Chain decompositions and independent trees in 4-connected graphs*

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Abstract

This work was motivated by the study of a multi-tree approach to reliability in distributed networks and by the study of non-separating paths and cycles in highly connected graphs. We first give a result on "non-separating chains" in 4-connected graphs. This result is then used to obtain a "non-separating chain decomposition" of a 4-connected graph G, and an $O(|V(G)|^2|E(G)|)$ algorithm for constructing such a decomposition. As an application of this decomposition, we show how to produce four "independent spanning trees" in a 4-connected graph in $O(|V(G)|^3)$ time.

1 Introduction.

In 1984, Itai and Rodeh [9] proposed a multi-tree approach to reliability in distributed networks. Let G be a graph and $r \in V(G)$. We may view G as a distributed network with a root r, and the vertices of G as processors. A fault-tolerant communication scheme can be designed for this network if we are able to find independent spanning trees of G [6, 9]. For a tree T and $x, y \in V(T)$, let $T[x, y]$ denote the unique path from x to y in T. A *rooted tree* T is a tree with a specified vertex called the *root* of T. Let T and T' be trees in a graph rooted at r. We say that T and T' are *independent* if for each vertex $x \in V(T) \cap V(T')$, the paths $T[r, x]$ and $T'[r, x]$ have no vertex in common except for r and x. For two subgraphs G and H of a graph, we use $G \cup H$ (respectively, $G \cap H$) to denote the union (respectively, intersection) of G and H. For any graph G and $S \subseteq V(G)$, let $G - S$ denote the graph obtained from G by deleting S and all edges of G with at least one end in S. A subgraph H of a graph G is called *non-separating* if $G - V(H)$ is connected.

Itai and Rodeh [9] developed a linear time algorithm that given any vertex r in a 2-connected graph G, finds two independent spanning trees of G rooted at r. Their algorithm is based on an ear decomposition of a 2-connected graph. An *ear decomposition* of a graph G is a collection $(P_0, P_1, \ldots, P_t)$ of subgraphs of G such that (i) P_0 is a cycle in G, (ii) for each $1 \le i \le t$, P_i is a path in G with only its ends in $V(\bigcup_{j=0}^{i-1} P_j)$, and (iii) $G = \bigcup_{i=0}^{t} P_i$. Later, Cheryian and Maheshwari [4] proved that for any vertex r in a 3-connected graph G, there exist three independent spanning trees of G rooted at r. Furthermore, they gave an $O(|V(G)|^2)$ algorithm for finding these trees. Again, their algorithm is based on an ear decomposition $(P_0, \ldots, P_t)$ of a 3-connected graph G, with one additional property: (iv) for each $1 \le i \le t$, $G - V(\bigcup_{j=0}^{i-1} P_j)$ is connected. (Such an ear decomposition is called a *non-separating* ear decomposition.)

Itai and Zehavi [10] proved independently that every 3-connected graph has three independent spanning trees rooted at any given vertex, and they conjectured that for any k-connected graph G and for any $r \in V(G)$, there exist k independent spanning trees of G rooted at r. Huck [7] proved this conjecture for planar 4-connected graphs. (A graph is *planar* if it can be drawn in the plane with no pairs of edges crossing, and such a drawing is called a *plane graph.*) Later, Miura *et al* [12] gave a linear algorithm for finding four independent rooted spanning trees in a planar 4-connected graph.

In this paper we prove the Itai-Zehavi conjecture for arbitrary 4-connected graphs.

THEOREM 1.1. *Let G be a 4-connected graph and let $r \in V(G)$. Then there exist four independent spanning trees in G rooted at r, and such trees can be found in $O(|V(G)|^3)$ time.*

Theorem 1.1 is important in terms of applications, since four independent spanning trees ensure at a reasonable cost a higher degree of reliability in distributed networks. Theorem 1.1 is of theoretical interest as well because when $k \ge 4$ there is not a good decomposition theory for k-connected graphs. (By "good" we mean that at each step of the decomposition, the graph is still highly connected.) In fact, the first step towards such a decomposition is a variation of a conjecture of Lovász [11]: for any integer $k > 0$, there is an integer $f(k) > 0$ with the property that for any $f(k)$-connected graph G and any edge e of G there exists an induced cycle C in G containing e such that $G - V(C)$ is k-connected.

For a 4-connected graph G we will be able to decompose G into subgraphs $H_0, \ldots, H_t$ as in an ear

*Second author was supported by CNPq (Proc. 200611/00-3) (Brazil), and third author was partially supported by NSF grant DMS 9970527

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decomposition. This time, however, the subgraphs H_i need not be paths or cycles, they are planar subgraphs of G and for each $1 \le i \le t$, $G - V(\bigcup_{j=0}^{i-1} H_j)$ is almost 2-connected. The planarity requirement on the H_i's is a natural one, for in the process of constructing such a decomposition, we need to find disjoint paths between given pairs of vertices. An algorithm of Shiloach [13] implies that given a graph G and distinct vertices a, b, c, d of G such that G is a $(4, \{a, b, c, d\})$-connected, one can in $O(|V(G)| + |E(G)|)$ time either find disjoint paths from a to c and from b to d, respectively, or certify that (G, a, b, c, d) is planar (in which case there exist no disjoint paths). Here, (G, a, b, c, d) is *planar* if G can be drawn in a closed disc in the plane with no pair of edges crossing such that a, b, c, d occur on the boundary of the disc in that cyclic order, and G is $(4, \{a, b, c, d\})$-*connected* if $|V(G)| \ge 5$ and for any $S \subset V(G)$ with $|S| \le 3$ every component of $G - S$ contains an element of $\{a, b, c, d\}$.

The proof of Theorem 1.1 is divided into three stages, and each one is of its own interest. First, we prove that if G is an "almost 4-connected" graph then G has a "non-separating planar chain" (a generalization of a path). Moreover, such a chain can be found in $O(|V(G)||E(G)|)$ time. This result is related to the above mentioned conjecture of Lovász. A precise description of this result is given in Section 2. Secondly, we use this non-separating chain result to show that if G is a 4-connected graph then G has a "non-separating chain decomposition" described in the above paragraph. Moreover, such a decomposition can be found in $O(|V(G)|^2|E(G)|)$ time. This is done in Section 3. We hope this decomposition will be useful for solving problems about 4-connected graphs. As an application, we use this decomposition to prove Theorem 1.1, and this is sketched in Section 4.

2 Non-separating planar chains

A *block* of a graph G is either a maximal 2-connected subgraph of G, or a subgraph of G induced by an edge not contained in any cycle. A block is *non-trivial* if it is 2-connected, and *trivial* otherwise. An *endblock* of a graph G is a block of G which contains at most one cut vertex of G.

We say that a connected graph is a *chain* if it contains at most two endblocks. Let H be a chain. Then the blocks of H can be labeled as $B_1, \ldots, B_k$ and the cut vertices of H can be labeled as $v_1, \ldots, v_{k-1}$ such that $V(B_i \cap B_{i+1}) = \{v_i\}$ and $V(B_i \cap B_j) = \emptyset$ if $j > i + 1$. We write $H := B_1v_1B_2v_2 \ldots v_{k-1}B_k$ to denote this situation. If $k \ge 2$, $v_0 \in V(B_1 - v_1)$ and $v_k \in V(B_k - v_{k-1})$, or, if $k = 1$, $v_0, v_k \in V(B_1)$ and $v_0 \ne v_k$, then we say that H is a v_0-v_k *chain*, and we denote this by $H := v_0B_1v_1 \ldots v_{k-1}B_kv_k$. See Figure 1 for an illustration with $k = 5$.

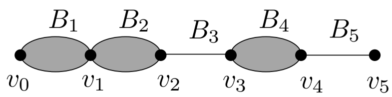

Figure 1: Example of a chain.

A *cyclic chain* is a graph H' which can be obtained from some chain $H := v_0B_1v_1 \ldots v_{k-1}B_kv_k$ with $k \ge 2$ by identifying v_0 and v_k. With no confusion, we also use the notation $H' := v_0B_1v_1 \ldots v_{k-1}B_kv_k$ to indicate this cyclic chain, and we say that each B_i is a *piece* of H' and $v_0 = v_k$ is a *root* of H'. See Figure 2 for an illustration.

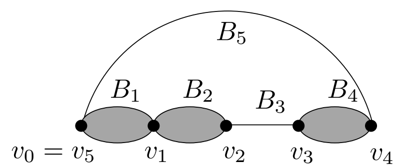

Figure 2: Example of a cyclic chain.

The chains we use in our decomposition result are special chains in the sense that their blocks are planar and contain Hamiltonian paths between their ends (hence, can be viewed as a generalization of a path). The latter result follows from a theorem of Thomassen [14].

Next we introduce the notation needed to define chains in a graph. For any $S \subseteq V(G)$, let $G[S]$ denote the subgraph of G with $V(G[S]) = S$ and $E(G[S])$ consisting of the edges of G with both ends in S; we say that $G[S]$ is the subgraph of G *induced* by S. For any $S \subseteq V(G)$, let $N_G(S)$ denote the set of vertices in $V(G) - S$ which are adjacent to some vertex in S; for a subgraph H of G, let $N_G(H) := N_G(V(H))$. For a graph G and $x, y \in V(G)$, let $G - xy$ denote the graph with vertex set $V(G)$ and edge set $E(G) - \{xy\}$. Note that if xy need is not an edge of G, then $E(G - xy) = E(G)$.

Let G be a graph and let $H := v_0B_1v_1 \ldots v_{k-1}B_kv_k$ be a chain (respectively, a cyclic chain). If H is an induced subgraph of G, then we say that H is a *chain in G* (respectively, a cyclic chain in G). We say that H is a *planar chain in G* (respectively, planar cyclic chain) if, for each 2-connected B_i, there exist distinct vertices $x_i, y_i \in V(G) - V(H)$ such that $(G[V(B_i) \cup \{x_i, y_i\}] - x_iy_i, x_i, v_{i-1}, y_i, v_i)$ is planar, and $B_i - \{v_{i-1}, v_i\}$ is a component of $G - \{x_i, y_i, v_{i-1}, v_i\}$. See Figure 3 for an illustration of a planar chain.

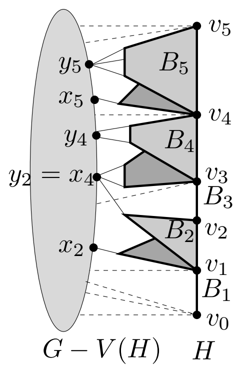

Figure 3: A planar chain in G.

Figure 4 illustrates a planar cyclic chain.

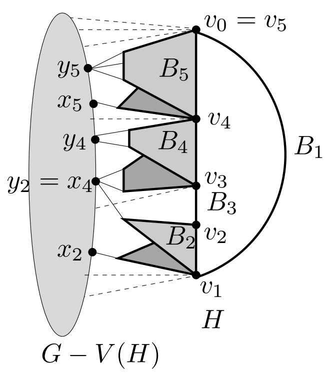

Figure 4: A planar cyclic chain in G.

Now we can precisely state our first result which is proved in [1]. See Figure 5 for an illustration of the statement of the result.

Theorem 2.1. *Let G be a graph, let a, a' be distinct vertices of G, let P be a non-separating induced path in G between a and a', let B_P be a nontrivial block of $G - V(P)$, and let $X_P := N_G(G - V(B_P))$. Suppose $G - (V(B_P) - X_P)$ is $(4, X_P \cup \{a, a'\})$-connected. Then there exists a planar a-a' chain H in G such that $G - V(H)$ is 2-connected and $B_P \subseteq G - V(H)$. Moreover, such a chain can be found in $O(|V(G)||E(G)|)$ time.*

Theorem 2.1 is fundamental for constructing a non-separating chain decomposition of a 4-connected graph. As an application of Theorem 2.1, we will show how to produce the first chain in such a decomposition.

In order to do this, we also need the following result proved by Tutte [15]: if G is a 3-connected graph, $e \in E(G)$, and $v \in V(G)$ not incident with e, then G has an induced cycle C such that $e \in E(C)$, $v \notin V(C)$, and $G - V(C)$ is connected. Cheriyan and Maheshwari [4]

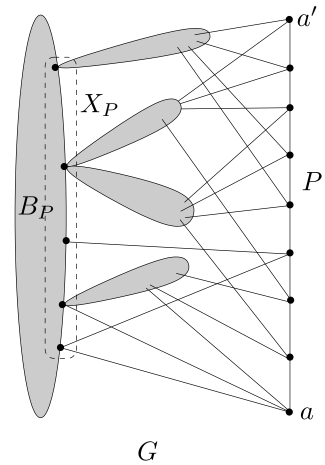

Figure 5: G, a, a', P, B_P, X_P in Theorem 2.1.

gave a linear algorithm for finding such a cycle C. We note that Tutte's result shows that $f(1) = 3$ for the above mentioned conjecture of Lovász, and the proof of Theorem 2.1 can be modified to show that $f(2) = 5$.

Corollary 2.1. *Let G be a 4-connected graph and let $ra \in E(G)$. Then there exists a planar cyclic chain H in G rooted at r such that $G - V(H - r)$ is 2-connected and ra is a piece of H, and such a chain can be found in $O(|V(G)||E(G)|)$ time.*

Proof. By Tutte's theorem, G has a non-separating induced cycle C such that $G - V(C)$ is connected. Let $P := C - r$, and let a' be the end of P other than a. Since C is induced, exactly two neighbors of r lie on P, namely a and a'. Thus, since $G - V(C)$ is connected, r is not a cut vertex of $G - V(P)$. Let B_P be the block of $G - V(P)$ containing r. Note that since G is 4-connected and P is induced in G, r has at least two neighbors in B_P, and hence, B_P is 2-connected.

Let $X_P := N_G(G - V(B_P))$. Then $X_P \subseteq V(B_P)$. Since G is 4-connected, $G - (V(B_P) - X_P)$ is $(4, X_P \cup \{a, a'\})$-connected. Thus, G, a, a', P, B_P, X_P satisfy the hypotheses of Theorem 2.1, and hence, one can find in $O(|V(G)||E(G)|)$ time a planar a-a' chain H' in G such that $G - V(H')$ is 2-connected and $B_P \subseteq G - V(H')$. Note that since $N_G(r) \subseteq V(B_P) \cup \{a, a'\}$, $r \notin N_G(H' - \{a, a'\})$. Let H be obtained from H' by adding r and the edges ra, ra'. Note that H' is an induced subgraph of G. Hence, H is a planar cyclic chain in G rooted at r such that ra induces a piece of H and $G - (V(H) - \{r\})$ is 2-connected.

By combining a result of Thomassen [14] and an algorithm in [5], Corollary 2.1 implies that if G is a 4-connected graph and $r \in V(G)$, then G has a cycle C

such that $r \in V(C)$ and $G - V(C - r)$ is 2-connected, and such a cycle can be found in $O(|V(G)||E(G)|)$ time.

3 Non-separating chain decomposition

Our goal is to decompose a 4-connected graph into subgraphs $H_0, \ldots, H_t$ such that H_0 and H_t are planar cyclic chains in G and each H_i ($1 \leq i \leq t-1$) is a planar chain or a "triangle chain". Planar chains and planar cyclic chains are natural generalization of paths and cycles, respectively. However, the "triangle chains" are not chains, but they arise in the proof and have a simple structure.

In the following definitions, let G be a graph, let $r \in V(G)$, and let F be an induced subgraph of G such that $r \in V(F)$.

We say that a planar x-y chain H in G is an *up F-chain* in G if $V(H) - \{x, y\} \subseteq V(G) - V(F)$, $\{x, y\} \subseteq V(F)$, and $N_G(H - \{x, y\}) \subseteq (V(G) - V(F)) \cup \{x, y\}$. We let $I(H) := V(H) - \{x, y\}$. See Figure 6 for an illustration.

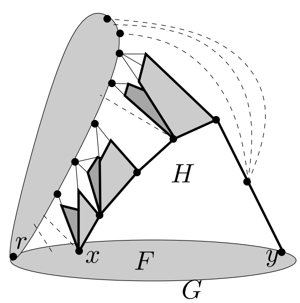

Figure 6: An up F-chain in G.

We say that a planar x-y chain H in G is a *down F-chain* in G if $V(H) - \{x, y\} \subseteq V(G) - V(F)$, $\{x, y\} \subseteq V(G) - V(F - r)$ and $N_G(H - \{x, y\}) \subseteq V(F) \cup \{x, y\}$. We let $I(H) := V(H) - \{x, y\}$. See Figure 7 for an illustration.

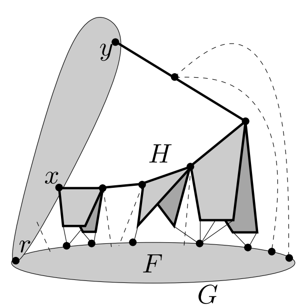

Figure 7: A down F-chain in G.

We say that a planar x-y chain H in G is an *elementary F-chain* in G if $\{x, y\} \subseteq V(F)$ and H is an x-y path of length two. We let $I(H) := V(H) - \{x, y\}$. See Figure 8 for an illustration.

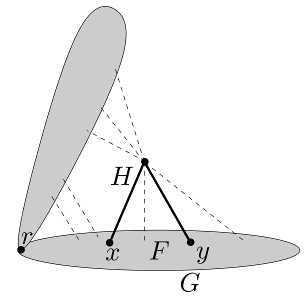

Figure 8: An elementary F-chain in G.

Next, we describe triangle chains that we allow in a non-separating chain decomposition. Suppose that $\{v_1, v_2, v_3\} \subseteq V(G) - V(F)$ induces a triangle T in G, and for each $i = 1, 2, 3$, v_i has exactly one neighbor x_i in $V(F - r)$ and exactly one neighbor y_i in $V(G) - (V(F) \cup V(T))$. Moreover, assume that x_1, x_2, x_3 are distinct and y_1, y_2, y_3 are distinct. Let H denote the subgraph of G obtained from T by adding vertices x_1, x_2, x_3 and by adding edges v_1x_1, v_2x_2, v_3x_3. We say that H is a *triangle F-chain* in G, and we let $I(H) := \{v_1, v_2, v_3\}$. See Figure 9 for an illustration.

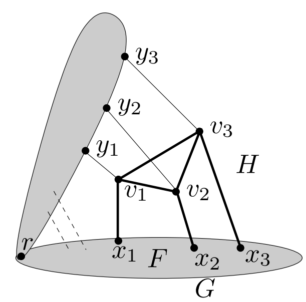

Figure 9: A triangle F-chain in G.

For convenience, we say that a chain H in G is a *good F-chain* in G if it is one of the following: an up F-chain, or a down F-chain, or an elementary F-chain, or a triangle F-chain. We also use the following notation: if H is a planar cyclic chain in G, then we let $I(H) := V(H)$. We are now ready to describe a non-separating chain decomposition.

Let G be a graph and let $r \in V(G)$. A *non-separating chain decomposition of G rooted at r* is a collection $(H_1, \ldots, H_t)$ of subgraphs of G, where $t \geq 2$, such that

(i) H_1 is planar cyclic chain in G rooted at r,

(ii) for each $i \in \{2, \ldots, t-1\}$, H_i is a good G_i-chain in G, where $G_i := G[\bigcup_{j=1}^{i-1} I(H_j)]$,

(iii) for each $i \in \{1, \ldots, t-1\}$, G_i and $G - V(G_i - r)$ are 2-connected, and

(iv) $H_t := G - (V(G_{t-1}) - \{r\})$ is planar cyclic chain in G rooted at r.

The chains $H_2, \ldots, H_{t-1}$ are called *internal chains* of the chain decomposition. If an edge ra induces a piece of H_1, then we say that $(H_1, \ldots, H_t)$ is a non-separating chain decomposition of G *starting at* ra.

The idea for constructing a non-separating chain decomposition is to find an internal chain one at a time. This is possible due to the following result proved in [2]. See Figure 10 for an illustration of its statement. We note that the proof of Theorem 3.1 relies on Theorem 2.1.

THEOREM 3.1. *Let G be a 4-connected graph, let F be a subgraph of G, and let $r \in V(F)$ such that $G_F := G - (V(F) - \{r\})$ is 2-connected. Suppose that G has a path P from a to a' such that*

(i) *$V(P) \cap V(F) = \{a, a'\}$ and P is an induced path in $G - aa'$,*

(ii) *$P(a, a')$ is a non-separating path in G_F,*

(iii) *r is contained in a nontrivial block B_P of $G_F - V(P(a, a'))$, and*

(iv) *if $r \in \{a, a'\}$, then r is not a cut vertex of $G_F - V(P(a, a'))$.*

Then there exists a good F-chain H in G such that $G_F - I(H)$ is 2-connected and $B_P \subseteq G_F - I(H)$. Moreover, such a chain can be found in $O(|V(G)||E(G)|)$ time.

With the help of Theorem 3.1, we are able to show the following result proved in [2].

THEOREM 3.2. *Let G be a 4-connected graph, let $r \in V(G)$, and let $ra \in E(G)$. Then G has a non-separating chain decomposition rooted at r and starting at ra, and such a decomposition can be found in $O(|V(G)|^2|E(G)|)$ time.*

Proof. (*Sketch*) We will describe an iterative algorithm to find a non-separating chain decomposition of G. Corollary 2.1 guarantees the existence of H_1. Thus, we may assume that so far we have obtained a partial non-separating chain decomposition $H_1, \ldots, H_s$ of G (that is, (i), (ii) and (iii) in the definition of non-separating chain decomposition are satisfied). Let $F := \bigcup_{i=1}^{s} I(H_i)$ and let $G_F := G - V(F - r)$. Using some auxiliary

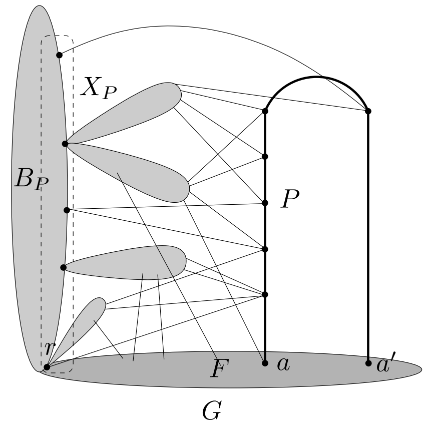

Figure 10: G, r, a, a', P, B_P, X_P in Theorem 3.1.

results in [2], one can show that either (1) there exist $a, a' \in V(F)$ and a non-separating path P from a to a' in G satisfying the hypothesis of Theorem 3.1, or (2) G_F is a planar cyclic chain rooted at r. Moreover, one can in $O(|V(G)| + |E(G)|)$ time either find a path as in (1) or certify that (2) holds.

If (2) holds, then taking $H_{s+1} := G_F$ and $t := s + 1$ we obtain a non-separating chain decomposition $H_1, \ldots, H_t$ as required.

So assume (1) holds. Thus, by Theorem 3.1, there exists a good F-chain H in G such that $G_F - I(H)$ is 2-connected. Let $H_{s+1} := H$. Then $H_1, \ldots, H_s, H_{s+1}$ satisfy (i), (ii) and (iii) in the definition of non-separating chain decomposition. Thus, this process can be repeated until a non-separating chain decomposition is found. One can also show that this algorithm runs in $O(|V(G)|^2|E(G)|)$ time.

4 Independent spanning trees

The decomposition in Theorem 3.2 can be used to construct four independent spanning trees in a 4-connected graph. To better explain our approach, let us first sketch the Itai-Rodeh algorithm for constructing two independent spanning trees in a 2-connected graph. Let G be a 2-connected graph.

First, they construct an ear decomposition of G starting from a cycle containing a given edge rt. Then they use this ear decomposition to find an *r-t numbering* which is a function $g : V(G) \mapsto \{1, \ldots, |V(G)|\}$ such that

(i) $g(r) = 1$ and $g(t) = |V(G)|$, and

(ii) every vertex $v \in V(G) - \{r, t\}$ has two neighbors u and w such that $g(u) < g(v) < g(w)$.

Using this numbering, they construct two independent spanning trees T_1 and T_2 of G rooted at r by defining for each vertex $v \in V(G) - \{r\}$ its parent in each tree. Tree T_1 has for each $v \in V(G) - \{r\}$ as parent a vertex u such that u is adjacent to v and $g(u) < g(v)$. Tree T_2 contains the edge rt, and has for each $v \in V(G) - \{r, t\}$ as parent a vertex w such that w is adjacent to v and $g(w) > g(v)$. It is not hard to show that T_1 and T_2 are independent spanning trees.

The idea for constructing four independent spanning trees in a 4-connected graph is similar to the 2-connected case. However, we need two numberings on the vertices of the graph. Let G be a 4-connected graph and let $r \in V(G)$. By Theorem 3.2, G has a non-separating chain decomposition $(H_0, \ldots, H_t)$ rooted at r. Recall that $G_i := G[\bigcup_{j=0}^{i-1} I(H_j)]$. We compute two numberings g, f defined on $V(G)$ which resemble r-t numberings. From g we compute two independent spanning trees T_1, T_2 such that the restrictions of T_1, T_2 to G_i are independent spanning trees in G_i rooted at r. Similarly, from f we compute two spanning trees T_3, T_4 such that the restrictions of T_3, T_4 to $G - V(G_i - r)$ are independent spanning trees in $G - V(G_i - r)$ rooted at r.

The main difficulty lies in the fact that it is not possible to number all vertices of G, because of 2-connected blocks (respectively, 2-connected pieces) in the planar chains (respectively, planar cyclic chains). To overcome this obstacle we compute four independent spanning trees in each one of these planar blocks (respectively, planar pieces) using (a linear algorithm in [12] for) Huck's result [7] mentioned in the introduction. These trees are then used to number the vertices in the planar blocks and pieces which have neighbors outside its chain. The numbering of vertices in triangle chains are less difficult. The details are somewhat complicated and due to the lack of space we cannot present them here, but they are given in [3]. By combining the partial numberings g, f with Huck's result, we can produce four independent spanning trees in G rooted at r, and therefore, prove Theorem 1.1.

We note that since the complexity of our algorithm in Theorem 3.2 for finding a non-separating chain decomposition is $O(|V(G)|^2|E(G)|)$, we cannot apply it directly to the input graph G, if we want to obtain an $O(|V(G)|^3)$ algorithm for finding four independent spanning trees. However, we can obtain such an $O(|V(G)|^3)$ algorithm by applying the following result of Ibaraki and Nagamochi [8] as a pre-processing step.

THEOREM 4.1. *Let G be a k-connected graph for some integer $k \geq 1$. Then one can find in $O(|V(G)| + |E(G)|)$ time a sparse spanning k-connected subgraph of G.*

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