

Reducing Hajós' coloring conjecture to 4-connected graphs

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Abstract

Hajós conjectured that, for any positive integer k , every graph containing no K_{k+1} -subdivision is k -colorable. This is true when $k \leq 3$, and false when $k \geq 6$. Hajós' conjecture remains open for $k = 4, 5$. In this paper, we show that any possible counterexample to this conjecture for $k = 4$ with minimum number of vertices must be 4-connected. This is a step in an attempt to reduce Hajós' conjecture for $k = 4$ to the conjecture of Seymour that any 5-connected non-planar graph contains a K_5 -subdivision.

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1 Introduction

Graphs considered in this paper are simple. The Four Color Theorem states that every planar graph is 4-colorable. The Kuratowski Theorem states that a graph is planar if, and only if, it contains neither a K_5 -subdivision nor a $K_{3,3}$ -subdivision. Also, a graph is planar if, and only if, it contains neither a K_5 -minor nor a $K_{3,3}$ -minor. Based on these characterizations of planar graphs, there are two conjectures which would generalize the Four Color Theorem. One of these was attributed to Hajós (see [1]) which states that, for any positive integer k , every graph containing no K_{k+1} -subdivision is k -colorable. The other is Hadwiger's conjecture [4]: For any positive integer k , every graph containing no K_{k+1} -minor is k -colorable. Both conjectures are easily seen to be true when $k = 1, 2$. It is also not hard to show that both conjectures are true for $k = 3$.

Hadwiger's conjecture for $k = 4$ is equivalent to the Four Color Theorem [10]. Hadwiger's conjecture for $k = 5$ can also be reduced to the Four Color Theorem [8], and it remains open for $k \geq 6$.

On the other hand, Catlin [1] showed that Hajós' conjecture fails when $k \geq 6$. In fact, Erdős and Fajtlowicz [3] showed that Hajós' conjecture fails for almost all graphs. However, Hajós' conjecture remains open for $k = 4$ and $k = 5$. It is therefore important to derive structural information about graphs containing no K_5 -subdivisions (respectively, K_6 -subdivisions). There has been considerable work concerning K_5 -subdivisions. Dirac [2] conjectured that every simple graph on n vertices with at least $3n - 5$ edges contains a K_5 -subdivision, which was proved by Mader [7]. The following conjecture of Seymour [9] remains open: Every 5-connected non-planar graph contains a K_5 -subdivision. A result in [5] shows that Seymour's conjecture implies Dirac's conjecture. Our aim is to establish a connection between Hajós' conjecture and Seymour's conjecture by looking at the connectivity of a minimum counterexample to Hajós' conjecture. More specifically, if a counterexample to Hajós' conjecture is 5-connected then, by the Four Color Theorem, Seymour's conjecture implies Hajós' conjecture for $k = 4$.

For convenience, we say a graph G is a *Hajós graph* if

- (i) G is not 4-colorable,
- (ii) G contains no K_5 -subdivision, and
- (iii) subject to (i) and (ii), $|V(G)|$ is minimum.

Note that any non-spanning subgraph of a Hajós graph is 4-colorable. The main result of this paper is the following.

(1.1) Theorem. *Every Hajós graph is 4-connected.*

Let G be a graph. A *separation* of G is a pair (G_1, G_2) of edge disjoint subgraphs of G such that $G = G_1 \cup G_2$ and $V(G_i) - V(G_{3-i}) \neq \emptyset$ for $i = 1, 2$. (Note that our definition of a separation is different from the usual one in which $V(G_i) - V(G_{3-i}) \neq \emptyset$ is not required.) We call (G_1, G_2) a k -*separation* if $|V(G_1 \cap G_2)| = k$. A set $S \subseteq V(G)$ is a k -*cut* in G , if $|S| = k$ and G has a separation (G_1, G_2) such that $V(G_1 \cap G_2) = S$.

To prove Theorem (1.1), we need to deal with k -cuts with $k \leq 3$. It is relatively easy to show that every Hajós graph admits no k -cuts with $k \leq 2$. The main work is to show that no Hajós graph admits a 3-cut. This can be outlined as follows. Suppose there is a Hajós graph G which admits a 3-cut. Choose a 3-separation (G_1, G_2) of G such that G_2 is minimal. Let G'_i denote the graph obtained from G_i by adding an edge between every pair of distinct vertices from $V(G_1 \cap G_2)$. To decide whether G'_i contains a K_5 -subdivision, we need to know whether G'_{3-i} has a cycle containing $V(G_1 \cap G_2)$. Therefore, we characterize those graphs in which there is no cycle through three specified vertices. This is done in Section 2. (It turns out that such a characterization was already obtained in [11]. However, we obtained our proof independently, and our proof is significantly shorter.) In Section 3, we prove a few lemmas concerning certain 3-separations (G_1, G_2) of G and show that G_1 and G_2 admit certain 4-colorings. In Section 4, we complete the proof of Theorem (1.1).

We remark that if any Hajós graph is 5-connected, then Seymour's conjecture if true would imply Hajós' conjecture for $k = 4$. However, to show that Hajós graphs are 5-connected, one needs to consider the much harder problem of characterizing graphs in which there is no K_4 -subdivision at specified locations.

We conclude this section with some notation. Let G be a graph. For $A, B \subseteq V(G)$, an A - B path in G is a path in G which has one end in A and the other in B and is otherwise disjoint from $A \cup B$. If $A = \{x\}$, we speak of x - B path instead of $\{x\}$ - B path, and if, in addition, $B = \{y\}$ then we write x - y path instead of $\{x\}$ - $\{y\}$ path. Two paths in G are said to be *internally disjoint* if no internal vertex of one is contained in the other. For any two sets $A, B \subseteq V(G)$, we say that a set $S \subseteq V(G)$ *separates A from B* if there is a separation (G_1, G_2) of G with $V(G_1 \cap G_2) = S$, $A \subseteq V(G_1)$, $B \subseteq V(G_2)$, $A - S \neq \emptyset$, and $B - S \neq \emptyset$. If $A = \{v\}$ we simply say S separates v from B , and if $A = \{v\}$ and $B = \{w\}$ then we simply say that S separates v from w .

Let H be a subgraph of a graph G , let $v_1, \dots, v_k \in V(G)$, and $\{u_i, w_i\} \subseteq V(H) \cup \{v_1, \dots, v_k\}$, $i = 1, \dots, m$. Then we let $H + \{v_1, \dots, v_k, u_1w_1, \dots, u_mw_m\}$ denote the graph with vertex set $V(H) \cup \{v_1, \dots, v_k\}$ and edge set $E(H) \cup \{u_1w_1, \dots, u_mw_m\}$.

Let G be a graph. For any $S \subseteq V(G)$, $G[S]$ denotes the subgraph of G induced by S . A path P in G is said to be a *branching path* in G if its internal vertices are of degree 2 in G and its ends are of degree at least 3 in G . Vertices of G with degree at least 3 are called *branching* vertices of G .

Given a graph G , we shall view a coloring of G as a mapping c from $V(G)$ to a set of colors such that $c(u) \neq c(v)$ whenever $uv \in E(G)$.

2 Cycles through three fixed vertices

In this section we characterize all 2-connected graphs in which there are three vertices not contained in any cycle. This was done without knowing that such a characterization was given by Watkins and Mesner [11, Theorem 2]. We give an alternative proof which makes use of a result of Lovász. Our proof is much shorter than that in [11], even including the proof of the following result of Lovász [6, Exercise 6.67].

(2.1) Lemma. *Let L be a set of three independent edges in a 3-connected graph H . Then there is a cycle in H containing L if, and only if, $H - L$ is connected.*

(2.2) Theorem. *Let G be a 2-connected graph and let x, y, z be three distinct vertices of G . Then there is no cycle through x, y and z in G if, and only if, one of the following statements holds.*

- (i) *There exists a 2-cut S in G and there exist three distinct components D_x, D_y, D_z of $G - S$ such that $u \in V(D_u)$ for each $u \in \{x, y, z\}$.*
- (ii) *There exist a vertex v of G , 2-cuts S_x, S_y, S_z in G , and components D_u of $G - S_u$ containing u , for all $u \in \{x, y, z\}$, such that $S_x \cap S_y \cap S_z = \{v\}$, $S_x - \{v\}, S_y - \{v\}, S_z - \{v\}$ are pairwise disjoint, and D_x, D_y, D_z are pairwise disjoint.*
- (iii) *There exist pairwise disjoint 2-cuts S_x, S_y, S_z in G and components D_u of $G - S_u$ containing u , for all $u \in \{x, y, z\}$, such that D_x, D_y, D_z are pairwise disjoint and $G - V(D_x \cup D_y \cup D_z)$ has exactly two components, each containing exactly one vertex from S_u , for all $u \in \{x, y, z\}$.*

Proof: It is straightforward to check that if (i) or (ii) or (iii) holds then G contains no cycle through x, y, z . Now assume that G contains no cycle through x, y, z .

First, let us assume that $\{x, y, z\}$ is not an independent set in G . Without loss of generality, let $xy \in E(G)$. Then, as G is 2-connected, there exists a z - x path P_x and a z - y path P_y in G such that $V(P_x \cap P_y) = \{z\}$. Hence, $(P_x \cup P_y) + xy$ is a cycle through x, y, z , a contradiction. Therefore,

- (1) $\{x, y, z\}$ is an independent set in G .

Next we show that

- (2) for any $u \in \{x, y, z\}$, u is not contained in any 2-cut in G separating the two vertices in $\{x, y, z\} - \{u\}$.

For otherwise, we may assume that there is a 2-separation (G_1, G_2) of G such that $x \in V(G_1 \cap G_2)$, $y \in V(G_1) - V(G_1 \cap G_2)$, and $z \in V(G_2) - V(G_1 \cap G_2)$. Since

G is 2-connected, G_1 (respectively, G_2) contains two internally disjoint paths from y (respectively, z) to $V(G_1 \cap G_2)$. These four paths form a cycle containing $\{x, y, z\}$, a contradiction.

(3) For any $u \in \{x, y, z\}$ there is a 2-cut S_u in G separating u from $\{x, y, z\} - \{u\}$.

Suppose the assertion is false. Without loss of generality, assume that G has no 2-cut separating x from $\{y, z\}$. By Menger's theorem, let P_1 and P_2 be two internally disjoint y - z paths, and let $C = P_1 \cup P_2$. Since x, y, z are not contained in any cycle in G , $x \notin V(C)$. By Menger's theorem, there must exist three paths R_1, R_2, R_3 from x to C , sharing only x . We may assume two of these paths, say R_1 and R_2 , end on P_1 . Then $C \cup R_1 \cup R_2$ contains a cycle through x, y and z , a contradiction. So we have (3).

For each $u \in \{x, y, z\}$, let D_u denote the component of $G - S_u$ containing u . Next, we show that we may choose S_x, S_y, S_z so that

(4) D_x, D_y, D_z are pairwise disjoint.

Since G is 2-connected, there is a cycle C in G through x and z . Then by (2), $S_x \cup S_z \subseteq V(C)$. Since $\{x, y, z\}$ is not contained in any cycle, $y \notin V(C)$. If there exist three y - $V(C)$ paths in G sharing only y , then two of these paths must end on an x - y paths in C , giving a cycle through x, y, z in G , a contradiction. Hence we may choose S_y to separate y from $V(C)$. By (2), $x, z \notin V(C)$. Thus, D_x and D_y are disjoint, and D_z and D_y are disjoint.

Again since G is 2-connected, there must be a cycle D through x and y in G , and $S_x \cup S_y \subseteq V(D)$ by (2). By a similar argument as above, we may choose S_z separating z from $V(D)$. By (2), $x, y \notin S_z$, and hence, D_z is disjoint from both D_x and D_y . So we have (4).

Case 1. For some choice of S_x, S_y, S_z satisfying (3) and (4), S_x, S_y, S_z are not pairwise disjoint.

Without loss of generality, let $S_x \cap S_y \neq \emptyset$. If $S_x = S_y$ then we can also choose $S_z = S_x$, and (i) holds.

So let $\{v\} = S_x \cap S_y$. Then there do not exist two paths in $G - v$ from z to $(S_x \cup S_y) - v$ sharing only z ; for otherwise G would contain a cycle through x, y, z . Hence there exists some $w \in V(G - v) - V(D_x \cup D_y)$ separating $(S_x \cup S_y) - \{v\}$ from z . Thus by choosing $S_z := \{w, v\}$ we see (ii) holds.

Case 2. For any choice of S_x, S_y, S_z satisfying (3) and (4), S_x, S_y, S_z are pairwise disjoint.

Choose $S_u = \{a_u, b_u\}$ for all $u \in \{x, y, z\}$ such that, subject to (3) and (4), D_x, D_y, D_z are maximal. Let $G' := (G - V(D_x \cup D_y \cup D_z)) + \{a_x b_x, a_y b_y, a_z b_z\}$. By the maximality of D_x, D_y, D_z , if G' is not 3-connected then, for every 2-separation (G_1, G_2) of G' , either

$\{a_x b_x, a_y b_y, a_z b_z\} \subseteq E(G_1)$ or $\{a_x b_x, a_y b_y, a_z b_z\} \subseteq E(G_2)$. Suppose $\{a_x b_x, a_y b_y, a_z b_z\} \subseteq E(G_1)$. Replace G_2 by an edge between the vertices in $V(G_1 \cap G_2)$. Repeating this operation until no such 2-separation exists, we obtain a 3-connected graph G'' in which $a_x b_x, a_y b_y$ and $a_z b_z$ are independent edges.

Suppose $G'' - \{a_x b_x, a_y b_y, a_z b_z\}$ is connected. Then by Lemma (2.1), there exists a cycle C'' in G'' through $a_x b_x, a_y b_y$ and $a_z b_z$. From C'' we can easily produce a cycle C through x, y, z in G by replacing the edges in $E(C'') - E(G)$ with paths in G , a contradiction. So assume that $G'' - \{a_x b_x, a_y b_y, a_z b_z\}$ is not connected, and hence, it has exactly two components. Then we see that $G - V(D_x \cup D_y \cup D_z)$ has exactly two components, each containing exactly one vertex from S_u for all $u \in \{x, y, z\}$. Hence (iii) holds. \square

3 3-Separations

The goal of this section is to show that every Hajós graph is 3-connected, and if a Hajós graph admits a 3-separation (G_1, G_2) chosen to minimize G_2 , then G_1 and G_2 admit special 4-colorings.

(3.1) Proposition. *Every Hajós graph is 3-connected.*

Proof: Let G be a Hajós graph. Obviously, G must be connected. Suppose G is not 2-connected, and let v be a cut vertex of G . Then there exists a separation (G_1, G_2) of G such that $V(G_1 \cap G_2) = \{v\}$ and G_1 and G_2 are proper subgraphs of G . Since G has no K_5 -subdivision, neither G_1 nor G_2 contains a K_5 -subdivision. Hence, G_1 and G_2 are 4-colorable. Let c_i denote a 4-coloring of G_i , $i = 1, 2$, using the same set of four colors. We may assume $c_1(v) = c_2(v)$ by permuting the colors of vertices of G_1 . Now c is a 4-coloring of G , where $c(u) = c_i(u)$ for all $u \in V(G_i)$, $i = 1, 2$. This shows that G is 4-colorable, a contradiction. Therefore, G is 2-connected.

Now, suppose G is not 3-connected. Let $\{x, y\}$ be a 2-cut of G , and let (G_1, G_2) be a separation of G such that $V(G_1 \cap G_2) = \{x, y\}$. Let $G'_i := G_i + xy$, $i = 1, 2$.

We claim that for each $i \in \{1, 2\}$, G'_i contains no K_5 -subdivision. For otherwise, let Σ be a K_5 -subdivision in G'_i . Then, $xy \in E(\Sigma)$, or else Σ would be a K_5 -subdivision in G . Since G is 2-connected, we may replace xy in Σ with an x - y path in G_{3-i} , and the result is a K_5 -subdivision in G , a contradiction.

Hence, since $|V(G'_i)| < |V(G)|$, both G'_1 and G'_2 are 4-colorable. Let c_i be a 4-coloring of G'_i , $i = 1, 2$, such that c_1 and c_2 use the same set of four colors. Note that $c_i(x) \neq c_i(y)$ for $i = 1, 2$. Hence by permuting the colors of vertices of G'_1 , we may assume that $c_1(x) = c_2(x)$ and $c_1(y) = c_2(y)$. Let c be a 4-coloring of G such that $c(u) = c_i(u)$ for all $u \in V(G'_i)$, $i = 1, 2$. Then G is 4-colorable, a contradiction. \square

For the remainder of this section, we choose a 3-separation (G_1, G_2) of a Hajós graph such that G_2 is minimal. We shall show two results concerning certain 4-colorings of G_1 . First, we need some structural information of G_2 .

(3.2) Lemma. *Let G be a Hajós graph, and let (G_1, G_2) be a 3-separation of G chosen to minimize G_2 . Then*

- (i) $|V(G_2)| \geq 5$,
- (ii) $G_2 - V(G_1 \cap G_2)$ is connected, and
- (iii) G_2 is 2-connected.

Proof: Suppose $|V(G_2)| \leq 4$. Then $|V(G_2)| = 4$. Let $v \in V(G_2) - V(G_1)$. Then by Proposition (3.1), v has degree 3 in G . Since $G - v$ does not contain a K_5 -subdivision, $G - v$ is 4-colorable. Because the degree of v in G is 3, any 4-coloring of $G - v$ can easily be extended to v to give a 4-coloring of G , a contradiction. Thus $|V(G_2)| \geq 5$ and (i) holds.

Now suppose $G_2 - V(G_1 \cap G_2)$ is not connected. Let D denote a component of $G_2 - V(G_1 \cap G_2)$. Then there is a 3-separation (G'_1, G'_2) of G with $V(G'_1 \cap G'_2) = V(G_1 \cap G_2)$ and $G'_2 - V(G_1 \cap G_2) = D$. This contradicts the choice of (G_1, G_2) , for G'_2 is properly contained in G_2 . So (ii) holds.

By (ii), $G_2 - V(G_1 \cap G_2)$ is connected, and by Proposition (3.1), every vertex from $V(G_1 \cap G_2)$ has a neighbor in $V(G_2) - V(G_1 \cap G_2)$. So G_2 is connected, and any possible cut vertex of G_2 must be contained in $V(G_2) - V(G_1 \cap G_2)$. Suppose G_2 is not 2-connected, and let v denote a cut vertex of G_2 . Then, $V(G_1 \cap G_2)$ cannot be contained in a component of $G_2 - v$, for otherwise v would be a cut vertex in G (contradicting Proposition (3.1)). So we may assume that some vertex x from $V(G_1 \cap G_2)$ is contained in the component of $G_2 - v$ which does not contain any other vertex from $V(G_1 \cap G_2)$. Then, since G is 3-connected, $xv \in E(G)$. Therefore, since $|V(G_2)| \geq 5$, $(V(G_1 \cap G_2) - \{x\}) \cup \{v\}$ is a cut in G , which yields a 3-separation (G'_1, G'_2) such that $G'_2 = G_2 - x$ is a proper subgraph of G_2 , a contradiction. Hence (iii) holds. \square

(3.3) Proposition. *Let G be a Hajós graph, let (G_1, G_2) be a 3-separation of G chosen to minimize G_2 , and let $V(G_1 \cap G_2) = \{x, y, z\}$. Then there is a 4-coloring c_1 of G_1 such that $c_1(x), c_1(y)$ and $c_1(z)$ are all distinct.*

Proof: Suppose this is not true, that is, $G'_1 := G_1 + \{xy, yz, zx\}$ is not 4-colorable. Then since $|V(G'_1)| < |V(G)|$, G'_1 contains a K_5 -subdivision, say Σ .

We claim that x, y, z are branching vertices of Σ . This is easy to see when $\{xy, xz, yz\} \subseteq E(\Sigma)$. So we may assume by symmetry that $yz \notin E(\Sigma)$. By (iii) of Lemma (3.2), there exist internally disjoint paths Y from x to y and Z from x to z in

G_2 . It is easy to see that $(\Sigma - \{xy, xz\}) \cup Y \cup Z$ (and hence G) contains a K_5 -subdivision, a contradiction.

Note that if G_2 contains a cycle C through x, y, z , then $(\Sigma - \{xy, yz, zx\}) \cup C$ (and hence G) contains a K_5 -subdivision, a contradiction. So there is no cycle through x, y, z in G_2 . Hence by applying Theorem (2.2) to G_2 , it suffices to consider the following three cases.

Case 1. There exists a 2-cut S in G_2 and there exist distinct components D_x, D_y, D_z of $G_2 - S$ such that $u \in V(D_u)$ for each $u \in \{x, y, z\}$.

Let $S = \{a, b\}$. Suppose $|V(D_x)| \geq 2$. Then $\{x, a, b\}$ is a 3-cut of G and $(G - V(D_x - x), G[V(D_x) \cup S])$ is a 3-separation of G . But $G[V(D_x) \cup S]$ is properly contained in G_2 , contradicting the choice of (G_1, G_2) . So $V(D_x) = \{x\}$. Similarly, we have $V(D_y) = \{y\}$, and $V(D_z) = \{z\}$. Hence, a and b are the only vertices of G not contained in G_1 .

Since G_1 is a non-spanning subgraph of G , G_1 is 4-colorable. Let c_1 be a 4-coloring of G_1 . If $c_1(x), c_1(y), c_1(z)$ are all distinct, then c_1 is a 4-coloring of G'_1 , a contradiction. So assume $c_1(x), c_1(y), c_1(z)$ are not all distinct. Define $c'_1(u) = c_1(u)$ for all $u \in V(G_1)$, and let $c'_1(a)$ and $c'_1(b)$ be two colors not in $\{c_1(x), c_1(y), c_1(z)\}$. Clearly, c'_1 is a 4-coloring of G , a contradiction.

Case 2. There exist a vertex v of G_2 , 2-cuts S_x, S_y, S_z in G_2 , and components D_u of $G_2 - S_u$ containing u , for all $u \in \{x, y, z\}$, such that $S_x \cap S_y \cap S_z = \{v\}$, $S_x - \{v\}, S_y - \{v\}, S_z - \{v\}$ are pairwise disjoint, and D_x, D_y, D_z are pairwise disjoint.

As in Case 1, we see that $V(D_x) = \{x\}$, $V(D_y) = \{y\}$ and $V(D_z) = \{z\}$.

Note that $G_1 + xy$ contains no K_5 -subdivision. For otherwise, by replacing xy in such a K_5 -subdivision with an x - y path in $G_2 - z$ (which exists by (iii) of Lemma (3.2)), we produce a K_5 -subdivision in G , a contradiction.

Thus, since $|V(G_1 + xy)| < |V(G)|$, $G_1 + xy$ is 4-colorable. Let c_1 be a 4-coloring of $G_1 + xy$. Then $c_1(x) \neq c_1(y)$. If $c_1(z) \neq c_1(x)$ and $c_1(z) \neq c_1(y)$, then c_1 is a 4-coloring of G'_1 , a contradiction. Therefore, we may assume (by symmetry between x and y) that $c_1(z) = c_1(y)$.

Since G_2 is a non-spanning subgraph of G , G_2 is 4-colorable. Let c_2 be a 4-coloring of G_2 . As y and z only have three neighbors in G_2 , we may choose c_2 so that $c_2(y) = c_2(z)$. Since x has only two neighbors in G_2 , we may further choose c_2 so that $c_2(x) \neq c_2(y)$. Now by permuting the colors of vertices of G_2 , we may assume that $c_2(u) = c_1(u)$ for all $u \in \{x, y, z\}$. Let c be a coloring of G defined by letting $c(u) = c_i(u)$ for all $u \in V(G_i)$, $i = 1, 2$. Then c is a 4-coloring of G , a contradiction.

Case 3. There exist pairwise disjoint 2-cuts S_x, S_y, S_z in G_2 and components D_u of $G_2 - S_u$ containing u , for all $u \in \{x, y, z\}$, such that D_x, D_y, D_z are pairwise disjoint and $G_2 - V(D_x \cup D_y \cup D_z)$ has exactly two components, each containing exactly one vertex from S_u , for all $u \in \{x, y, z\}$.

As in Case 1, we can show $V(D_x) = \{x\}$, $V(D_y) = \{y\}$ and $V(D_z) = \{z\}$.

Let $S_x := \{a_x, b_x\}$, $S_y := \{a_y, b_y\}$, and $S_z := \{a_z, b_z\}$, and assume that $\{a_x, a_y, a_z\}$ (respectively, $\{b_x, b_y, b_z\}$) is contained in the component A (respectively, B) of $G_2 - V(D_x \cup D_y \cup D_z)$. Then $|V(A)| = 3 = |V(B)|$; for otherwise, $(G - V(A - \{a_x, a_y, a_z\}), A)$ or $(G - V(B - \{b_x, b_y, b_z\}), B)$ is a 3-separation of G and A or B is properly contained in G_2 , contradicting the choice of (G_1, G_2) .

Now $G_1 + \{xy, yz\}$ contains no K_5 -subdivision. For otherwise, let Σ be a K_5 -subdivision in $G_1 + \{xy, yz\}$. By (iii) of Lemma (3.2), there are internally disjoint paths X, Z from y to x, z , respectively, in G_2 . Now $(\Sigma - \{xy, yz\}) \cup X \cup Z$ (and hence G) contains a K_5 -subdivision, a contradiction.

Since $|V(G_1 + \{xy, yz\})| < |V(G)|$, $G_1 + \{xy, yz\}$ is 4-colorable. Let c_1 be a 4-coloring of $G_1 + \{xy, yz\}$. Then $c_1(x) \neq c_1(y) \neq c_1(z)$. If $c_1(x) \neq c_1(z)$, then G_1' is 4-colorable, a contradiction. So assume that $c_1(x) = c_1(z)$. For convenience, assume that the colors we use are $\{\alpha, \beta, \gamma, \delta\}$ and $c_1(x) = \alpha$ and $c_1(y) = \beta$. Let c be a coloring of G such that $c(u) = c_1(u)$ for all $u \in V(G_1)$, $c(a_x) = c(b_z) = \gamma$, $c(b_x) = c(a_z) = \beta$, $c(a_y) = \delta$, and $c(b_y) = \alpha$. It is easy to check that c is a 4-coloring of G , a contradiction. \square

(3.4) Proposition. *Let G be a Hajós graph, let (G_1, G_2) be a 3-separation of G chosen to minimize G_2 , and let $V(G_1 \cap G_2) = \{x, y, z\}$. Suppose there is a vertex $x' \in V(G_1) - \{x, y, z\}$ separating x from $\{y, z\}$ in G_1 . Then there exist 4-colorings c_1 and c_2 of G_1 such that $c_1(x) = c_1(y) \neq c_1(z)$ and $c_2(x) = c_2(z) \neq c_2(y)$.*

Proof: Note that $xy, xz \notin E(G)$ for otherwise x' does not separate x from $\{y, z\}$ in G_1 . Also $xx' \in E(G)$, for otherwise $\{x, x'\}$ would be a 2-cut in G , contradicting Proposition (3.1). Let $G_1^* := (G_1 - x) + \{x'y, yz\}$.

We claim that G_1^* contains no K_5 -subdivision. For otherwise, let Σ be a K_5 -subdivision in G_1^* . Since G contains no K_5 -subdivision, $\{x'y, yz\} \cap E(\Sigma) \neq \emptyset$. By (iii) of Lemma (3.2), we see that G_2 contains two internally disjoint paths X, Z from y to x, z , respectively. Now $(\Sigma - \{x'y, yz\}) \cup (X + \{x', xx'\}) \cup Z$, and hence G , contains a K_5 -subdivision, a contradiction.

Therefore, since $|V(G_1^*)| < |V(G)|$, G_1^* is 4-colorable. Let c_1^* be a 4-coloring of G_1^* . Then $c_1^*(x') \neq c_1^*(y) \neq c_1^*(z)$. Define a coloring c_1 of G_1 by letting $c_1(x) = c_1^*(y)$ and $c_1(u) = c_1^*(u)$ for all $u \in V(G_1) - \{x\}$. It is easy to see that c_1 gives the desired 4-coloring of G_1 .

Similarly, by defining $G_1^* := (G_1 - x) + \{x'z, yz\}$, we can show that G_1 has the desired 4-coloring c_2 . \square

Next, we show that if (G_1, G_2) is a 3-separation of a Hajós graph chosen to minimize G_2 , then G_2 admits certain 4-colorings. First, we need the following lemma.

(3.5) Lemma. *Let G be a Hajós graph, and let (G_1, G_2) be a 3-separation of G chosen to minimize G_2 . Then G_1 is 2-connected.*

Proof: Suppose G_1 is not 2-connected. Since G is 3-connected (by Proposition (3.1)), there must exist vertices $x \in V(G_1 \cap G_2)$ and $x' \in V(G_1) - V(G_1 \cap G_2)$ such that x' separates x from $V(G_1 \cap G_2) - \{x\}$. Let y, z denote the other two vertices in $V(G_1 \cap G_2) - \{x\}$. By Proposition (3.4), there exists a 4-coloring c_1 of G_1 such that $c_1(x) = c_1(y) \neq c_1(z)$, and there exists a 4-coloring c'_1 of G_1 such that $c'_1(x) = c'_1(z) \neq c'_1(y)$.

Note that $G_2 + yz$ contains no K_5 -subdivision. For otherwise, let Σ be a K_5 -subdivision in $G_2 + yz$. By Proposition (3.1), $G_1 - x$ has a y - z path P . Now $(\Sigma - yz) \cup P$ (and hence G) contains a K_5 -subdivision, a contradiction. Since $|V(G_2 + yz)| < |V(G)|$, $G_2 + yz$ is 4-colorable. Let c_2 be a 4-coloring of $G_2 + yz$. Then $c_2(y) \neq c_2(z)$.

First, assume that $c_2(y) \neq c_2(x) \neq c_2(z)$. Then c_2 is a 4-coloring of $G_2 + \{xy, xz, yz\}$. By Proposition (3.3), G_1 has a 4-coloring c_1 such that $c_1(x), c_1(y)$ and $c_1(z)$ are all distinct. We may assume c_1 and c_2 use the same set of four colors, and by permuting colors of vertices of G_1 , we have $c_1(u) = c_2(u)$ for all $u \in \{x, y, z\}$. Now define a coloring c of G with $c(u) = c_i(u)$ for all $u \in V(G_i)$, $i = 1, 2$. This shows that G is 4-colorable, a contradiction.

Now by symmetry between y and z (with respect to c_1 and c'_1), we may assume that $c_2(x) = c_2(y) \neq c_2(z)$. We may also assume that c_1 and c_2 use the same set of four colors, and by permuting colors if necessary, $c_1(u) = c_2(u)$ for all $u \in \{x, y, z\}$. Define $c(u) = c_i(u)$ for all $u \in V(G_i)$, $i = 1, 2$. Then it is easy to see that c is a 4-coloring of G , a contradiction. \square

(3.6) Proposition. *Let G be a Hajós graph, let (G_1, G_2) be a 3-separation of G chosen to minimize G_2 , and let $V(G_1 \cap G_2) = \{x, y, z\}$. Let $F \subseteq \{xy, xz, yz\}$. Then $G_2 + F$ is 4-colorable if, and only if, $|F| \leq 2$.*

Proof: First, assume that $|F| = 3$. Then $G_2 + F = G_2 + \{xy, xz, yz\}$. Suppose $G_2 + F$ is 4-colorable, then there is a 4-coloring c_2 of G_2 such that $c_2(x), c_2(y)$ and $c_2(z)$ are all distinct. By Proposition (3.3), let c_1 be a 4-coloring of G_1 such that $c_1(x), c_1(y)$ and $c_1(z)$ are all distinct. Assume that c_1 and c_2 use the same set of four colors. By permuting colors if necessary, we may assume that $c_1(u) = c_2(u)$ for all $u \in \{x, y, z\}$. Let $c(u) = c_i(u)$ for all $u \in V(G_i)$, $i = 1, 2$. Then we see that c is a 4-coloring of G , a contradiction. Hence $G_2 + F$ is not 4-colorable when $|F| = 3$.

Now assume $|F| = 1$. By symmetry, consider $F = \{xy\}$. If $G_2 + xy$ has no K_5 -subdivision, then by the choice of G , we see that $G_2 + xy$ is 4-colorable. So assume that $G_2 + xy$ has a K_5 -subdivision, say Σ . By Proposition (3.5), we see that $G_1 - z$ has an x - y path P . Now $(\Sigma - xy) \cup P$ (and hence G) contains a K_5 -subdivision, a contradiction.

Finally, assume $|F| = 2$. By symmetry, we consider $F = \{xy, xz\}$. If $G_2 + \{xy, xz\}$ contains no K_5 -subdivision then, by the choice of G , we see that $G_2 + \{xy, xz\}$ is 4-

colorable. So we may assume that $G_2 + \{xy, xz\}$ does contain a K_5 -subdivision, and denote it by Σ . By Lemma (3.5), G_1 contains internally disjoint paths Y, Z from x to y, z , respectively. Hence $(\Sigma - \{xy, yz\}) \cup Y \cup Z$ (and hence G) contains a K_5 -subdivision, a contradiction. \square

We conclude this section with a useful observation.

(3.7) Lemma. *Let G be a Hajós graph, and let (G_1, G_2) be a 3-separation of G chosen to minimize G_2 . Then there is no cycle in G_1 containing $V(G_1 \cap G_2)$, and $V(G_1 \cap G_2)$ is an independent set in G_1 .*

Proof: Let $V(G_1 \cap G_2) = \{x, y, z\}$. By Proposition (3.6), $G_2 + \{xy, xz, yz\}$ is not 4-colorable. Hence, $G_2 + \{xy, xz, yz\}$ has a K_5 -subdivision Σ . If there is a cycle C in G_1 through x, y, z , then $(\Sigma - \{xy, yz, zx\}) \cup C$ (and hence G) contains a K_5 -subdivision, a contradiction. So G_1 contains no cycle through x, y, z . Therefore, by Lemma (3.5), $\{x, y, z\}$ must be independent in G_1 . \square

4 4-Connectivity

In this section, we prove Theorem (1.1). First, we prove a lemma.

(4.1) Lemma. *Let G be a Hajós graph, let (G_1, G_2) be a 3-separation of G chosen to minimize G_2 , and let $V(G_1 \cap G_2) = \{x, y, z\}$. Let E_x (respectively, E_y) denote the set of edges of G_1 incident with x (respectively, y), and let G_1^* denote the graph obtained from G_1 by adding the edge yz and identifying x and y as x^* (and deleting multiple edges). Then $E_x \cap E_y = \emptyset$, G_1^* contains a K_5 -subdivision, and for any K_5 -subdivision Σ in G_1^* ,*

- (i) x^* is a branching vertex of Σ ,
- (ii) $yz \notin E(\Sigma)$,
- (iii) $|E_x \cap E(\Sigma)| = 2 = |E_y \cap E(\Sigma)|$, and
- (iv) for any two branching vertices u, v of Σ , there are four internally disjoint u - v paths in Σ .

Proof: For convenience, vertices and edges of G_1 are also viewed as vertices and edges of G_1^* , except for x and y . By Lemma (3.7), $E_x \cap E_y = \emptyset$.

Suppose G_1^* contains no K_5 -subdivision. Then since $|V(G_1^*)| < |V(G)|$, G_1^* is 4-colorable. Hence G_1 has a 4-coloring c_1 such that $c_1(x) = c_1(y) \neq c_1(z)$. By Proposition (3.6), $G_2 + \{xz, yz\}$ is 4-colorable. Let c_2 be a 4-coloring of $G_2 + \{xz, yz\}$. Then $c_2(x) \neq c_2(z) \neq c_2(y)$. If $c_2(x) \neq c_2(y)$ then $G_2 + \{xy, yz, zx\}$ is 4-colorable, contradicting Proposition (3.6). So $c_2(x) = c_2(y)$. We may assume that c_1 and c_2 use the same set

of four colors, and we may permute the colors of vertices of G_1 so that $c_1(u) = c_2(u)$ for all $u \in \{x, y, z\}$. Let $c(u) = c_i(u)$ for all $u \in V(G_i)$, $i = 1, 2$. Then c is a 4-coloring of G , a contradiction.

Now let Σ be a K_5 -subdivision in G_1^* . By (iii) of Lemma (3.2), let P_x (respectively, P_y, P_z) denote a y - z path (respectively, z - x path, x - y path) in $G - x$ (respectively, $G - y, G - z$). For the same reason, G_2 contains internally disjoint paths X_y, X_z from x to y, z , respectively, and internally disjoint paths Y_x, Y_z from y to x, z , respectively.

Suppose x^* is not a branching vertex of Σ . Then since G_1 has no K_5 -subdivision, exactly one branching path of Σ , say R , uses x^* . Let q, r be the neighbors of x^* in R . First assume that $z \in \{q, r\}$, say $z = r$. If $qy \in E(G_1)$ then $((\Sigma - x^*) + \{y, qy\}) \cup P_x$ is a K_5 -subdivision in G , a contradiction. So assume $qx \in E(G_1)$ then $((\Sigma - x^*) + \{x, qx\}) \cup P_y$ is a K_5 -subdivision in G , a contradiction. Now assume that $z \notin \{q, r\}$. If $qx, rx \in E(G_1)$ then $(\Sigma - x^*) + \{x, qx, rx\}$ is a K_5 -subdivision in G_1 , a contradiction. If $qy, ry \in E(G_1)$ then $(\Sigma - x^*) + \{y, qy, ry\}$ is a K_5 -subdivision in G_1 , a contradiction. So assume by symmetry that $qx, ry \in E(G_1)$. Then $((\Sigma - x^*) + \{x, y, qx, ry\}) \cup P_z$ is a K_5 -subdivision in G , a contradiction. Thus x^* is a branching vertex of Σ , and (i) holds.

Suppose $yz \in E(\Sigma)$. Then either $|E_x \cap E(\Sigma)| \leq 1$ or $|E_y \cap E(\Sigma)| \leq 1$. By symmetry, assume that $|E_x \cap E(\Sigma)| \leq 1$. If $|E_x \cap E(\Sigma)| = 0$ then let $yy_1, yy_2, yy_3 \in E_y \cap E(\Sigma)$, and we see that $((\Sigma - x^*) + \{y, yy_1, yy_2, yy_3\}) \cup P_x$ is a K_5 -subdivision in G , a contradiction. So assume $|E_x \cap E(\Sigma)| = 1$. Let $yy_1, yy_2 \in E_y \cap E(\Sigma)$ and $xx' \in E_x \cap E(\Sigma)$. Then $((\Sigma - x^*) + \{x, y, yy_1, yy_2, xx'\}) \cup Y_x \cup Y_z$ is a K_5 -subdivision in G , a contradiction. So $yz \notin E(\Sigma)$, and (ii) holds.

If $|E_x \cap E(\Sigma)| = 0$ or $|E_y \cap E(\Sigma)| = 0$, then by (ii), Σ gives a K_5 -subdivision in G (by simply renaming x^* as y or x), a contradiction. Suppose (iii) fails, and assume by symmetry that $|E_x \cap E(\Sigma)| = 1$ and $|E_y \cap E(\Sigma)| = 3$. Let $xx' \in E_x \cap E(\Sigma)$, $yy_1, yy_2, yy_3 \in E_y \cap E(\Sigma)$. Then $((\Sigma - x^*) + \{x, y, xx', yy_1, yy_2, yy_3\}) \cup P_z$ is a K_5 -subdivision in G , a contradiction. So (iii) must hold.

Clearly, (iv) holds. □

Proof of Theorem (1.1). Suppose the assertion of Theorem (1.1) is not true. Let G be Hajós graph, and assume that G is not 4-connected. By Proposition (3.1), G is 3-connected. Let (G_1, G_2) be a 3-separation of G chosen to minimize G_2 , and let $V(G_1 \cap G_2) = \{x, y, z\}$.

By Lemma (3.7), $\{x, y, z\}$ is not contained in any cycle in G_1 , and $\{x, y, z\}$ is an independent set in G_1 . Let E_x (respectively, E_y) denote the set of edges of G_1 incident with x (respectively, y). Let G_1^* denote the graph obtained from G_1 by adding the edge yz and identifying x and y as x^* (and deleting multiple edges). Then by Lemma (4.1), $E_x \cap E_y = \emptyset$, and G_1^* contains a K_5 -subdivision, say Σ . Note that Σ satisfies (i)–(iv) of Lemma (4.1).

Note that G_1 is 2-connected (by Lemma (3.5)) and G_1 has no cycle containing $\{x, y, z\}$ (by Lemma (3.7)). By applying Theorem (2.2) to G_1 , we consider the following three cases.

Case 1. There exists a 2-cut S in G_1 and there exist three distinct components D_x, D_y, D_z of $G_1 - S$ such that $u \in V(D_u)$ for each $u \in \{x, y, z\}$.

Let $S := \{a, b\}$. By (i) of Lemma (4.1), x^* is a branching vertex of Σ . Therefore, D_z contains no branching vertex of Σ because S and the edge zx^* show that G_1^* contains at most three internally disjoint paths between x^* and D_z , contradicting (iv) of Lemma (4.1). Similarly, either $D_x - x$ or $D_y - y$ has no branching vertex of Σ since $S \cup \{x^*\}$ is a 3-cut in G_1^* separating $D_x - x$ from $D_y - y$.

Therefore, we may assume that all branching vertices of Σ are in $D_x \cup S \cup \{x^*\}$. Then G has at most one a - b path which is contained in Σ , and we denote it by P if it exists. We shall derive a contradiction by either constructing a K_5 -subdivision in G or giving a 4-coloring of G .

By (iii) of Lemma (4.1), $|E_y \cap E(\Sigma)| = 2$. Hence, let P_a, P_b be the paths in Σ from y to a, b , respectively. If $G[V(D_y) \cup S]$ (respectively, $G[V(D_z) \cup S]$) contains internally disjoint paths Y from a to y (respectively, z) and B from a to b , then we can produce a K_5 -subdivision in G as follows: replace P_a, P by Y, B , respectively, replace P_b by a path in $G[V(D_z) \cup \{b\}]$ (respectively, $G[V(D_y) \cup \{b\}]$) from z (respectively, y) to b , and add two internally disjoint paths from x to $\{y, z\}$ in G_2 (which exist by (iii) of Lemma (3.2)). This gives a contradiction. So we may assume that such paths Y, B do not exist in $G[V(D_y) \cup S]$ (respectively, $G[V(D_z) \cup S]$). Then there is a cut vertex a_y (respectively, a_z) of $G[V(D_y) \cup S]$ (respectively, $G[V(D_z) \cup S]$) separating a from $\{y, b\}$ (respectively, $\{z, b\}$). Since G is 3-connected, we see that a_y (respectively, a_z) is the only neighbor of a in $G[V(D_y) \cup S]$ (respectively, $G[V(D_z) \cup S]$).

Similarly, we conclude that b has only one neighbor b_y in $G[V(D_y) \cup S]$, and b has only one neighbor b_z in $G[V(D_z) \cup S]$.

Next we use the above structural information to color vertices of G . By Proposition (3.3), G_1 has a 4-coloring c_1 such that $c_1(x), c_1(y)$ and $c_1(z)$ are all distinct. We shall obtain a new 4-coloring c'_1 of G_1 such that x, y, z use exactly two colors. For convenience, let $\{\alpha, \beta, \gamma, \delta\}$ denote the four colors used by c_1 , and let H_{ij} denote the subgraph of G_1 induced by vertices of color i or j , for all $\{i, j\} \subseteq \{\alpha, \beta, \gamma, \delta\}$. Let $c_1(x) = \alpha$, $c_1(y) = \beta$, and $c_1(z) = \gamma$. Note that $\{y, z\}$ must be contained in a component of $H_{\beta\gamma}$, as otherwise we could switch colors in the component of $H_{\beta\gamma}$ containing y , yielding the desired 4-coloring c'_1 of G_1 . Therefore by symmetry between a and b , we may assume that $c_1(a_y) = \beta = c_1(a_z)$ and $c_1(a) = \gamma$ or $c_1(a_y) = \gamma = c_1(a_z)$ and $c_1(a) = \beta$. By the same argument, $\{x, z\}$ must be contained in a component of $H_{\alpha\gamma}$, and $\{x, y\}$ must be contained in a component of $H_{\alpha\beta}$. Therefore, $c_1(b_y) = \beta$, $c_1(b) = \alpha$, and $c_1(b_z) = \gamma$. But then, neither x nor z can be in the component of $H_{\beta\delta}$ containing y , and neither y nor z is

in the component of $H_{\alpha\delta}$ containing x . Thus we can switch the colors in the component of $H_{\beta\delta}$ containing y and in the component of $H_{\alpha\delta}$ containing x . This yields the desired 4-coloring c'_1 of G_1 , with $c'_1(x) = c'_1(y) = \delta$ and $c'_1(z) = \gamma$.

Now by symmetry, assume that $c'_1(x) = c'_1(y) \neq c'_1(z)$. By Proposition (3.6), $G_2 + \{xz, yz\}$ is 4-colorable. Let c_2 be a 4-coloring of $G_2 + \{xz, yz\}$ using the colors from $\{\alpha, \beta, \gamma, \delta\}$. If $c_2(x) \neq c_2(y)$ then c_2 is a 4-coloring of $G_2 + \{xy, yz, zx\}$, contradicting Proposition (3.6). So $c_2(x) = c_2(y)$. By permuting colors if necessary, we may assume that $c_2(u) = c'_1(u)$ for all $u \in \{x, y, z\}$. Now let $c(u) = c'_1(u)$ for all $u \in V(G_1)$ and $c(u) = c_2(u)$ for all $u \in V(G_2)$. Then c is a 4-coloring of G , a contradiction.

Case 2. There exist a vertex v of G_1 , 2-cuts S_x, S_y, S_z in G_1 , and components D_u of $G_1 - S_u$ containing u , for all $u \in \{x, y, z\}$, such that $S_x \cap S_y \cap S_z = \{v\}$, $S_x - \{v\}, S_y - \{v\}, S_z - \{v\}$ are pairwise disjoint, and D_x, D_y, D_z are pairwise disjoint.

By (i) of Lemma (4.1), x^* is a branching vertex of Σ . Therefore, D_z contains no branching vertex of Σ because S_z and the edge zx^* shows that G_1^* contains at most three internally disjoint paths between x^* and D_z , contradicting (iv) of Lemma (4.1). In fact, all branching vertices of Σ must be contained in $R := V(D_x - x) \cup V(D_y - y) \cup S_x \cup S_y \cup \{x^*\}$. For otherwise, Σ has a branching vertex $v \notin R$, and Σ must have four disjoint path leaving R . But this forces $yz \in E(\Sigma)$, contradicting (ii) of Lemma (4.1).

We claim that, for each $u \in \{x, y\}$, not all branching vertices of Σ are contained in $V(D_u) \cup S_u \cup \{x^*\}$. For otherwise, suppose by symmetry that all branching vertices of Σ are contained in $V(D_x) \cup S_x \cup \{x^*\}$. By (iii) of Lemma (4.1), let x^*s, x^*t be the two edges in $E(\Sigma) \cap E_x$, let x^*q, x^*r be the two edges in $E(\Sigma) \cap E_y$, and let B_q, B_r be the branching paths in Σ containing x^*q, x^*r , respectively. Since $yz \notin E(\Sigma)$ (by (ii) of Lemma (4.1)), both B_q and B_r have an x^* - S_y subpath whose internal vertices are all contained in D_y . Let P_{xy}, P_{xz} be two internally disjoint paths in G_2 from x to y, z , respectively, which exist by (iii) of Lemma (3.2). Note that there exists an $(S_z - \{v\})$ - $(S_x - \{v\})$ path Q_{xz} in $(G_1 - v) - V(D_x \cup D_y \cup D_z)$; for otherwise, one of $\{v, x\}, \{v, z\}$ is a 2-cut in G , contradicting Proposition (3.1). Let Y be a y - v path in D_y and let Z be a z - $(S_z - \{v\})$ path in D_z . Then

$$(((\Sigma - x^*) + \{x, xs, xt\}) - (V(B_q \cup B_r) - (V(D_x) \cup S_x))) \cup (P_{xy} \cup Y) \cup (P_{xz} \cup Z \cup Q_{xz})$$

is a K_5 -subdivision in G , a contradiction.

Since $|\{x^*\} \cup S_x \cup S_y| = 4$, there must exist a branching vertex x' of Σ such that $x' \in V(D_x - x) \cup V(D_y - y)$. By symmetry, we may assume that $x' \in V(D_x - x)$. Hence by the above claim, there is also a branching vertex y' of Σ such that $y' \in V(D_y - y) \cup (S_y - \{v\})$. Now $S_x \cup \{x^*\}$ is a 3-cut in Σ separating x' from y' , contradicting (iv) of Lemma (4.1).

Case 3. There exist pairwise disjoint 2-cuts S_x, S_y, S_z in G_1 and components D_u of $G_1 - S_u$ containing u , for all $u \in \{x, y, z\}$, such that D_x, D_y, D_z are pairwise disjoint and

$G_1 - V(D_x \cup D_y \cup D_z)$ has exactly two components, each containing exactly one vertex from S_u , for all $u \in \{x, y, z\}$.

Let $S_x = \{a_x, b_x\}$, $S_y = \{a_y, b_y\}$, and $S_z = \{a_z, b_z\}$ such that $\{a_x, a_y, a_z\}$ is contained in a component A of $G_1 - V(D_x \cup D_y \cup D_z)$, and $\{b_x, b_y, b_z\}$ is contained in another component B of $G_1 - V(D_x \cup D_y \cup D_z)$.

As in Case 1 and Case 2, we can show that all branching vertices of Σ are in $R \cup S_z$, where $R := V(D_x - x) \cup V(D_y - y) \cup S_x \cup S_y \cup \{x^*\}$. In fact, all branching vertices of Σ must be in R . For otherwise, assume by symmetry that a_z is a branching vertex of Σ . Then, since $yz \notin E(\Sigma)$ (by (ii) of Lemma (4.1)), $\{b_z, a_x, a_y\}$ shows that Σ cannot contain four internally disjoint paths between a_z and x^* , contradicting (iv) of Lemma (4.1).

We claim that, for each $u \in \{x, y\}$, not all branching vertices of Σ are contained in $V(D_u) \cup S_u \cup \{x^*\}$. For otherwise, we may assume that all branching vertices of Σ are contained in $V(D_x) \cup S_x \cup \{x^*\}$. By (iii) of Lemma (4.1), let x^*s, x^*t be the two edges in $E(\Sigma) \cap E_x$, let x^*q, x^*r be the two edges in $E(\Sigma) \cap E_y$, and let A_q, B_r be the branching paths in Σ containing x^*q, x^*r , respectively. Since $yz \notin E(\Sigma)$, both A_q and B_r have an x^*-S_y subpath whose internal vertices are all contained in D_y . Let P_{xy}, P_{xz} be two internally disjoint paths in G_2 from x to y, z , respectively, which exist by (iii) of Lemma (3.2). Note that there exists an a_y-a_x path Q_{xy} in A (since A is connected) and there exists a b_z-b_x path Q_{xz} in B (since B is connected). Let Y be an $y-a_y$ path in $G[V(D_y) \cup \{a_y\}]$ and let Z be an $z-b_z$ path in $G[V(D_z) \cup \{b_z\}]$. Then,

$$(((\Sigma - x^*) + \{x, xs, xt\}) - (V(A_q \cup B_r) - (V(D_x) \cup S_x))) \cup (P_{xy} \cup Y \cup Q_{xy}) \cup (P_{xz} \cup Z \cup Q_{xz})$$

is a K_5 -subdivision in G , a contradiction.

We further claim that the set of branching vertices of Σ is $S_x \cup S_y \cup \{x^*\}$. For otherwise, there must be a branching vertex x' of Σ such that $x' \in V(D_x - x) \cup V(D_y - y)$. By symmetry, we may assume that $x' \in V(D_x - x)$. Then by the above claim, there is a branching vertex y' of Σ such that $y' \in V(D_y - y) \cup S_y$. Now $S_x \cup \{x^*\}$ is a 3-cut in Σ separating x' from y' , contradicting (iv) of Lemma (4.1).

Since $yz \notin E(\Sigma)$, we see that Σ must contain two branching paths from $\{a_x, a_y\}$ to $\{b_y, b_z\}$ which are also contained in $G_1 - V(D_x \cup D_y)$. But this is impossible, because a_z separates $\{a_x, a_y\}$ from $\{b_y, b_z\}$ in $G_1 - V(D_x \cup D_y)$, a contradiction. \square

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