Disjoint Paths in Graphs I, 3-planar graphs and basic obstructions

Xingxing Yu*
School of Mathematics
Georgia Institute of Technology
Atlanta, GA 30332

Abstract

Let G be a graph, $\{a, b, c\} \subseteq V(G)$, and $\{a', b', c'\} \subseteq V(G)$ such that $\{a, b, c\} \neq \{a', b', c'\}$. We say that $(G, \{a, c\}, \{a', c'\}, (b, b'))$ is an *obstruction* if, for any three vertex disjoint paths from $\{a, b, c\}$ to $\{a', b', c'\}$ in G, one path is from b to b'. Robertson and Seymour asked the problem of characterizing all obstructions. In this paper, we present a list of "basic" obstructions and show how to produce other obstructions from these basic ones. We also prove results about disjoint paths in graphs. Results in this paper will be used in subsequent papers to characterize all obstructions.

*Partially supported by NSF grant DMS-9970527 and NSA grant MSPF-02G-074

1 Introduction and notation

Hajós (see [4]) conjectured that any graph containing no subdivision of K_{k+1} is k-colorable. While this is true for $k \leq 3$, Catlin [2] disproved it for all $k \geq 6$. However, Hajós' conjecture remains open for $k = 4$ and $k = 5$. For the case $k = 4$, it is essential to know the structure of graphs containing no subdivisions of K_5. Seymour [8] conjectured in 1975 that every 5-connected non-planar graph contains a subdivision of K_5. In [5], it is shown that Seymour's conjecture implies a conjecture of Dirac [3]. Although Dirac's conjecture was solved by Mader [6], Seymour's conjecture remains open.

One possible approach to the aforementioned conjectures is to study the following *rooted-K_4* problem.

(1.1) Problem. *Given a graph G and $\{u, v, w, x\} \subseteq V(G)$, determine whether G contains a subdivision of K_4 in which u, v, w, x are the vertices of degree three.*

In [11], the rooted-K_4 problem is solved for planar graphs. It seems possible to generalize the approach in [11] to all graphs. In order to extend a key result ([11], Theorem (2.1)) to general graphs, we need to characterize "obstructions". An *obstruction* consists of a graph G and two sets of vertices $\{a, b, c\}$ and $\{a', b', c'\}$ of G such that $\{a, b, c\} \neq \{a', b', c'\}$ and, for any three vertex disjoint paths between $\{a, b, c\}$ and $\{a', b', c'\}$ in G, one path is from b to b'.

In connection with their graph minors project, Robertson and Seymour [7] studied the following problem.

(1.2) Problem. *Characterize obstructions.*

In a series of three papers, we will solve (1.2). The main purpose of this paper is to give a list of basic obstructions and describe how they can be used to produce all obstruction. We will also prove a few results about disjoint paths in graphs. The second paper of this series characterizes a special class of obstructions, which is then used in the third paper to complete the characterization of all obstructions. We mention that some basic obstructions and constructions were already known to Robertson and Seymour.

This paper is organized as follows. In the remainder of this section, we introduce necessary notation and terminology. In Section 2, we describe a class of graphs, called 3-planar graphs, and use it to re-formulate a known result about disjoint paths. In Section 3, we prove several results about disjoint paths in graphs. In Section 4, we give a list of basic obstructions and describe how to produce other obstructions from these basic ones.

Only finite simple graphs are considered. We use $\emptyset$ to denote both the empty set and the null graph. Let G be a graph. Then $V(G)$ and $E(G)$ denote the vertex set and edge set of G, respectively. By $H \subseteq G$, we mean that H is a subgraph of G. Let $Y \subseteq V(G)$;

then $N_G(Y)$, or $N(Y)$ if G is understood, denotes the set of vertices in $V(G) - Y$ each of which is adjacent to a vertex in Y. If $Y = \{y\} \subseteq V(G)$, then we use $N_G(y)$ or $N(y)$ instead of $N_G(\{y\})$ or $N(\{y\})$.For any $H \subseteq G$, we write $N(H)$ instead of $N(V(H))$. For $X \subseteq V(G)$ or $X \subseteq E(G)$, $G[X]$ denotes the subgraph of G induced by X. For $X \subseteq E(G)$, $G - X$ denotes the graph obtained from G by deleting X. For $X \subseteq V(G)$ or $X \subseteq G$, $G - X$ denotes the graph obtained from G by deleting the vertices in X and those edges of G incident with vertices in X. If $x \in V(G) \cup E(G)$, we write $G - x$ instead of $G - \{x\}$. If there is no danger of confusion, we write $S \subseteq G$ instead of $S \subseteq V(G)$ or $S \subseteq E(G)$, and write $x \in G$ instead of $x \in V(G)$ or $x \in E(G)$. Let T be a set of 2-element subsets of $V(G)$; then $G + T$ denotes the graph with vertex set $V(G)$ and edge set $E(G) \cup T$, and if $T = \{\{x, y\}\}$, then we write $G + xy$ instead of $G + T$.

For subgraphs G and H of a graph, $G \cup H$ and $G \cap H$ denote the union and intersection of G and H, respectively. We say that G and H are *disjoint* if $V(G) \cap V(H) = \emptyset$. We use $G - H$ instead of $G - (G \cap H)$.

The *ends* of a path P are the vertices of minimum degree in P. A path P with ends u and v is also said to be *from u to v* or *between u and v*. Given a path P in a graph and $x, y \in V(P)$, we use xPy to denote the subpath of P between x and y (inclusive). A path P from x to y in a graph G is said to be *internally disjoint* from $H \subseteq G$ if $(P - \{x, y\}) \cap H = \emptyset$.

A *separation* in a graph G is a pair (G_1, G_2) of subgraphs of G such that $G = G_1 \cup G_2$, $E(G_1 \cap G_2) = \emptyset$, and $E(G_i) \cup V(G_i - G_{3-i}) \neq \emptyset$ for $i \in \{1, 2\}$. If $|V(G_1 \cap G_2)| = k$, then (G_1, G_2) is a *k-separation* in G. A set $S \subseteq V(G)$ is a *k-cut* or a *cut set* of size k in G, where k is a positive integer, if $|S| = k$ and G has a separation (G_1, G_2) such that $V(G_1 \cap G_2) = S$ and $V(G_i - S) \neq \emptyset$ for $i \in \{1, 2\}$. If $v \in V(G)$ and $\{v\}$ is a cut set of G, then v is said to be a *cut vertex* of G.

Finally, we use $A := B$ to rename B as A.

2 3-Planar graphs

The goal of this section is to give an equivalent formulation of a theorem about disjoint paths in graphs, which is, for our purpose, more convenient to use.

A *plane graph* is a graph drawn in the plane with no pair of edges crossing, its vertices are points in the plane and edges are polygonal arcs in the plane. Let G be a plane graph. A *face* of G is a connected component (in topological sense) of the complement of G in the plane. The *infinite* face of G is its unbounded face. A cycle in G is *facial* if it bounds a face of G. If G is 2-connected, then the cycle of G bounding its infinite face is its *outer cycle*. For a cycle C in G and distinct $x, y \in V(C)$, xCy denotes the clockwise path in C from x to y. A *plane subgraph* of G is a subgraph of G inheriting the drawing of G.

A *closed disc* in the plane and its *boundary* are the homeomorphic images of $\{(x, y) :$

$x^2 + y^2 \leq 1\}$ and $\{(x, y) : x^2 + y^2 = 1\}$, respectively. If D is a closed disc in the plane, then ∂D denotes its boundary.

For convenience, we introduce the following.

(2.1) Definition. Let G be a graph with two pairs of vertices $\{s_i, t_i\}$, $i = 1, 2$. An $(\{s_1, t_1\}, \{s_2, t_2\})$-*linkage* in G consists of paths S_i, $i = 1, 2$, in G with ends s_i and t_i such that $(S_1 - \{s_1, t_1\}) \cap (S_2 - \{s_2, t_2\}) = \emptyset$.

Seymour proved the following result ([9], Theorem 4.1). In [9], $N(A)$ is denoted by $\Delta(A)$.

(2.2) Theorem. *Let G be a graph with distinct vertices s_1, s_2, t_1, and t_2. Then G contains no $(\{s_1, t_1\}, \{s_2, t_2\})$-linkage iff, for some $k \geq 0$, there are pairwise disjoint sets $A_1, \ldots, A_k \subseteq V(G) - \{s_1, t_1, s_2, t_2\}$ such that*

(a) for $1 \leq i \neq j \leq k$, $N(A_i) \cap A_j = \emptyset$,

(b) for $1 \leq i \leq k$, $|N(A_i)| \leq 3$, and

(c) if G' is the graph obtained from G by (for each i) deleting A_i and adding new edges joining every pair of distinct vertices in $N(A_i)$, and also for $j = 1, 2$, adding an edge e_j from s_j to t_j, then G' may be drawn in the plane with no pair of edges crossing except e_1 and e_2 which cross once.

We make two observations about Seymour's theorem.

First, (c) can be re-formulated as follows: (c$'$) the graph $G'' := G' - \{e_1, e_2\}$ can be drawn in a closed disc D with no pair of edges crossing so that s_1, s_2, t_1, t_2 occur on ∂D in that cyclic order.

Second, let $\mathcal{A} = \{A_1, \ldots, A_k\}$ ($\mathcal{A} = \emptyset$ when $k = 0$). We can choose $\mathcal{A}$ so that, subject to (a), (b), and (c$'$), the following property holds: if $|N(A_i)| = 3$ then $N(A_i)$ induces a facial triangle in G''.

To see the second observation, we may choose $\mathcal{A}$ such that, subject to (a), (b) and (c$'$), the number of non-facial triangles in G'' induced by members of $\mathcal{A}$ is minimum. Suppose, without loss of generality, that $|N(A_1)| = 3$ and $N(A_1)$ induces a triangle T_1 in G'' which is not facial. Let $D_1 \subseteq V(G'')$ such that, for each $x \in V(G'')$, $x \in D_1$ iff x is contained in the closed disc bounded by T_1. Define $A_1' \subseteq V(G)$ such that, for each $x \in V(G)$, $x \in A_1'$ iff $x \in D_1 - N(A_1)$ or $x \in A_j$ for some A_j with $N(A_j) \subseteq D_1$. Let $\mathcal{A}' = (\mathcal{A} - \{A_j : N(A_j) \subseteq D_1\}) \cup \{A_1'\}$. Then (a), (b), and (c$'$) hold with $\mathcal{A}'$ in place of $\mathcal{A}$. But the number of non-facial triangles in the new G'' induced by the members of $\mathcal{A}'$ is smaller, contradicting the choice of $\mathcal{A}$.

With these observations, we introduce the following definition.

(2.3) Definition. A *3-planar graph* $(G, \mathcal{A})$ consists of a graph G and a set $\mathcal{A} = \{A_1, \ldots, A_k\}$ of pairwise disjoint subsets of $V(G)$ (possibly $\mathcal{A} = \emptyset$) such that

(a) for $1 \le i \ne j \le k$, $N(A_i) \cap A_j = \emptyset$,

(b) for $1 \le i \le k$, $|N(A_i)| \le 3$, and

(c) if $p(G, \mathcal{A})$ denotes the graph obtained from G by (for each i) deleting A_i and adding new edges joining every pair of distinct vertices in $N(A_i)$, then $p(G, \mathcal{A})$ can be drawn in a closed disc D with no pair of edges crossing such that, for each A_i with $|N(A_i)| = 3$, $N(A_i)$ induces a facial triangle in $p(G, \mathcal{A})$.

If, in addition, $b_0, b_1, \ldots, b_n$ are some vertices in G such that $b_i \notin A_j$ for any $A_j \in \mathcal{A}$ and $b_0, b_1, \ldots, b_n$ occur on ∂D in that cyclic order, then we say that $(G, \mathcal{A}, b_0, b_1, \ldots, b_n)$ is *3-planar*. If there is no need to specify $\mathcal{A}$, we will simply say that $(G, b_0, b_1, \ldots, b_n)$ is 3-planar.

One may think of $p(G, \mathcal{A})$ as a "projection" of $(G, \mathcal{A})$. In later proofs, we will often need to go back and forth between $(G, \mathcal{A})$ and $p(G, \mathcal{A})$. Hence, it will be convenient for us to identify the vertices in $p(G, \mathcal{A})$ and those vertices in G which are not contained in any member of $\mathcal{A}$.

The aforementioned result of Seymour can now be stated as follows; different versions of this result were proved independently by Chakravarty and Robertson [1] and Thomassen [10].

(2.4) Theorem. *Let G be a graph and $\{s_1, s_2, t_1, t_2\} \subseteq V(G)$. Then G contains no $(\{s_1, s_2\}, \{t_1, t_2\})$-linkage iff (G, s_1, s_2, t_1, t_2) is 3-planar.*

When dealing with 3-planar graphs $(G, \mathcal{A})$, it will be convenient to choose $\mathcal{A}$ to be "minimal".

(2.5) Definition. Let $(G, \mathcal{A})$ be 3-planar, let $A \in \mathcal{A}$ such that $N(A) = \{a_1, \ldots, a_m\}$ (where $m \le 3$), and let $H = G[A \cup N(A)]$. We say that A is *minimal* if there is no collection $\mathcal{H}$ of pairwise disjoint subsets of A such that $\mathcal{H} \ne \{A\}$ and $(H, \mathcal{H}, a_1, \ldots, a_m)$ is 3-planar. We say that $\mathcal{A}$ is *minimal* if every member of $\mathcal{A}$ is minimal.

(2.6) Proposition. *If $(G, b_0, \ldots, b_n)$ is 3-planar, then there is a collection $\mathcal{A}$ of pairwise disjoint subsets of $V(G) - \{b_0, \ldots, b_n\}$ such that $(G, \mathcal{A}, b_0, \ldots, b_n)$ is 3-planar and $\mathcal{A}$ is minimal.*

Proof. To see this, we define the following relation: for any two collections $\mathcal{A}_i$, $i = 1, 2$, of subsets of $V(G) - \{b_0, \ldots, b_n\}$ such that $(G, \mathcal{A}_i, b_0, \ldots, b_n)$ is 3-planar, we say that $\mathcal{A}_1 \le \mathcal{A}_2$ if every member of $\mathcal{A}_1$ is a subset of a member of $\mathcal{A}_2$. It is easy to see that this is a partial order relation. We choose $\mathcal{A}$ to be minimal with respect to this partial order.

We claim that $\mathcal{A}$ is minimal. For otherwise, there exists some $A \in \mathcal{A}$ such that A is not minimal. Let $N(A) = \{a_1, \ldots, a_m\}$ where $m \leq 3$, and let $H := G[A \cup N(A)]$. Then there is a collection $\mathcal{H}$ of pairwise disjoint subsets of $V(H) - \{a_1, \ldots, a_m\} = A$ such that $\mathcal{H} \neq \{A\}$ and $(H, \mathcal{H}, a_1, \ldots, a_m)$ is 3-planar. Let $\mathcal{A}' = (\mathcal{A} - \{A\}) \cup \mathcal{H}$. Clearly, $\mathcal{A}'$ satisfies (a) and (b) of (2.3). It is also easy to see that $\mathcal{A}'$ satisfies (c) of (2.3), because $p(G, \mathcal{A}')$ can be drawn in a closed disc in the plane with no pair of edges crossing (by drawing $p(H, \mathcal{H})$ in place of the complete subgraph of $p(G, \mathcal{A})$ induced by $N(A)$). This can be done because if $|N(A)| = 3$ then $N(A)$ induces a facial triangle of $p(G, \mathcal{A})$. Thus $(G, \mathcal{A}', b_0, \ldots, b_n)$ is 3-planar. But $\mathcal{A}' \leq \mathcal{A}$ and $\mathcal{A}' \neq \mathcal{A}$, contradicting the minimality of $\mathcal{A}$. □

Therefore, for any 3-planar graph $(G, \mathcal{A})$, we may always choose $\mathcal{A}$ to be minimal.

3 Disjoint paths

The purpose of this section is to prove several results to be used in later papers to characterize obstructions. Also, the proofs of these results should serve as a warm-up for arguments involving $(G, \mathcal{A})$ and $p(G, \mathcal{A})$. The first result of this section says that if $(G, \mathcal{A})$ is 3-planar and $A \in \mathcal{A}$ is minimal, then $G[A \cup N(A)]$ has certain connectivity properties.

(3.1) Proposition. *Let $(G, \mathcal{A})$ be 3-planar, let $A \in \mathcal{A}$ with $N(A) = \{u, v, w\}$, and let $H := G[A \cup \{u, v, w\}]$. Suppose that A is minimal. Then*

(1) for any proper subset T of $N(A)$, $H - T$ is connected,

(2) for any $x \in A$, H contains an $(\{x, u\}, \{v, w\})$-linkage, and

(3) for any $x \in A$, $N(A)$ is contained in a component of $H - x$.

Proof. Suppose (1) fails. Then there exists a proper subset T of $N(A)$ such that $H - T$ is not connected. Hence, H has a separation (H_1, H_2) such that $V(H_1 \cap H_2) = T$ and $V(H_i) - T \neq \emptyset$ for $i = 1, 2$. In this case, let $S_i := V(H_i) - N(A)$. Because $T \subseteq N(A)$, $N(S_i) \subseteq N(A)$. Also $N(S_1) \cap N(S_2) \subseteq T$. Let $\mathcal{H} = \{S_1, S_2\}$. Then $\mathcal{H} \neq \{A\}$ and $(H, \mathcal{H}, u, v, w)$ is 3-planar, contradicting the minimality of A.

Now assume that (2) fails. Then there is some $x \in A$ such that H has no $(\{x, u\}, \{v, w\})$-linkage. So by (2.4), there is a collection $\mathcal{H}$ of pairwise disjoint subsets of $V(H) - \{x, u, v, w\}$ such that $(H, \mathcal{H}, x, v, u, w)$ is 3-planar. Clearly, $\mathcal{H} \neq \{A\}$, contradicting the minimality of A.

Finally, suppose (3) fails. Then there is some $x \in A$ such that $N(A)$ is not contained in any component of $H - x$. Hence, H has a separation (H_1, H_2) such that $V(H_1 \cap H_2) = \{x\}$ and $V(H_i) \cap N(A) \neq \emptyset$ for $i = 1, 2$. Let $S_i := V(H_i) - (N(A) \cup \{x\})$, $i = 1, 2$, and let

$\mathcal{H}$ be the set consisting of the non-empty S_i's. Then $\mathcal{H} \neq \{A\}$ and $|N(S_i)| \leq 3$. Clearly, $(H, \mathcal{H}, u, v, w)$ is 3-planar, contradicting the minimality of A. $\square$

The next result is a technical lemma which shows a way to extend a linkage in $p(G, \mathcal{A})$ to a linkage in $(G, \mathcal{A})$.

(3.2) Proposition. *Let $(G, \mathcal{A})$ be 3-planar and let $b, b' \in V(G)$ be distinct. Let $u \in V(G)$, let $u^* = u$ when $u \in p(G, \mathcal{A})$, and let u^* be an arbitrary vertex in $N(U_u)$ when $u \in U_u$ for some $U_u \in \mathcal{A}$. Let $v \in V(G)$, let $v^* = v$ when $v \in p(G, \mathcal{A})$, and let v^* be an arbitrary vertex in $N(U_v)$ when $v \in U_v$ for some $U_v \in \mathcal{A}$. Suppose*

- *(i) $u^* \neq v^*$, and $U_u \neq U_v$ when both U_u and U_v are defined,*
- *(ii) for any $A \in \mathcal{A}$ and for any proper subset T of $N(A)$, $G[A \cup N(A)] - T$ is connected,*
- *(iii) for any $A \in \mathcal{A}$ with $|N(A)| = 3$, A is minimal, and*
- *(iv) $p(G, \mathcal{A})$ contains a $(\{u^*, b\}, \{v^*, b'\})$-linkage L^*.*

Then G contains a $(\{u, b\}, \{v, b'\})$-linkage L such that, for any vertex z of $p(G, \mathcal{A})$, z is contained in a path in L iff z is contained in a path in L^*.

Proof. Let P^* and Q^* be the paths in L^* such that $u^* \in P^*$ and $v^* \in Q^*$. We will find P and Q by adding a path from u to u^* in G and a path from v to v^* in G, and by replacing each edge $xy \in E(P^* \cup Q^*)$ with a path from x to y in G.

First, we describe paths P_u and P'_u in G. If $u = u^*$, let $P'_u = \emptyset$ and let P_u consist of the vertex u. Now assume that $u \neq u^*$. Then $u \in U_u \in \mathcal{A}$. Suppose $|N(U_u)| = 2$. Let P'_u be the path consisting of the only vertex in $N(U_u) - \{u^*\}$. By (ii), $G[U_u \cup N(U_u)] - P'_u$ contains a path from u to u^*, and let P_u denote one such path. So assume $|N(U_u)| = 3$, and let $N(U_u) = \{u', u'', u^*\}$. By (iii), U_u is minimal. Hence by (2) of (3.1) (with U_u, u^*, u, u', u'' as A, x, u, v, w, respectively), $G[U_u \cup N(U_u)]$ contains disjoint paths P_u, P'_u from u^*, u' to u, u'', respectively. It is easy to see that $P_u \cap P'_u = \emptyset$

We now describe paths Q_v and Q'_v in G. If $v = v^*$, let $Q'_v = \emptyset$ and let Q_v consist of the vertex v. Now assume that $v \neq v^*$. Then $v \in U_v \in \mathcal{A}$. Suppose $|N(U_v)| = 2$. Let Q'_v denote the path consisting of the only vertex in $N(U_v) - \{v^*\}$. By (ii), $G[U_v \cup N(U_v)] - Q'_v$ contains a path from v to v^*, and let Q_v denote one such path. So assume that $|N(U_v)| = 3$, and let $N(U_v) = \{v', v'', v^*\}$. By (iii), U_v is minimal. Then by (2) of (3.1) (with U_v, v^*, v, v', v'' as A, x, u, v, w, respectively), $G[U_v \cup N(U_v)]$ contains disjoint paths Q_v, Q'_v from v^*, v' to v, v'', respectively. It is easy to see that $Q_v \cap Q'_v = \emptyset$.

Next we describe P_{xy} (respectively, Q_{xy}) for each $xy \in E(P^*)$ (respectively, $xy \in E(Q^*)$). If $xy \in E(G)$, let P_{xy} (respectively, Q_{xy}) be the path such that xy is its only edge. Now assume $xy \notin E(G)$. Then $\{x, y\} \subseteq N(U)$ for some $U \in \mathcal{A}$. If $U = U_u$ then let $P_{xy} = P'_u$ (respectively, $Q_{xy} = P'_u$), and if $U = U_v$ then let $P_{xy} = Q'_v$ (respectively, $Q_{xy} = P'_v$). Suppose $U \neq U_u$ and $U \neq U_v$. If $|N(U)| = 2$, then by (ii), $G[U \cup N(U)]$ has a path P_{xy} (respectively, Q_{xy}) from x to y. So assume $|N(U)| = 3$. By (iii), U is minimal. Hence by (2) of (3.1) (with U, x, y as A, v, w, respectively), $G[U \cup N(U)] - (N(U) - \{x, y\})$ contains a path P_{xy} (respectively, Q_{xy}) from x to y.

Let P be obtained from $P^* \cup P_u$ by replacing all $xy \in E(P^*)$ with P_{xy}. Let Q be obtained from $Q^* \cup Q_v$ by replacing all $xy \in E(Q^*)$ with Q_{xy}. Then P is a path in G from u to b and Q is a path in G from v to b'. It is clear that, for any vertex z of $p(G, \mathcal{A})$, $z \in P \cup Q$ iff $z \in P^* \cup Q^*$.

Note that $P \cap Q = \emptyset$ (because $|N(U)| \leq 3$ for all $U \in \mathcal{A}$, $P^* \cap Q^* = \emptyset$, $P_u \cap P'_u = \emptyset$, and $Q_v \cap Q'_v = \emptyset$). Hence P and Q form the desired $(\{u, b\}, \{v, b'\})$-linkage L in G. $\square$

Remark. We note that in most cases when (3.2) is applied, $\mathcal{A}$ is minimal. In such cases, condition (ii) of (3.2) is satisfied because of (1) of (3.1), and condition (iii) of (3.2) is obviously satisfied.

The next result characterizes 2-connected plane graphs which do not contain certain linkages.

(3.3) Theorem. *Let G be a 2-connected plane graph with outer cycle C, let $b, b' \in V(C)$ be distinct, let $x \in V(G) - V(C)$, and let $y \in V(G) - \{b, b', x\}$. Suppose that*

(i) G contains a path from x to $bCb' - \{b, b'\}$ and a path from x to $b'Cb - \{b, b'\}$, both internally disjoint from C, and

(ii) G contains no $(\{x, b\}, \{y, b'\})$-linkage or G contains no $(\{x, b'\}, \{y, b\})$-linkage.

Then there is a vertex $z \in V(G)$ such that (1) either x and b belong to different components of $G - \{z, b'\}$ or x and b' belong to different components of $G - \{z, b\}$, and (2) either y and b belong to different components of $G - \{z, b'\}$ or y and b' belong to different components of $G - \{z, b\}$.

Proof. By (i), let P and Q be paths in G from x to $p \in V(bCb') - \{b, b'\}$ and $q \in V(b'Cb) - \{b, b'\}$, respectively, such that $(P - p) \cap C = \emptyset$ and $(Q - q) \cap C = \emptyset$. Let $u \in V(P \cap Q)$ such that $V(uPp \cap uQq) = \{u\}$. Without loss of generality, we may assume that $xPu = xQu$. Clearly, $u \notin C$. Let $C_b := bCp \cup pPu \cup uQq \cup qCb$, and let $C'_b := pCb' \cup b'Cq \cup qQu \cup uPp$. Let G_b (respectively, G'_b) denote the subgraph of G contained in the closed disc in the plane bounded by C_b (respectively, C'_b).

Case 1. $x \in G_b$ and $y \in G'_b$ or $x \in G'_b$ and $y \in G_b$.

By symmetry, we may assume that $x \in G_b$ and $y \in G'_b$. Let $z := u$.

First, assume that y and b are contained in a component of $G - \{z, b'\}$. Then $G'_b - \{z, b'\}$ contains a path Y from y to some $y' \in V(C'_b) - \{z, b'\}$ such that $V(Y \cap C'_b) = \{y'\}$. By symmetry between zC'_bb' and $b'C'_bz$, we may assume that $y' \in zC'_bb' - \{z, b'\}$. Then $Q \cup qCb$ and $Y \cup y'C'_bb'$ form an $(\{x, b\}, \{y, b'\})$-linkage, and $Q \cup b'Cq$ and $Y \cup bCy'$ (if $y' \in pC'_bb'$) or $Y \cup bC_by'$ (if $y' \in zC'_bp$) form an $(\{x, b'\}, \{y, b\})$-linkage, contradicting (ii).

Therefore, y and b must belong to different components of $G - \{z, b'\}$, and we have (2). Because G is 2-connected and $y \neq b'$, G'_b contains a path Y from y to z internally disjoint from C_b.

Now assume that x and b' are contained in a component of $G - \{z, b\}$. Then $G_b - \{z, b\}$ contains a path X from x to some $x' \in V(C_b) - \{z, b\}$. By symmetry between bC_bz and zC_bb, we may assume that $x' \in bC_bz - \{z, b\}$. Then $X \cup bC_bx'$ and $Y \cup b'C'_bz$ form an $(\{x, b\}, \{y, b'\})$-linkage, and $Y \cup zC_bb$ and $X \cup x'Cb'$ (if $x' \in bC_bp$) or $X \cup x'C'_bb'$ (if $x' \in pC_bz$) form an $(\{x, b'\}, \{y, b\})$-linkage, contradicting (ii).

Hence x and b' must belong to different components of $G - \{z, b\}$, and we have (1).

Case 2. $\{x, y\} \subseteq G_b$ or $\{x, y\} \subseteq G'_b$.

By symmetry, we may assume that $\{x, y\} \subseteq G_b$. We may further assume that there are disjoint paths in $G_b - b$ from $\{x, y\}$ to $C_b - b$. For otherwise, there is $z \in V(G_b - b)$ such that any component of $G_b - \{z, b\}$ containing a vertex of $C_b - \{z, b\}$ contains neither x nor y. Therefore, x and b' belong to different components of $G - \{z, b\}$ and y and b' belong to different components of $G - \{z, b\}$, and so, (1) and (2) hold.

Thus, let X, Y denote two disjoint paths in $G_b - b$ from x, y to $x', y' \in V(C_b - b)$, respectively, such that X and Y are internally disjoint from C_b. Note that $x = x'$ if $x \in C_b$ and $y = y'$ if $y \in C_b$.

We claim that X and Y can be chosen such that $\{x', y'\} \not\subseteq bC_bp$ and $\{x', y'\} \not\subseteq qC_bb$. Suppose by symmetry that $\{x', y'\} \subseteq bC_bp$. Let $r \in V(xPu) \cap V(X \cup Y)$ such that rPu is minimal. Note that $V(xPu \cap C_b) = \{u\}$. If $r \in X$, then $xXr \cup rPu$ and Y are disjoint paths in $G_b - b$ from x, y to u, y', respectively, such that $\{u, y'\} \not\subseteq bC_bp$ and $\{u, y'\} \not\subseteq qC_bb$. Similarly, if $r \in Y$, then X and $yYr \cup rPu$ are disjoint paths in $G_b - b$ from x, y to x', u, respectively, such that $\{u, x'\} \not\subseteq bC_bp$ and $\{u, x'\} \not\subseteq qC_bb$.

Therefore, we have three cases.

If $\{x', y'\} \subseteq pC_bq$, then we may assume (by symmetry between X and Y) that p, x', y', q occur on C_b in that clockwise order. In this case, $X \cup bC_bx'$ and $Y \cup b'C'_by'$ form an $(\{x, b\}, \{y, b'\})$-linkage, and $X \cup x'C'_bb'$ and $Y \cup y'C_bb$ form an $(\{x, b'\}, \{y, b\})$-linkage, contradicting (ii).

If one of $\{x', y'\}$ is contained in $pC_bq - \{p, q\}$ and the other is contained in qC_bp, then (by symmetry between bC_bp and qC_bb and between X and Y) we may assume that $x' \in bC_bp$ and $y' \in pC_bq - p$. In this case, $X \cup bC_bx'$ and $Y \cup b'C'_by'$ form an $(\{x, b\}, \{y, b'\})$-linkage, and $X \cup x'Cb'$ and $Y \cup y'C_bb$ form an $(\{x, b'\}, \{y, b\})$-linkage, contradicting (ii).

So we may assume (by symmetry between X and Y) that $x' \in bC_bp$ and $y' \in qC_bb$. Then $X \cup bCx'$ and $Y \cup b'Cy'$ form an $(\{x, b\}, \{y, b'\})$-linkage, and $X \cup x'Cb'$ and $Y \cup y'Cb$ form an $(\{x, b'\}, \{y, b\})$-linkage, contradicting (ii). $\square$

The final two results of this section deal with linkages in general graphs. These results will be crucial in our effort to characterize obstructions.

(3.4) Theorem. *Let H be a 2-connected graph and $\{u, v, w, c, c'\} \subseteq V(H)$ such that*

(i) *H contains neither a $(\{c, u\}, \{c', v\})$-linkage nor a $(\{c, w\}, \{c', v\})$-linkage, and*

(ii) *$(H, c', c, a_1, a_2, a_3)$ is not 3-planar for any sequence $a_1a_2a_3 \in \{wvu, vwu, vuw\}$.*

Then

(1) *there are vertices p, q of H and a separation (H_1, H_2) in H such that $V(H_1 \cap H_2) = \{p, q\}$, $\{c, c', v\} \subseteq H_1$, $\{u, w\} \subseteq H_2$, and (H_1, c', c, v, p, q) is 3-planar, and*

(2) *H has a $(\{c, u\}, \{c', w\})$-linkage such that v is not used by this linkage unless in (1) we can choose $p = v$.*

Proof. Let H be a counter example to (3.4) such that $|V(H)| + |E(H)|$ is minimum. Because H contains no $(\{c, u\}, \{c', v\})$-linkage and by (2.4), there is a collection $\mathcal{H}$ of pairwise disjoint subsets of $V(H) - \{c, c', u, v\}$ such that $(H, \mathcal{H}, c', c, v, u)$ is 3-planar. Choose $\mathcal{H}$ to be minimal. Because H is 2-connected, $p(H, \mathcal{H})$ is 2-connected. For convenience, let $H' := p(H, \mathcal{H})$. Without loss of generality, we may assume that H' is a 2-connected plane graph such that c', c, v, u occur on its outer cycle C in that clockwise order. Define $W = \{w\}$ if $w \notin U$ for any $U \in \mathcal{H}$, and otherwise, $w \in U_w$ for some $U_w \in \mathcal{H}$, and let $W = N_H(U_w)$. In any case, $W \subseteq H'$, and if $|W| = 3$ then W induces a facial triangle in H'.

Claim 1. H' has a 2-separation (A_1, A_2) such that $V(A_1 \cap A_2) \subseteq cCc'$, $c'Cc \subseteq A_1$, and $W \subseteq A_2$.

Suppose on the contrary that such a separation does not exist. Then by the planarity of H', $H' - cCc'$ contains a path from some $w^* \in W$ to $c'Cc - \{c, c'\}$. Hence, H' contains a $(\{c, w^*\}, \{c', v\})$-linkage. We view $H, \mathcal{H}, c, c', w, w^*, v$ as $G, \mathcal{A}, b, b', u, u^*, v^* = v$ in (3.2), respectively. Since $\mathcal{H}$ is minimal, conditions (ii) and (iii) of (3.2) are satisfied (see the remark following (3.2)), and since $w^* \neq v$, condition (i) of (3.2) is also satisfied. Thus, by (3.2), H contains a $(\{c, w\}, \{c', v\})$-linkage, contradicting (i).

Claim 2. For any 2-separation (A_1, A_2) of H' as in Claim 1 (that is, $V(A_1 \cap A_2) \subseteq cCc'$, $c'Cc \subseteq A_1$, and $W \subseteq A_2$), one element of $V(A_1 \cap A_2)$ is contained in $cCv - v$ and the other is contained in $uCc' - u$.

Let (A_1, A_2) be a 2-separation of H' such that $V(A_1 \cap A_2) \subseteq cCc'$, $c'Cc \subseteq A_1$, and $W \subseteq A_2$. Because $\mathcal{H}$ is minimal and $|N_H(U)| \leq 3$ for every $U \in \mathcal{H}$, $N_H(U) \subseteq A_1$ or $N_H(U) \subseteq A_2$.

First, assume that $V(A_1 \cap A_2) \subseteq cCv$. Let $S \subseteq V(H)$ such that, for each $r \in V(H)$, $r \in S$ iff $r \notin V(A_1 \cap A_2) \cup \{w\}$, and either $r \in A_2$ or $r \in U$ for some $U \in \mathcal{H}$ with $N_H(U) \subseteq A_2$. Then if $S \neq \emptyset$, $N_H(S) = V(A_1 \cap A_2) \cup \{w\}$. Let $\mathcal{H}' = (\mathcal{H} - \{U \in \mathcal{H} : N_H(U) \subseteq A_2\}) \cup \{S\}$ if $S \neq \emptyset$; and otherwise, let $\mathcal{H}' := \mathcal{H}$. It is easy to check that $(H, \mathcal{H}', c', c, w, v, u)$ is 3-planar, contradicting (ii).

Hence $V(A_1 \cap A_2) \not\subseteq cCv$. Similarly, $V(A_1 \cap A_2) \not\subseteq vCu$ (since by (ii), (H, c', c, v, w, u) is not 3-planar) and $V(A_1 \cap A_2) \not\subseteq uCc'$ (since by (ii), (H, c', c, v, u, w) is not 3-planar).

Now assume that one element p of $V(A_1 \cap A_2)$ is contained in $vCu - u$ and the other, q, is contained in $uCc' - u$. Let H_2 be the subgraph of H induced by $\{x \in V(H) : x \in A_2$ or $x \in U$ for some $U \in \mathcal{H}$ such that $N_H(U) \subseteq A_2\}$. Then $w \in H_2$. Let $H_1 := H - (H_2 - \{p, q\})$. Then (H_1, H_2) is a separation of H such that $V(H_1 \cap H_2) = \{p, q\}$ and $(H_1, \mathcal{H}_1, c', c, v, p, q)$ is 3-planar, where $\mathcal{H}_1 = \{U \in \mathcal{H} : U \subseteq V(H_1)\}$. Hence (1) of (3.4) holds. Note that H_2 contains a $(\{p, u\}, \{q, w\})$-linkage; otherwise, (H_2, p, w, u, q) is 3-planar by (2.4), and hence, $(H, c', c, v, p, w, u, q)$ is 3-planar, contradicting (ii). By planarity of A_1, A_1 contains a $(\{c, p\}, \{c', q\})$-linkage such that v is not used by this linkage unless we can choose $p = v$. Note that $p(H_1, \mathcal{H}_1) = A_1$ or $p(H_1, \mathcal{H}_1) = A_1 - pq$. Hence, $p(H_1, \mathcal{H}_1)$ also has a $(\{c, p\}, \{c', q\})$-linkage. We view $H_1, \mathcal{H}_1, p, q, c, c'$ as $G, \mathcal{A}, u^* = u, v^* = v, b, b'$ in (3.2), respectively. Since $\mathcal{H}$ is minimal, $\mathcal{H}_1$ is also minimal. Hence conditions (ii) and (iii) of (3.2) are satisfied. Since $p \neq q$, condition (i) of (3.2) is also satisfied. By (3.2), H_1 contains a $(\{c, p\}, \{c', q\})$-linkage. Hence, H contains a $(\{c, u\}, \{c', w\})$-linkage, and v is not used by this linkage unless we can choose $p = v$. Therefore, (2) of (3.4) follows. This contradicts the assumption that H is a counter example to (3.4).

Therefore, if Claim 2 fails, then the only remaining case is when one element$p$ of $V(A_1 \cap A_2)$ is contained in $cCv - v$ and the other, q, is contained in $vCu - v$. Let H_2 denote the subgraph of H induced by $\{x \in V(H) : x \in A_2$ or $x \in U$ for some $U \in \mathcal{H}$ such that $N_H(U) \subseteq A_2\}$. Then $w \in H_2$. Let $H_1 := H - (H_2 - \{p, q\})$. Then $(H_1, \mathcal{H}_1, c', c, p, q, u)$ is 3-planar, where $\mathcal{H}_1 = \{U \in \mathcal{H} : U \subseteq V(H_1)\}$. Moreover, H_2 has a $(\{p, w\}, \{q, v\})$-linkage; for otherwise, (H_2, p, v, w, q) is 3-planar by (2.4), and so, $(H, c', c, p, v, w, q, u)$ is 3-planar, contradicting (ii). Note that A_1 contains a $(\{c, p\}, \{c', q\})$-linkage, and $p(H_1, \mathcal{H}_1) = A_1$ or $p(H_1, \mathcal{H}_1) = A_1 - pq$. Hence, $p(H_1, \mathcal{H}_1)$ also has a $(\{c, p\}, \{c', q\})$-linkage. By the same argument as in the previous paragraph, we can use (3.2) to show that H_1 has a $(\{c, p\}, \{c', q\})$-linkage. Thus H contains a $(\{c, w\}, \{c', v\})$-linkage, contradicting (i). This completes the proof of Claim 2.

By Claim 1, let (A_1, A_2) be a 2-separation of H' such that $V(A_1 \cap A_2) \subseteq cCc'$, $c'Cc \subseteq A_1$, and $W \subseteq A_2$. Let $V(A_1 \cap A_2) = \{s, s'\}$. By Claim 2, let $s \in cCv - v$ and $s' \in uCc' - u$. Let H_2 denote the subgraph of H induced by $\{x \in V(H) : x \in A_2$ or $x \in U$ for some $U \in \mathcal{H}$ such that $N_H(U) \subseteq A_2\}$. Then $w \in H_2$. Let $H_1 := H - (H_2 - \{s, s'\})$.

Claim 3. We may choose (A_1, A_2) such that $V(H_1) \neq \{c, c'\}$.

Suppose $V(H_1) = \{c, c'\}$ for any 2-separation (A_1, A_2) such that $V(A_1 \cap A_2) \subseteq cCc'$, $c'Cc \subseteq A_1$, and $W \subseteq A_2$. Then $H' - (cCc' - c)$ has a path from some $w^* \in W$ to c. Hence, H' has a $(\{c, w^*\}, \{c', v\})$-linkage. We view $H, \mathcal{H}, c, c', w, w^*, v$ as $G, \mathcal{A}, b, b', u, u^*, v = v^*$ in (3.2), respectively. Because $\mathcal{H}$ is minimal, conditions of (3.2) are satisfied. By (3.2), H has a $(\{c, w\}, \{c', v\})$-linkage, contradicting (i). This completes the proof of Claim 3.

Let $\mathcal{H}_1 = \{U \in \mathcal{H} : U \subseteq H_1\}$. Then $(H_1, \mathcal{H}_1, c, s, s', c')$ is 3-planar. Note that A_1 (or $A_1 - ss'$ if $ss' \in E(A_1)$) contains an $(\{s, c\}, \{s', c'\})$-linkage, and $p(H_1, \mathcal{H}_1) = A_1$

or $p(H_1, \mathcal{H}_1) = A_1 - ss'$. Hence, $p(H_1, \mathcal{H}_1)$ has an $(\{s, c\}, \{s', c'\})$-linkage. We view $H_1, \mathcal{H}_1, s, s', c, c'$ as $G, \mathcal{A}, u^* = u, v^* = v, b, b'$ in (3.2), respectively. Because $\mathcal{H}$ is minimal, $\mathcal{H}_1$ is also minimal. Hence conditions of (3.2) are satisfied. By (3.2), H_1 contains an $(\{s, c\}, \{s', c'\})$-linkage.

Then $H_2 + ss'$ contains neither an $(\{s, u\}, \{s', v\})$-linkage nor an $(\{s, w\}, \{s', v\})$-linkage. For otherwise, such a linkage and an $(\{s, c\}, \{s', c'\})$-linkage in H_1 form a $(\{c, u\}, \{c', v\})$-linkage or a $(\{c, w\}, \{c', v\})$-linkage in H, contradicting (i).

Also $(H_2, s', s, a_1, a_2, a_3)$ is not 3-planar for any $a_1a_2a_3 \in \{wvu, vwu, vuw\}$. For otherwise, $(H, c', c, a_1, a_2, a_3)$ is also 3-planar, contradicting (ii).

Thus (i) and (ii) of (3.4) hold for $H_2 + ss'$ (with $H_2 + ss', s, s'$ replacing H, c, c', respectively). Because H is connected and by Claim 3, $|V(H_2 + ss')| + |E(H_2 + ss')| < |V(H)| + |E(H)|$. By the choice of H, (1) and (2) of (3.4) hold for $H_2 + ss'$ (with $H_2 + ss', s, s'$ replacing H, c, c', respectively). Hence, (a) there are vertices p, q of $H_2 + ss'$ and a separation (A_1', A_2') in $H_2 + ss'$ such that $V(A_1' \cap A_2') = \{p, q\}$, $\{s, s', v\} \subseteq A_1'$, $\{u, w\} \subseteq A_2'$, and (A_1', s', s, v, p, q) is 3-planar, and (b) $H_2 + ss'$ contains an $(\{s, u\}, \{s', w\})$-linkage such that v is not used by this linkage unless in (a) we can choose $p = v$.

Let $H_1 := H_1 \cup A_1'$ and let $H_2 := A_2'$. Because H_1 contains an $(\{s, c\}, \{s', c'\})$-linkage and since (H_1, c', c, s, s') is 3-planar, it follows from (a) and (b) above, (1) and (2) of (3.4) hold for H. But this contradicts the assumption that H is a counter example to (3.4). □

(3.5) Theorem. *Let H be a 2-connected graph and $\{u, v, w, c, c'\} \subseteq V(H)$ such that*

(i) H contains no $(\{c, u\}, \{c', v\})$-linkage,

(ii) (H, c', c, v, w, u) is not 3-planar,

(iii) H has no 2-separation (K, K') such that $V(K \cap K')$ contains u or v, $\{v, c\} \subseteq K$ and $\{u, c'\} \subseteq K'$, and

(iv) $H - \{u, v\}$ is connected and $uv \notin E(H)$.

Then one of the following holds:

(1) $H - v$ has a $(\{c, u\}, \{c', w\})$-linkage or $H - u$ has a $(\{c, w\}, \{c', v\})$-linkage; or

(2) there are a vertex p of H and a separation (H_1, H_2) in H such that $V(H_1 \cap H_2) = \{p, u, v\}$, $\{c, c'\} \subseteq H_1$, $w \in H_2$, (H_1, c', c, v, p, u) is 3-planar.

Proof. By (i) and by (2.4), there is a collection $\mathcal{H}$ of pairwise disjoint subsets of $V(H) - \{c, c', u, v\}$ such that $(H, \mathcal{H}, c', c, v, u)$ is 3-planar. Choose $\mathcal{H}$ to be minimal. Because H is 2-connected, $p(H, \mathcal{H})$ is 2-connected. For convenience, let $H' := p(H, \mathcal{H})$. Without loss of generality, we may assume that H' is a 2-connected plane graph such that c', c, v, u occur on its outer cycle C in that clockwise order. Define $W = \{w\}$ if $w \notin U$ for any $U \in \mathcal{H}$, and otherwise, $w \in U_w$ for some $U_w \in \mathcal{H}$, and let $W := N_H(U_w)$. In any case, $W \subseteq H'$, and if $|W| = 3$ then W induces a facial triangle in H'.

Claim 1. We may assume that for any $w^* \in W$, $H' - v$ has no $(\{c, u\}, \{c', w^*\})$-linkage and $H' - u$ has no $(\{c, w^*\}, \{c', v\})$-linkage.

Suppose that $H' - v$ has a $(\{c, u\}, \{c', w^*\})$-linkage. We view $H, \mathcal{H}, c, c', u, w, w^*$ as $G, \mathcal{A}, b, b', u = u^*, v, v^*$ in (3.2), respectively. Because $\mathcal{H}$ is minimal, conditions of (3.2) hold. Hence, $H - v$ has a $(\{c, u\}, \{c', w\})$-linkage, and so, (1) holds. By a similar argument, if $H' - u$ has a $(\{c, w^*\}, \{c', v\})$-linkage, then $H - u$ has a $(\{c, w\}, \{c', v\})$-linkage, and (1) holds.

Claim 2. We may assume that $(\{c, c'\} \cup W) - \{u, v\}$ is contained in a block B of $H' - \{u, v\}$.

Suppose $(\{c, c'\} \cup W) - \{u, v\}$ is not contained in any block of $H' - \{u, v\}$. Because W induces a complete subgraph of H', and by planarity and by (iii), H' has a separation (A_1, A_2) such that $V(A_1 \cap A_2) = \{u, v, p\}$, $\{c, c'\} \subseteq A_1$, and $W \subseteq A_2$. Let H_2 denote the subgraph of H induced by $\{x \in V(H) : x \in A_2$ or $x \in U$ for some $U \in \mathcal{H}$ with $N_H(U) \subseteq A_2\}$. Let $H_1 := H - (H_2 - \{u, v, p\})$. Then (H_1, H_2) satisfies (2) of (3.5).

Let D denote the outer cycle of B. Let $v', u' \in V(cDc')$ such that H' has a path from v (respectively, u) to v' (respectively, u') internally disjoint from B. Then c, v', u', c' occur on D in this clockwise order. Select v', u' so that $v'Du'$ is minimal. Because of (iv), $v'Du' = v'Cu'$.

Claim 3. We may assume that either $W = \{w\}$ and $w \in v'Du'$ or B has a 2-separation (B_1, B_2) such that $V(B_1 \cap B_2) \subseteq v'Du'$, $\{c, c'\} \subseteq B_1$, and $W \subseteq B_2$.

For otherwise, there is some $w^* \in W$ such that B contains a $(\{v', c'\}, \{w^*, c\})$-linkage or a $(\{u', c\}, \{w^*, c'\})$-linkage. Then $H' - u$ contains a $(\{v, c'\}, \{w^*, c\})$-linkage or $H' - v$ contains a $(\{u, c\}, \{w^*, c'\})$-linkage. We view $H, \mathcal{H}, c, c', v, w$ (or u, w) as $b, b', \{u, v\}$ in (3.2), respectively. Because $\mathcal{H}$ is minimal, the conditions of (3.2) hold. Hence, $H - u$ contains a $(\{v, c'\}, \{w, c\})$-linkage or $H - v$ contains a $(\{u, c\}, \{w, c'\})$-linkage. Hence (1) of (3.5) holds.

If $W = \{w\}$ and $w \in v'Du'$, then by (iv), (H, c', c, v, w, u) is 3-planar, contradicting (ii). So by Claim 3, B has a 2-separation (B_1, B_2) such that $V(B_1 \cap B_2) \subseteq v'Du'$, $\{c, c'\} \subseteq B_1$, and $W \subseteq B_2$. Let $S \subseteq V(H)$ such that, for each $x \in V(H)$, $x \in S$ iff $x \notin V(B_1 \cap B_2) \cup \{w\}$, and either $x \in B_2$ or $x \in U$ for some $U \in \mathcal{H}$ with $N_H(U) \subseteq B_2$. Let $\mathcal{H}' = (\mathcal{H} - \{U \in \mathcal{H} : N_H(U) \subseteq B_2\}) \cup \{S\}$. Then by (iv), $(H, \mathcal{H}', c', c, v, w, u)$ is 3-planar, contradicting (ii). $\square$

4 Basic obstructions

In this section, we will describe the basic obstructions, called rungs. We will also show that rungs can be used to construct another type of obstructions, called ladders.

We then state a characterization of obstructions in terms of ladders. First, we define the basic obstructions.

(4.1) Definition. Let G be a graph, $\{a,b,c\} \subseteq V(G)$, and $\{a',b',c'\} \subseteq V(G)$. Suppose $\{a,b,c\} \neq \{a',b,'c'\}$, and assume that G has no 3-separation (G_1,G_2) such that $\{a,b,c\} \subseteq G_1$ and $\{a',b',c'\} \subseteq G_2$. We say that $(G,(a,b,c),(a',b',c'))$ is a *rung* if one of the following conditions is satisfied:

(1) $b=b'$ or $\{a,c\}=\{a',c'\}$;

(2) $a=a'$ and $(G-a,c,c',b',b)$ is 3-planar;

(2′) $c=c'$ and $(G-c,a,a',b',b)$ is 3-planar;

(3) $\{a,b,c\} \cap \{a',b',c'\} = \emptyset$ and (G,a',b',c',c,b,a) is 3-planar;

(4) $\{a,b,c\} \cap \{a',b',c'\} = \emptyset$, G has a 1-separation (G_1,G_2) such that $\{a,a',b,b'\} \subseteq G_1$, $\{c,c'\} \subseteq G_2$, and (G_1,a,a',b',b) is 3-planar;

(4′) $\{a,b,c\} \cap \{a',b',c'\} = \emptyset$, G has a 1-separation (G_1,G_2) such that $\{c,c',b,b'\} \subseteq G_1$, $\{a,a'\} \subseteq G_2$, and (G_1,c,c',b',b) is 3-planar;

(5) $\{a,b,c\} \cap \{a',b',c'\} = \emptyset$, (G,a,a',b',b) is 3-planar, and G has a separation (G_1,G_2) such that $V(G_1 \cap G_2) = \{z,b\}$ (or $V(G_1 \cap G_2) = \{z,b'\}$), $\{a,a',b,b'\} \subseteq G_1$, $\{c,c'\} \subseteq G_2$, and (G_2,c,c',z,b) (or (G_2,c,c',b',z)) is 3-planar;

(5′) $\{a,b,c\} \cap \{a',b',c'\} = \emptyset$, (G,c,c',b',b) is 3-planar, and G has a separation (G_1,G_2) such that $V(G_1 \cap G_2) = \{z,b\}$ (or $V(G_1 \cap G_2) = \{z,b'\}$), $\{c,c',b,b'\} \subseteq G_1$, $\{a,a'\} \subseteq G_2$, and (G_2,a,a',z,b) (or (G_2,a,a',b',z)) is 3-planar;

(6) $\{a,b,c\} \cap \{a',b',c'\} = \emptyset$, and there are pairwise edge disjoint subgraphs G_a,G_c,M of G such that $G = G_a \cup G_c \cup M$, $V(G_a \cap M) = \{u,w\}$, $V(G_c \cap M) = \{p,q\}$, $V(G_a \cap G_c) = \emptyset$, $\{a,a',b'\} \subseteq G_a$, $\{c,c',b\} \subseteq G_c$, (G_a,a,a',b',w,u) is 3-planar, and (G_c,c',c,b,p,q) is 3-planar;

(6′) $\{a,b,c\} \cap \{a',b',c'\} = \emptyset$, and there are pairwise edge disjoint subgraphs G_a,G_c,M of G such that $G = G_a \cup G_c \cup M$, $V(G_a \cap M) = \{u,w\}$, $V(G_c \cap M) = \{p,q\}$, $V(G_a \cap G_c) = \emptyset$, $\{a,a',b\} \subseteq G_a$, $\{c,c',b'\} \subseteq G_c$, (G_a,b,a,a',w,u) is 3-planar, and (G_c,b',c',c,p,q) is 3-planar; or

(7) $\{a,b,c\} \cap \{a',b',c'\} = \emptyset$, and there are pairwise edge disjoint subgraphs G_a,G_c,M of G such that $G = G_a \cup G_c \cup M$, $V(G_a \cap M) = \{b,b',w\}$, $V(G_c \cap M) = \{b,b',p\}$, $V(G_a \cap G_c) = \{b,b'\}$, $\{a,a'\} \subseteq G_a$, $\{c,c'\} \subseteq G_c$, (G_a,a,a',b',w,b) is 3-planar, and (G_c,c',c,b,p,b') is 3-planar.

Remark. Note the symmetry between $\{a, a'\}$ and $\{c, c'\}$. Also note that rungs need not be connected. For example, if $V(G) = \{a, b, c\} \cup \{a', b', c'\}$, $E(G) = \emptyset$, $a = a'$, $c = c'$, and $b \neq b'$, then G has no 3-separation (G_1, G_2) such that $\{a, b, c\} \subseteq G_1$ and $\{a', b', c'\} \subseteq G_2$, and so, $(G, (a, b, c), (a', b', c'))$ is a rung as in (1) of (4.1). Also there was no intention to make all cases in (4.1) disjoint.

(4.2) Proposition. *If $(G, (a, b, c), (a', b', c'))$ is a rung, then $(G, \{a, c\}, \{a', c'\}, (b, b'))$ is an obstruction.*

Proof. Let A, B, C be disjoint paths in G from $\{a, b, c\}$ to $\{a', b', c'\}$ such that $a \in A, b \in B$ and $c \in C$. We need to show that $b' \in B$. This is obvious if (1) of (4.1) holds.

If (2) of (4.1) holds, then $B \cup C \subseteq G - a$, and so, $b' \in B$ by (2.4) (because $(G - a, c, c', b', b)$ is 3-planar). Similarly, if (2′) of (4.1) holds, then $b' \in B$. Also, if (3) of (4.1) holds, then $b' \in B$ by (2.4) (because (G, a', b', c', c, b, a) is 3-planar).

Assume (4) of (4.1) holds. Then $A \cup B \subseteq G_1$ and $C \subseteq G_2$. Because (G_1, a, a', b', b) is 3-planar, $b' \in B$ by (2.4). Similarly, if (4′) of (4.1) holds, then $b' \in B$.

Assume that (5) of (4.1) holds. By symmetry, assume that $V(G_1 \cap G_2) = \{z, b'\}$. Because (G_2, c, c', b', z) is 3-planar, $c' \in C$ by (2.4). Therefore, A and B are disjoint paths in G from $\{a, b\}$ to $\{a', b'\}$. Because (G, a, a', b', b) is 3-planar, $b' \in B$ by (2.4). Similarly, $b' \in B$ if (5′) of (4.1) holds.

Now assume that (6) of (4.1) holds. Then $c' \notin A$, for otherwise, $\{u, w\} \cap V(X) \neq \emptyset$ for every $X \in \{A, B, C\}$, a contradiction. Further $\{u, w\} \not\subseteq A$; otherwise, $A \cap B \neq \emptyset$ or $A \cap C \neq \emptyset$, a contradiction. Hence $A \subseteq G_a$. Then $a' \in A$; otherwise, the path of B or C containing a' must intersect A (because (G_a, a, a', b', w, u) is 3-planar), a contradiction. By a symmetric argument, we can prove that $C \subseteq G_c$ and $c' \in C$. Hence, $b' \in B$. Similarly, if (6′) of (4.1) holds, then $b' \in B$.

Finally, assume that (7) of (4.1) holds. Suppose $a' \notin A$. Then either $w \in A$ or $b' \in A$, and hence $a' \notin B$ and $a' \notin C$ (because (G_a, a, a', b', w, b) is 3-planar), a contradiction. Hence $a' \in A$. Similarly, $c' \in C$. Therefore, $b' \in B$. □

Next, we define ladders.

(4.3) Definition. Let L be a graph and let $R_1, \ldots, R_m$ be edge disjoint subgraphs of L such that

(i) $(R_i, (x_{i-1}, v_{i-1}, y_{i-1}), (x_i, v_i, y_i))$ is a rung for each $1 \leq i \leq m$,

(ii) $V(R_i \cap R_j) = \{x_i, v_i, y_i\} \cap \{x_{j-1}, v_{j-1}, y_{j-1}\}$ for $1 \leq i < j \leq k$,

(iii) if $x_i = x_j$ (or $v_i = v_j$, or $y_i = y_j$) then $x_k = x_i$ (or $v_k = v_i$, or $y_k = y_i$) for all $i \leq k \leq j$, and

(iv) $L = (\bigcup_{i=1}^m R_i) + S$, where S consists of edges of L with both ends in some $\{x_i, v_i, y_i\}$, $i = 1, \ldots, m$.

Then we say that $(L, (x_0, v_0, y_0), (x_m, v_m, y_m))$ is a *ladder with rungs* $(R_i, (x_{i-1}, v_{i-1}, y_{i-1}), (x_i, v_i, y_i))$, $i = 1, \ldots, m$, or simply, a *ladder along* $v_0 \ldots v_m$.

Remark. Intuitively, L in (4.3) is obtained from the disjoint union of rungs $R_1, \ldots, R_m$ by identifying the vertices with the same labels x_i, y_i, v_i, and by adding edges with both ends in some $\{x_i, v_i, y_i\}$.

(4.4) Proposition. *Let* $(L, (x_0, v_0, y_0), (x_m, v_m, y_m))$ *be a ladder with rungs* $(R_i, (x_{i-1}, v_{i-1}, y_{i-1}), (x_i, v_i, y_i))$, $i = 1, \ldots, m$. *Then* $(L, \{x_0, y_0\}, \{x_m, y_m\}, (v_0, v_m))$ *is an obstruction.*

Proof. Let S denote the set of edges in L with both ends in some $\{x_i, v_i, y_i\}$. Suppose that A, B, C are disjoint paths in L from $\{x_0, v_0, y_0\}$ to $\{x_m, v_m, y_m\}$ such that $x_0 \in A$, $v_0 \in B$, and $y_0 \in C$. Then $E(X) \cap S = \emptyset$ for every $X \in \{A, B, C\}$. Hence, for each $i \in \{1, \ldots, m\}$, $A \cap R_i$, $B \cap R_i$, and $C \cap R_i$ are disjoint paths in R_i from $\{x_{i-1}, v_{i-1}, y_{i-1}\}$ to $\{x_i, v_i, y_i\}$. By (4.2), $v_{i-1} \in B$ implies $v_i \in B$ for all $i \in \{0, \ldots, m-1\}$. Hence $v_m \in B$, and so, (4.4) follows. □

We conclude this paper by stating a characterization of obstructions, to be proved in subsequent papers. For a sequence S, the *reduced sequence* of S is the sequence obtained from S by removing all but one consecutive identical elements. For example, the reduced sequence of *aaabcca* is *abca*.

(4.5) Theorem. *Let* $(G, \{a, c\}, \{a', c'\}, (b, b'))$ *be an obstruction. Then one of the following statements holds.*

(1) *G has a separation (G_1, G_2) of order at most 2 such that $\{a, b, c\} \subseteq V(G_1)$ and $\{a', b', c'\} \subseteq V(G_2)$.*

(2) *There is a subset $T \subseteq V(G)$ such that $|T| \leq 3$ and some component of $G - T$ contains no element of $\{a, b, c\} \cup \{a', b', c'\}$.*

(3) *$(G, (a, b, c), (a', b', c'))$ is a ladder.*

(4) *G has a separation (J, L) such that $V(J \cap L) = \{w_0, \ldots, w_n\}$, $(J, w_0, \ldots, w_n)$ is 3-planar, $(L, (a, b, c), (a', b', c'))$ is a ladder along a sequence $v_0 \ldots v_m$, where $v_0 = b$, $v_m = b'$, and $w_0 \ldots w_n$ is the reduced sequence of $v_0 \ldots v_m$.*

References

[1] K. Chakravarti and N. Robertson, Covering three edges with a bond in a nonseparable graph. *Annals of Discrete Math.* (Deza and Rosenberg eds) (1979) 247.

[2] P. A. Catlin, Hajós' graph-coloring conjecture: variations and counterexamples. J. Combin. Theory *Ser. B* **26** (1979), no. 2, 268–274. 05C15

[3] G. A. Dirac, Homomorphism theorems for graphs. *Math. Ann.* **153** (1964) 69–80.

[4] T. R. Jensen and B. Toft, *Graph Coloring Problems*. Wiley-Interscience Series in Discrete Mathematics and Optimization. 1995.

[5] A. E. Kézdy and P. J. McGuinness, Do 3n-5 edges suffice for subdivision of K_5. *J. Graph Theory* **15** (1991) 289–406.

[6] W. Mader, $3n-5$ edges do force a subdivision of K_5. *Combinatorica*, **18** (1998), no. 4, 569–595.

[7] N. Robertson and P. D. Seymour, private communication with R. Thomas.

[8] P. D. Seymour, private communication.

[9] P. D. Seymour, Disjoint paths in graphs. *Discrete Math.* **29** (1980) 293–309.

[10] C. Thomassen, 2-Linked graphs. *Europ. J. Combin.* **1** (1980) 371–378.

[11] X. Yu, Subdivisions in planar graphs. *J. Combin. Theory Ser. B* **72** (1998) 10–52.