v \in W} \tilde{w}(v)^{\log_3 2} + \sum_{i \in I} \tilde{w}(a_i)^{\log_3 2} < \frac{1}{2} \sum_{v \in T} \tilde{w}(v)^{\log_3 2}.$$

So we must have

$$\sum_{j \in J} \tilde{w}(a_j)^{\log_3 2} > \frac{1}{2} \sum_{v \in T} \tilde{w}(v)^{\log_3 2}.$$

In this case, we will show that H contains a cycle T' through all a_j with $j \in J$ such that a_j is good with respect to T' . That is, if $u_j a_j v_j \subset T'$, then $\exists x_j, y_j \in V(C_j^1)$ (or $V(C_{j-g}^4)$ if $j \geq g+1$) such that $x_j \neq y_j$ and $u_j x_j, y_j v_j \in E(G)$.

For each $j \in J$, consider the bipartite graph G_j induced by $[C_j^0, C_j^1]$ (or by $[C_{j-g}^4, C_{j-g}^5]$ if $j \geq g+1$). Let V_j be a vertex cover for G_j . Then we can find a homotopically non-trivial closed curve Γ_j lying between C_j^0 and C_j^1 (or C_{j-g}^4 and C_{j-g}^5) in S_g with $\Gamma_j \cap G = V_j$. The face-width of G is at least $f(g, 6)$, so $|V_j| \geq f(g, 6)$. Applying Konig's theorem we find that there must be a matching M_j in G_j with $|M_j| \geq f(g, 6)$. Let H' be obtained from H by deleting edges in $[C_j^0, C_j^1] - M_j$ (or $[C_{j-g}^4, C_{j-g}^5] - M_j$ if $j \geq g+1$).

Note that $|J| \leq 2g$ and $f(g, 6)$ as specified in Lemma 1 is greater than $4g$. Note also that H' has no set S with $|S| < f(g, 6)$ separating two vertices of $\{a_j : j \in J\}$, otherwise the face-width of G is $< f(g, 6)$. Hence H' contains a cycle T' through all $a_j, j \in J$. By the construction of H' , T' is a cycle of H through all $a_j, j \in J$, and the edges of T' at $a_j, j \in J$, correspond to edges in M_j . Since M_j is a matching in G , every a_j is good with respect to T' .

Now, let u_j, v_j be the neighbors of a_j in T' , and let $x_j \neq y_j \in V(C_j^1)$ (or $V(C_{j-g}^4)$ if $j \geq g+1$) such that $x_j u_j, y_j v_j \in E(G)$. For $j \in J$, applying Lemma 7 to A_j we find an $x_j - y_j$ path R_j in A_j such that

$$\sum_{v \in R_j - y_j} w(v)^{\log_3 2} \geq \left[\frac{1}{4} w(A_j - y_j) \right]^{\log_3 2}.$$

Define the cycle R' in G as $(T' - \{a_j : j \in J\}) \cup (\bigcup_{j \in J} u_j x_j R_j y_j v_j)$.

We then have, again with an application of Lemma 2,

$$\begin{aligned} \sum_{v \in R'} w(v)^{\log_3 2} &\geq \sum_{j \in J} \sum_{v \in R_j - y_j} w(v)^{\log_3 2} + \sum_{j \in J} w(y_j)^{\log_3 2} \\ &\geq \sum_{j \in J} \left[\frac{1}{4} w(A_j - y_j) \right]^{\log_3 2} + \sum_{j \in J} \left[\frac{1}{4} w(y_j) \right]^{\log_3 2} \\ &\geq \sum_{j \in J} \left[\frac{1}{4} \tilde{w}(a_j) \right]^{\log_3 2} \\ &\geq \frac{1}{2} \left[\frac{1}{4} w(G) \right]^{\log_3 2} \quad \square \end{aligned}$$

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