Disjoint Paths in Graphs II, a special case

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Abstract

Let G be a graph, $\{a, b, c\} \subseteq V(G)$, and $\{a', b', c'\} \subseteq V(G)$ such that $\{a, b, c\} \neq \{a', b', c'\}$. We say that $(G, \{a, c\}, \{a', c'\}, (b, b'))$ is an *obstruction* if, for any three vertex disjoint paths from $\{a, b, c\}$ to $\{a', b', c'\}$ in G, one path is from b to b'. In this paper we characterize a special class of obstructions. This will be used to characterize all obstructions.

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1 Introduction

We will use the notation and terminology introduced in [4]. Let G be a graph, $\{a,b,c\} \subseteq V(G)$, and $\{a',b',c'\} \subseteq V(G)$, and assume that $\{a,b,c\} \neq \{a',b',c'\}$. We say that $(G,\{a,c\},\{a',c'\},(b,b'))$ is an *obstruction* if, for any three vertex disjoint paths from $\{a,b,c\}$ to $\{a',b',c'\}$ in G, one path is from b to b'.

In connection with their graph minors project, Robertson and Seymour asked the problem of characterizing all obstructions. In this paper, we solve this problem of Robertson and Seymour under the following assumption: (i) $G-\{b,b'\}$ contains disjoint paths A, C from $\{a,c\}$ to $\{a',c'\}$, (ii) $A \cup C$ is an induced subgraph of G, and (iii) $G-(A\cup C)$ is connected. This result will be used in the next paper to characterize all obstructions.

To state our result more precisely and for convenience, we repeat the following concepts and notation from [4]: 3-planar graphs, rungs, and ladders.

A *3-planar graph* $(G,\mathcal{A})$ consists of a graph G and a set $\mathcal{A}=\{A_1,\ldots,A_k\}$ of pairwise disjoint subsets of $V(G)$ (possibly $\mathcal{A}=\emptyset$) such that

(a) for $1 \le i \neq j \le k$, $N(A_i)\cap A_j=\emptyset$,

(b) for $1 \le i \le k$, $|N(A_i)| \le 3$, and

(c) if $p(G,\mathcal{A})$ denotes the graph obtained from G by (for each i) deleting A_i and adding edges joining every pair of distinct vertices in $N(A_i)$, then $p(G,\mathcal{A})$ can be drawn in a closed disc D with no pair of edges crossing such that, for each A_i with $|N(A_i)|=3$, $N(A_i)$ induces a facial triangle in $p(G,\mathcal{A})$.

If, in addition, $b_0,b_1,\ldots,b_n$ are vertices of G such that $b_i \notin A_j$ for any $A_j \in \mathcal{A}$ and $b_0,b_1,\ldots,b_n$ occur on the boundary of D in that cyclic order, then we say that $(G,\mathcal{A},b_0,b_1,\ldots,b_n)$ is *3-planar*. If there is no need to specify $\mathcal{A}$, we will simply say that $(G,b_0,b_1,\ldots,b_n)$ is 3-planar.

(1.1) Definition. Let G be a graph, $\{a,b,c\} \subseteq V(G)$, and $\{a',b',c'\} \subseteq V(G)$. Suppose $\{a,b,c\} \neq \{a',b,'c'\}$, and assume that G has no 3-separation (G_1,G_2) such that $\{a,b,c\} \subseteq G_1$ and $\{a',b',c'\} \subseteq G_2$. Then we call $(G,(a,b,c),(a',b',c'))$ a *rung* if one of the following conditions is satisfied:

(1) $b=b'$ or $\{a,c\}=\{a',c'\}$;

(2) $a=a'$ and $(G-a,c,c',b',b)$ is 3-planar;

(2$'$) $c=c'$ and $(G-c,a,a',b',b)$ is 3-planar;

(3) $\{a,b,c\}\cap\{a',b',c'\}=\emptyset$ and (G,a',b',c',c,b,a) is 3-planar;

(4) $\{a,b,c\}\cap\{a',b',c'\}=\emptyset$, G has a 1-separation (G_1,G_2) such that $\{a,a',b,b'\}\subseteq G_1$, $\{c,c'\}\subseteq G_2$, and (G_1,a,a',b',b) is 3-planar;

(4′) $\{a,b,c\}\cap\{a',b',c'\}=\emptyset$, G has a 1-separation (G_1,G_2) such that $\{c,c',b,b'\}\subseteq G_1$, $\{a,a'\}\subseteq G_2$, and (G_1,c,c',b',b) is 3-planar;

(5) $\{a,b,c\}\cap\{a',b',c'\}=\emptyset$, (G,a,a',b',b) is 3-planar, and G has a separation (G_1,G_2) such that $V(G_1\cap G_2)=\{z,b\}$ (or $V(G_1\cap G_2)=\{z,b'\}$), $\{a,a',b,b'\}\subseteq G_1$, $\{c,c'\}\subseteq G_2$, and (G_2,c,c',z,b) (or (G_2,c,c',b',z)) is 3-planar;

(5′) $\{a,b,c\}\cap\{a',b',c'\}=\emptyset$, (G,c,c',b',b) is 3-planar, and G has a separation (G_1,G_2) such that $V(G_1\cap G_2)=\{z,b\}$ (or $V(G_1\cap G_2)=\{z,b'\}$), $\{c,c',b,b'\}\subseteq G_1$, $\{a,a'\}\subseteq G_2$, and (G_2,a,a',z,b) (or (G_2,a,a',b',z)) is 3-planar;

(6) $\{a,b,c\}\cap\{a',b',c'\}=\emptyset$, and there are pairwise edge disjoint subgraphs G_a,G_c,M of G such that $G=G_a\cup G_c\cup M$, $V(G_a\cap M)=\{u,w\}$, $V(G_c\cap M)=\{p,q\}$, $V(G_a\cap G_c)=\emptyset$, $\{a,a',b'\}\subseteq G_a$, $\{c,c',b\}\subseteq G_c$, (G_a,a,a',b',w,u) is 3-planar, and (G_c,c',c,b,p,q) is 3-planar;

(6′) $\{a,b,c\}\cap\{a',b',c'\}=\emptyset$, and there are pairwise edge disjoint subgraphs G_a,G_c,M of G such that $G=G_a\cup G_c\cup M$, $V(G_a\cap M)=\{u,w\}$, $V(G_c\cap M)=\{p,q\}$, $V(G_a\cap G_c)=\emptyset$, $\{a,a',b\}\subseteq G_a$, $\{c,c',b'\}\subseteq G_c$, (G_a,b,a,a',w,u) is 3-planar, and (G_c,b',c',c,p,q) is 3-planar;

(7) $\{a,b,c\}\cap\{a',b',c'\}=\emptyset$, and there are pairwise edge disjoint subgraphs G_a,G_c,M of G such that $G=G_a\cup G_c\cup M$, $V(G_a\cap M)=\{b,b',w\}$, $V(G_c\cap M)=\{b,b',p\}$, $V(G_a\cap G_c)=\{b,b'\}$, $\{a,a'\}\subseteq G_a$, $\{c,c'\}\subseteq G_c$, (G_a,a,a',b',w,b) is 3-planar, and (G_c,c',c,b,p,b') is 3-planar.

Let L be a graph and let $R_1,\dots,R_m$ be edge disjoint subgraphs of L such that

(i) $(R_i,(x_{i-1},v_{i-1},y_{i-1}),(x_i,v_i,y_i))$ is a rung for each $1\le i\le m$,

(ii) $V(R_i\cap R_j)=\{x_i,v_i,y_i\}\cap\{x_{j-1},v_{j-1},y_{j-1}\}$ for $1\le i<j\le k$,

(iii) if $x_i=x_j$ then $x_k=x_i$ for all $i\le k\le j$, if $v_i=v_j$ then $v_k=v_i$ for all $i\le k\le j$, and if $y_i=y_j$ then $y_k=y_i$ for all $i\le k\le j$.

(iv) $L=(\bigcup_{i=1}^m R_i)+S$, where S consists of edges of L with both ends in $\{x_i,v_i,y_i\}$ for some $1\le i\le m$.

Then we call $(L,(x_0,v_0,y_0),(x_m,v_m,y_m))$ a *ladder with rungs* $(R_i,(x_{i-1},v_{i-1},y_{i-1}),(x_i,v_i,y_i))$, $i=1,\dots,m$, or simply, a *ladder along* $v_0\dots v_m$.

In order to describe the main result of this paper, we also need the following concepts of good rung and good ladder.

(1.2) Definition. Let $(G,(a,b,c),(a',b',c'))$ be a rung. Suppose that $G-\{b,b'\}$ has disjoint paths A,C from a,c to a',c', respectively, and assume that the following conditions are satisfied:

(1) If G is connected and $b\ne b'$, then $G-((A-a')\cup(C-c))$ contains a $(\{b,a'\},\{c,b'\})$-linkage, and $G-((A-a)\cup(C-c'))$ contains an $(\{a,b'\},\{b,c'\})$-linkage;

(1′) if G is connected and $b = b'$, then $G - ((A - a) \cup C)$ has a path from a to b', $G - ((A - a') \cup C)$ has a path from a' to b, $G - (A \cup (C - c))$ has a path from c to b', and $G - (A \cup (C - c'))$ has a path from c' to b;

(2) if G is not connected and $\{a, b\} \cup \{a', b'\}$ is contained in a component of G, then $G - ((A - a) \cup C)$ contains a path from a to b' (and not using b if $b \neq b'$) and $G - ((A - a') \cup C)$ contains a path from b to a' (and not using b' if $b \neq b'$); and

(2′) if G is not connected and $\{b, c\} \cup \{b', c'\}$ is contained in a component of G, then $G - (A \cup (C - c))$ contains a path from c to b' (and not using b if $b \neq b'$), and $G - (A \cup (C - c'))$ contains a path from b to c' (and not using b' if $b \neq b'$).

Then we call $(G, (a, b, c), (a', b', c'), A, C)$ a *good* rung. Let $(L, (x_0, v_0, y_0), (x_m, v_m, y_m))$ be a ladder with good rungs $(R_i, (x_{i-1}, v_{i-1}, y_{i-1}), (x_i, v_i, y_i), X_i, Y_i)$, $i = 1, \ldots, m$. Then we say that $(L, (x_0, v_0, y_0), (x_m, v_m, y_m), X, Y)$ is a *good* ladder along $v_0 \ldots v_m$, where $X = \bigcup_{i=1}^m X_i$ and $Y = \bigcup_{i=1}^m Y_i$.

For a sequence S, the *reduced sequence* of S is the sequence obtained from S by removing all but one consecutive identical elements. For example, the reduced sequence of *aaabcca* is *abca*. We can now state the main result of this paper.

(1.3) Theorem. *Let $(G, \{a, c\}, \{a', c'\}, (b, b'))$ be an obstruction satisfying the following conditions:*

(i) $G - \{b, b'\}$ contains disjoint paths A, C from a, c to a', c', respectively,

(ii) $A \cup C$ is an induced subgraph of G, and

(iii) $G - (A \cup C)$ is connected.

Then one of the following statements holds:

(1) There is a separation (G_1, G_2) in G of order at most 2 such that $\{a, b, c\} \subseteq V(G_1)$ and $\{a', b', c'\} \subseteq V(G_2)$.

(2) There is a subset $T \subseteq V(G)$ such that $|T| \leq 3$ and some component of $G - T$ contains no element of $\{a, b, c\} \cup \{a', b', c'\}$.

(3) $(G, (a, b, c), (a', b', c'), A, C)$ is a good ladder.

(4) G has a separation (J, L) such that $V(J \cap L) = \{w_0, \ldots, w_n\}$, $(J, w_0, \ldots, w_n)$ is 3-planar, $(L, (a, b, c), (a', b', c'), A, C)$ is a good ladder along a sequence $v_0 \ldots v_m$, where $v_0 = b$, $v_m = b'$, and $w_0 \ldots w_n$ is the reduced sequence of $v_0 \ldots v_m$.

To prove (1.3), we need several results from [4], which will be stated in Section 2. The proof of (1.3) is by way of contradiction. Assume the theorem is false, and let $(G, \{a, c\}, \{a', c'\}, (b, b'))$ be a counterexample with $|V(G)| + |E(G)|$ minimum. In Section 3, we will derive properties of G, and the proof of (1.3) will be completed in Section 4.

2 Preliminary results

We begin this section by a reformulation of a result, proved independently in [1], [2], and [3]. Also, see (2.4) of [4].

(2.1) Theorem. *Let G be a graph and $\{s_1, s_2, t_1, t_2\} \subseteq V(G)$. Then G contains no $(\{s_1, t_1\}, \{s_2, t_2\})$-linkage iff (G, s_1, s_2, t_1, t_2) is 3-planar.*

When dealing with a 3-planar graph $(G, \mathcal{A})$, it will be convenient to choose $\mathcal{A}$ to be "minimal". Let $(G, \mathcal{A})$ be 3-planar, let $A \in \mathcal{A}$ with $N(A) = \{a_1, \ldots, a_m\}$ (where $m \leq 3$), and let $H = G[A \cup N(A)]$. We say that A is *minimal* if there is no collection $\mathcal{H}$ of pairwise disjoint subsets of A such that $\mathcal{H} \neq \{A\}$ and $(H, \mathcal{H}, a_1, \ldots, a_m)$ is 3-planar. We say that $\mathcal{A}$ is *minimal* (in G) if every member of $\mathcal{A}$ is minimal.

The following result is (2.6) in [4] which says that, for any 3-planar graph $(G, \mathcal{A})$, we may choose $\mathcal{A}$ to be minimal.

(2.2) Proposition. *If $(G, b_0, \ldots, b_n)$ is 3-planar, then there is a collection $\mathcal{A}$ of pairwise disjoint subsets of $V(G) - \{b_0, \ldots, b_n\}$ such that $(G, \mathcal{A}, b_0, \ldots, b_n)$ is 3-planar and $\mathcal{A}$ is minimal.*

The next threes results are (3.1), (3.2) and (3.3) in [4], concerning disjoint paths in 3-planar graphs.

(2.3) Proposition. *Let $(G, \mathcal{A})$ be 3-planar, let $A \in \mathcal{A}$, and let $H := G[A \cup N(A)]$. Suppose that A is minimal. Then*

(1) for any proper subset T of $N(A)$, $H - T$ is connected,

(2) if $N(A) = \{u, v, w\}$, then for any $x \in A$, H contains an $(\{x, u\}, \{v, w\})$-linkage, and

(3) for any $x \in A$, $N(A)$ is contained in a component of $H - x$.

(2.4) Proposition. *Let $(G, \mathcal{A})$ be 3-planar and let $b, b' \in V(G)$ be distinct. Let $u \in V(G)$, let $u^* = u$ when $u \in p(G, \mathcal{A})$, and let u^* be a vertex in $N(U_u)$ when $u \in U_u$ for some $U_u \in \mathcal{A}$. Let $v \in V(G)$, let $v^* = v$ when $v \in p(G, \mathcal{A})$, and let v^* be a vertex in $N(U_v)$ when $v \in U_v$ for some $U_v \in \mathcal{A}$. Suppose*

(i) $u^* \neq v^*$, and $U_u \neq U_v$ when both U_u and U_v are defined,

(ii) for any $A \in \mathcal{A}$ and for any proper subset T of $N(A)$, $G[A \cup N(A)] - T$ is connected,

(iii) for any $A \in \mathcal{A}$ with $|N(A)| = 3$, A is minimal, and

(iv) $p(G, \mathcal{A})$ contains a $(\{u^*, b\}, \{v^*, b'\})$-linkage L^*.

Then G contains a $(\{u, b\}, \{v, b'\})$-linkage L such that, for any vertex z of $p(G, \mathcal{A})$, z is contained in a path in L iff z is contained in a path in L^*.

We note that if $A \in \mathcal{A}$ and A is minimal then (ii) follows from (1) of (2.3) and (iii) is obvious. We also note that, most of the time, A is minimal when (2.4) is applied.

(2.5) Theorem. *Let G be a 2-connected plane graph with outer cycle C, let $b, b' \in V(C)$ be distinct, let $x \in V(G) - V(C)$, and let $y \in V(G) - \{b, b', x\}$. Suppose that*

(i) G contains a path from x to $bCb' - \{b, b'\}$ and a path from x to $b'Cb - \{b, b'\}$, both internally disjoint from C, and

(ii) G contains no $(\{x, b\}, \{y, b'\})$-linkage or G contains no $(\{x, b'\}, \{y, b\})$-linkage.

Then there is a vertex $z \in V(G)$ such that

(1) either x and b belong to different components of $G - \{z, b'\}$ or x and b' belong to different components of $G - \{z, b\}$, and

(2) either y and b belong to different components of $G - \{z, b'\}$ or y and b' belong to different components of $G - \{z, b\}$.

The next two results are (3.4) and (3.5) in [4], concerning disjoint paths in 2-connected graphs.

(2.6) Theorem. *Let H be a 2-connected graph and $\{u, v, w, c, c'\} \subseteq V(H)$, and assume that*

(i) H contains neither a $(\{c, u\}, \{c', v\})$-linkage nor a $(\{c, w\}, \{c', v\})$-linkage, and

(ii) $(H, c', c, a_1, a_2, a_3)$ is not 3-planar for any sequence $a_1a_2a_3 \in \{wvu, vwu, vuw\}$.

Then

(1) there are vertices p, q of H and a separation (H_1, H_2) in H such that $V(H_1 \cap H_2) = \{p, q\}$, $\{c, c', v\} \subseteq H_1$, $\{u, w\} \subseteq H_2$, and (H_1, c', c, v, p, q) is 3-planar, and

(2) there exists a $(\{c, u\}, \{c', w\})$-linkage in H such that v is not used by this linkage unless in (1) we can choose $p = v$.

(2.7) Theorem. *Let H be a 2-connected graph and $\{u, v, w, c, c'\} \subseteq V(H)$, and assume that*

(i) H contains no $(\{c, u\}, \{c', v\})$-linkage,

(ii) (H, c', c, v, w, u) is not 3-planar,

(iii) H has no 2-separation (K, K') such that $V(K \cap K')$ contains u or v, $\{v, c\} \subseteq K$ and $\{u, c'\} \subseteq K'$, and

(iv) $H - \{u, v\}$ is connected and $uv \notin E(H)$.

Then one of the following holds:

(1) $H - v$ has a $(\{c, u\}, \{c', w\})$-linkage or $H - u$ has a $(\{c, w\}, \{c', v\})$-linkage; or

(2) there are a vertex p of H and a separation (H_1, H_2) in H such that $V(H_1 \cap H_2) = \{p, u, v\}$, $\{c, c'\} \subseteq H_1$, $w \in H_2$, (H_1, c', c, v, p, u) is 3-planar.

Finally, we introduce the following notation which will be used, for convenience, in the proof of (1.3).

(2.8) Notation. Let H be a block of a connected graph H'. Let $\sigma_{H',H} : V(H') \to V(H)$ be defined as follows: if $x \in V(H)$ then $\sigma_{H',H}(x) = x$, and if $x \in V(H'-H)$ then $\sigma_{H',H}(x)$ is the cut vertex of H' contained in all paths in H' from x to H. For $S \subseteq V(H')$, let $\sigma_{H',H}(S) = \{\sigma_{H',H}(x) : x \in S\}$. If there is no confusion, we simply write $\sigma(x)$ and $\sigma(S)$.

3 Technical lemmas

In this section, we will prove fifteen technical lemmas about a hypothetical minimum counter example to (1.3).

Assume (1.3) is false, and let $(G, \{a, c\}, \{a', c'\}, (b, b'))$ be a counter example to (1.3) such that $|V(G)| + |E(G)|$ is minimum. Then we have the following:

(C1) $G - \{b, b'\}$ contains disjoint paths A, C from a, c to a', c', respectively.

(C2) $A \cup C$ is an induced subgraph of G.

(C3) $G - (A \cup C)$ is connected.

(C4) There does not exist a separation (G_1, G_2) in G of order at most 2 such that $\{a, b, c\} \subseteq V(G_1)$ and $\{a', b', c'\} \subseteq V(G_2)$.

(C5) There does not exist a subset $T \subseteq V(G)$ such that $|T| \leq 3$ and some component of $G - T$ contains no element of $\{a, b, c\} \cup \{a', b', c'\}$.

(C6) $(G, (a, b, c), (a', b', c'), A, C)$ is not a good ladder.

(C7) There does not exist a separation (J, L) in G such that $V(J \cap L) = \{w_0, \dots, w_n\}$, $(J, w_0, \dots, w_n)$ is 3-planar, $(L, (a, b, c), (a', b', c'), A, C)$ is a good ladder along a sequence $v_0 \dots v_m$, where $v_0 = b$, $v_m = b'$, and $w_0 \dots w_n$ is the reduced sequence of $v_0 \dots v_m$.

For convenience, let $H' := G - (A \cup C)$. We will use $N(X)$ and $N(x)$ instead of $N_G(X)$ and $N_G(x)$ for any $X \subseteq G$ and $x \in V(G)$.

(3.1) Lemma. *G contains no 3-separation (G_1, G_2) such that $\{a, b, c\} \subseteq G_1$ and $\{a', b', c'\} \subseteq G_2$.*

Proof. First, we claim that $\{ab, cb, a'b', c'b'\} \cap E(G) = \emptyset$. For otherwise, let $G' := G - \{ab, cb, a'b', c'b'\}$, and we have $|V(G')| + |E(G')| < |V(G)| + |E(G)|$. Note that $(G', \{a, c\}, \{a', c'\}, (b, b'))$ is an obstruction, and (C1)-(C3) hold for G'. So by the choice of G and by (C4) and (C5), either (1) $(G', (a, b, c), (a', b', c'), A, C)$ is a good ladder along a sequence $v_0 \dots v_l$, where $v_0 = b$ and $v_l = b'$, or (2) G' has a separation (J', L') such that

$V(J' \cap L') = \{w_0, \ldots, w_k\}$, $(J', w_0, \ldots, w_k)$ is 3-planar, $(L', (a, b, c), (a', b', c'), A, C)$ is a good ladder along a sequence $v_0 \ldots v_l$, where $v_0 = b$, $v_l = b'$, and $w_0 \ldots w_k$ is the reduced sequence of $v_0 \ldots v_l$. It is then easy to see that (C6) or (C7) holds for G, a contradiction.

Now assume that there is a 3-separation (G_1, G_2) in G such that $\{a, b, c\} \subseteq G_1$ and $\{a', b', c'\} \subseteq G_2$. Let $V(G_1 \cap G_2) = \{a'', b'', c''\}$, where $a'' \in A$, $b'' \in H'$, and $c'' \in C$. By (C2) and since $\{ab, cb, a'b', c'b'\} \cap E(G) = \emptyset$, $\{a'', b'', c''\} \neq \{a, b, c\}$ and $\{a'', b'', c''\} \neq \{a', b', c'\}$. Because $(G, \{a, c\}, \{a', c'\}, (b, b'))$ is an obstruction, it is easy to see that $(G_1, \{a, c\}, \{a'', c''\}, (b, b''))$ and $(G_2, \{a'', c''\}, \{a', c'\}, (b'', b'))$ are obstructions.

By conditions (C1) - (C3), we have (a) $G_1 - \{b, b''\}$ contains disjoint paths aAa'' and cCc'' from a, a'' to c, c'', respectively, (b) $aAa'' \cup cCc''$ is an induced subgraph of G_1, and (c) $G_1 - (aAa'' \cup cCc'')$ is connected. By (C4) and (C5), and since $|V(G_1)| + |E(G_1)| < |V(G)| + |E(G)|$, one of the following holds: (1) $(G_1, (a, b, c), (a'', b'', c''), aAa'', cCc'')$ is a good ladder along a sequence $v_0 \ldots v_l$, where $v_0 = b$ and $v_l = b''$ (and let $w_0 \ldots w_k$ be the reduced sequence of $v_0 \ldots v_l$, let $J_1 = \{w_0, \ldots, w_k\}$, and let $L_1 = G_1$); or (2) G_1 has a separation (J_1, L_1) such that $V(J_1 \cap L_1) = \{w_0, \ldots, w_k\}$, $(J_1, w_0, \ldots, w_k)$ is 3-planar, $(L_1, (a, b, c), (a'', b'', c''), aAa'', cCc'')$ is a good ladder along a sequence $v_0 \ldots v_l$, where $v_0 = b$, $v_l = b''$, and $w_0 \ldots w_k$ is the reduced sequence of $v_0 \ldots v_l$.

Similarly, one of the following holds: (1) $(G_2, (a'', b'', c''), (a', b', c'), a''Aa', c''Cc')$ is a good ladder along a sequence $v_l \ldots v_m$, where $v_l = b''$ and $v_m = b'$ (and let $w_k \ldots w_n$ be the reduced sequence of $v_l \ldots v_m$, let $J_2 = \{w_k, \ldots, w_n\}$, and let $L_2 = G_2$); or (2) G_2 has a separation (J_2, L_2) such that $V(J_2 \cap L_2) = \{w_k, \ldots, w_n\}$, $(J_2, w_k, \ldots, w_n)$ is 3-planar, $(L_2, (a'', b'', c''), (a', b', c'), a''Aa', c''Cc')$ is a good ladder along $v_l \ldots v_m$, where $v_l = b''$, $v_m = b'$, and $w_k \ldots w_n$ is the reduced sequence of $v_l \ldots v_m$.

Now let $J = J_1 \cup J_2$ and $L = L_1 \cup L_2$. If $E(J) = \emptyset$, then $(G, (a, b, c), (a', b', c'), A, C)$ is a good ladder, contradicting (C6). If $E(J) \neq \emptyset$, then (J, L) contradicts (C7). $\square$

(3.2) Lemma. $b \neq b'$.

Proof. Suppose $b = b'$. Then by (3.1), $(G, (a, b, c), (a', b', c'))$ is a rung as in (1) of (1.1).

Assume first that $N(x) \cap V(H') \neq \emptyset$ for every $x \in \{a, c\} \cup \{a', c'\}$. In this case, G is connected. Because H' is connected, $G - ((A - a) \cup C)$ has a path from a to b', $G - ((A - a') \cup C)$ has a path from a' to b, $G - (A \cup (C - c))$ has a path from c to b', and $G - (A \cup (C - c'))$ has a path from c' to b. By (1′) of (1.2), $(G, (a, b, c), (a', b', c'), A, C)$ is a good rung, contradicting (C6).

So assume by symmetry that $N(a) \cap V(H') = \emptyset$.

Then $a = a'$. Otherwise, a has a unique neighbor a^* and $a^* \in A$. Hence by (3.1), G has no separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{a^*, b, c\}$, $\{a, b, c\} \subseteq G_1$ and $\{a', b', c'\} \subseteq G_2$. So $|V(G)| = 4$ and $E(G) = \emptyset$. By (1.2), $(G, (a, b, c), (a', b', c'), A, C)$ is a good rung, contradicting (C6).

If $N(c) \cap V(H') = \emptyset$ or $N(c') \cap V(H') = \emptyset$, then $c = c'$ by a similar argument as in the previous paragraph, contradicting the fact that $\{a, b, c\} \neq \{a', b', c'\}$.

Therefore, $N(c) \cap V(H') \neq \emptyset$ and $N(c') \cap V(H') \neq \emptyset$. Hence $G-(A \cup (C-c))$ contains a path from c to b' and $(G-(A \cup (C-c'))$ contains a path from c' to b. By $(2')$ of (1.2), $(G, (a, b, c), (a', b', c'), A, C)$ is a good rung, contradicting (C6). $\square$

(3.3) Lemma. *The following statements hold:*

(1) Both $(G-C, a, a', b', b)$ and $(G-A, c, c', b', b)$ are 3-planar.

(2) Neither b nor b' is a cut vertex of H', b and b' are contained in a block H of H', and $\sigma(N(A \cup C)) \subseteq H - \{b, b'\}$, where $\sigma := \sigma_{H',H}$ is defined as in (2.8).

(3) For each $x \in \{a, c\} \cup \{a', c'\}$, $\emptyset \neq \sigma(N(x) \cap V(H')) \subseteq H - \{b, b'\}$.

Proof. Suppose $(G-C, a, a', b', b)$ is not 3-planar. By (2.1), $G-C$ contains an $(\{a, b'\}, \{b, a'\})$-linkage. Hence, G contains disjoint paths from a, b, c to b', a', c', respectively, contradicting the assumption that $(G, \{a, c\}, \{a', c'\}, (b, b'))$ is an obstruction. Hence, $(G-C, a, a', b', b)$ is 3-planar. Similarly, $(G-A, c, c', b', b)$ is 3-planar. Thus, (1) holds.

Suppose on the contrary that (2) fails. By symmetry, we may assume that one of the following holds:

(a) b is a cut vertex of H';

(b) b and b' are not contained in any block of H'; or

(c) $b \in \sigma(N(A \cup C))$, but not (a) or (b).

If (a) holds, then let $u = b$ and let (H_1, H_2) be a separation of H' such that $V(H_1 \cap H_2) = \{u\}$ and $b' \in H_2$. We can further choose (H_1, H_2) so that $H_2 - u$ is connected. If (b) holds, then let (H_1, H_2) be a separation of H' such that $V(H_1 \cap H_2) = \{u\}$, $b \in H_1$, and $b' \in H_2$. We can further choose (H_1, H_2) so that $H_2 - u$ is connected. If (c) holds, then let $u = b$, $H_1 = \{u\}$, and $H_2 = H'$. In this case, $H_2 - u$ is connected.

Let $v \in (V(A) \cap N(H_1)) \cup \{a\}$ such that vAa' is minimal, and let $w \in (V(C) \cap N(H_1)) \cup \{c\}$ such that wCc' is minimal. By the definition of H_1, either $H_1 - u \neq \emptyset$ or $\{w, v\} \cap N(H_1) \neq \emptyset$. Then $N((aAv - v) \cup (cCw - w)) \cap V(H_2 - u) \neq \emptyset$; for otherwise, G has a separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{u, v, w\}$, $\{a, b, c\} \cup H_1 \subseteq G_1$, and $\{a', b', c'\} \cup H_2 \subset G_2$, contradicting (3.1). By symmetry, let $z \in V(aAv - v)$ with a neighbor $z' \in V(H_2 - u)$, and let $v' \in V(H_1) \cap N(v)$. Then H_1 contains a path P from v' to b, and $H_2 - u$ contains a path Q from z' to b'. It is then easy to see that $(aAz \cup Q) + zz'$ and $(P \cup vAa') + vv'$ form an $(\{a, b'\}, \{b, a'\})$-linkage in $G - C$, contradicting (1).

Finally, suppose on the contrary that (3) fails. By symmetry, assume that $\sigma(N(a) \cap V(H')) = \emptyset$. Then $a = a'$; otherwise, a has a unique neighbor a^* and $a^* \in A$, and so, G has a separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{a^*, b, c\}$, $\{a, b, c\} \subseteq G_1$, and $\{a', b', c'\} \subseteq G_2$, contradicting (3.1). By (1) and by (3.1), $(G, (a, b, c), (a', b', c'))$ is a rung as in (2) of (1.1). If $\sigma(N(c) \cap V(H')) = \emptyset$ or $\sigma(N(c') \cap V(H')) = \emptyset$, then $c = c'$

(by the same argument as for proving $a = a'$), and in this case, A, C, H' are distinct components of G, and so, $(G, (a, b, c), (a', b', c'), A, C)$ is a good rung, contradicting (C6). So $\sigma(N(x) \cap V(H')) \neq \emptyset$ for $x \in \{c, c'\}$. By (2), $\sigma(N(x) \cap V(H')) \subseteq H - \{b, b'\}$ for $x \in \{c, c'\}$, and both $H - b$ and $H - b'$ are connected. So $G - (A \cup (C - c))$ has a path from c to b' and not using b, and $G - (A \cup (C - c'))$ has a path from c' to b and not using b'. By (2′) of (1.2), $(G, (a, b, c), (a', b', c'), A, C)$ is a good rung, contradicting (C6). □

(3.4) Lemma. $\{a, c\} \neq \{a', c'\}$.

Proof. Suppose $\{a, c\} = \{a', c'\}$. Then by (3.1), $(G, (a, b, c), (a', b', c'))$ is a rung as in (1) of (1.1). In this case, $H' = G - \{a, c\}$, and by (C5), $H' = H$. If G contains both a $(\{b, a'\}, \{c, b'\})$-linkage and a $(\{b, c'\}, \{a, b'\})$-linkage, then by (1) of (1.2), $(G, (a, b, c), (a', b', c'), A, C)$ is a good rung, contradicting (C6). Therefore, we may assume by symmetry that G does not contain any $(\{b, a'\}, \{c, b'\})$-linkage. By (2.1) and by (C5), G can be drawn in a closed disc in the plane with no pair of edges crossing such that b, c, a, b' occur on the boundary of the disc in that clockwise order. Because $H' = H = G - \{a, c\}$ and by (3) of (3.3), $|V(H)| \geq 3$. Hence, H is a 2-connected plane subgraph of G.

Let $w_1, \ldots, w_{n-1}$ be the vertices of H such that $w_i \in N(\{a, c\})$. Let $w_0 = b$ and $w_n = b'$, and assume that $w_0, w_1, \ldots, w_n$ occur on the outer cycle of H in that clockwise order from b to b'. Let S denote the set of edges of G with one end in $\{a, c\}$ and the other in $\{w_0, \ldots, w_n\}$. For each $i \in \{1, \ldots, n\}$, let R_i be the graph with $V(R_i) = \{a, c, w_{i-1}, w_i\}$ and $E(R_i) = \emptyset$. Then $(R_i, (a, w_{i-1}, c), (a', w_i, c'), A, C)$ is a good rung. Let $L = (\bigcup_{i=1}^{n} R_i) + S$; then $(L, (a, b, c), (a', b', c'), A, C)$ is a good ladder along $w_0 \ldots w_n$. Now let $J = H$. Then (J, L) contradicts (C6). □

(3.5) Lemma. *H contains no separation (S, T) with $V(S \cap T) = \{b, b'\}$.*

Proof. By (3.4) and by symmetry, we may assume $a \neq a'$. Suppose H has a separation (S, T) with $V(S \cap T) = \{b, b'\}$. Because H is 2-connected, $S - b$, $S - b'$, $T - b$, and $T - b'$ are all connected.

By (3) of (3.3), let $v \in \sigma(N(a) \cap V(H'))$, $v' \in \sigma(N(a') \cap V(H'))$, $w \in \sigma(N(c) \cap V(H'))$, and $w' \in \sigma(N(c') \cap V(H'))$. Then $\{v, v', w, w'\} \subseteq H - \{b, b'\}$. Without loss of generality, we may assume that $v \in S - \{b, b'\}$.

We claim that $\sigma(N(A)) \subseteq S$. For otherwise, since $a \neq a'$, there are distinct $x, y \in V(A)$, $x' \in \sigma(N(x) \cap V(H'))$, and $y' \in \sigma(N(y) \cap V(H'))$ such that $x' \in S - \{b, b'\}$ and $y' \in T - \{b, b'\}$. Because $S - b$ and $T - b'$ are connected, H contains an $(\{x', b'\}, \{y', b\})$-linkage. Because $S - b'$ and $T - b$ are connected, H contains an $(\{x', b\}, \{y', b'\})$-linkage. Hence, $G - C$ contains an $(\{a, b'\}, \{b, a'\})$-linkage, contradicting (1) of (3.3).

Next we claim that $\sigma(N(C)) \subseteq S$ or $\sigma(N(C)) \subseteq T$. If $c \neq c'$, then this claim follows from the same argument for proving $\sigma(N(A)) \subseteq S$. So assume that $c = c'$. By (1) of (3.3) and by (3.1), $(G, (a, b, c), (a', b', c'))$ is a rung as in $(2')$ of (1.1). Suppose $\sigma(N(C)) \not\subseteq S$. Then we can choose $w \in \sigma(N(c) \cap V(H'))$ so that $w \in T - \{b, b'\}$. Because $S - b$ and $T - b'$ are connected, H has a $(\{v, b'\}, \{w, b\})$-linkage. Hence, $G - ((A - a) \cup (C - c'))$ contains an $(\{a, b'\}, \{b, c'\})$-linkage. Because $S - b'$ and $T - b$ are connected, H has a $(\{v', b\}, \{w, b'\})$-linkage. Hence $G - ((A - a') \cup (C - c))$ has a $(\{b, a'\}, \{c, b'\})$-linkage. By (1) of (1.2), $(G, (a, b, c), (a', b', c'), A, C)$ is a good rung, contradicting (C6).

Therefore, we have two cases.

Case 1. $\sigma(N(C)) \subseteq T$.

Then G has a separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{b, b'\}$, $A \subseteq G_1$, and $C \subseteq G_2$. Note that (G_1, a, a', b', b) and (G_2, c, c', b', b) are 3-planar; for otherwise, by (2.1), G_1 (and hence $G - C$) has an $(\{a, b'\}, \{b, a'\})$-linkage or G_2 (and hence $G - A$) has a $(\{b, c'\}, \{c, b'\})$-linkage, contradicting (1) of (3.3). Hence, (G, a', b', c', c, b, a) is 3-planar. By (3.1), $(G, (a, b, c), (a', b', c'))$ is a rung as in (3) of (1.1). Because $S - b'$ and $T - b$ are connected, H has a $(\{v', b\}, \{w, b'\})$-linkage. Hence $G - ((A - a') \cup (C - c))$ has a $(\{b, a'\}, \{c, b'\})$-linkage. Because $S - b$ and $T - b'$ are connected, H has a $(\{v, b'\}, \{w', b\})$-linkage. Hence, $G - ((A - a) \cup (C - c'))$ has an $(\{a, b'\}, \{b, c'\})$-linkage. By (1) of (1.2), $(G, (a, b, c), (a', b', c'), A, C)$ is a good rung, contradicting (C6).

Case 2. $\sigma(N(C)) \subseteq S$.

Let G' be obtained from G by deleting edges and vertices in T except $\{b, b'\}$. Then $(G', \{a, c\}, \{a', c'\}, (b, b'))$ is an obstruction. Because S is connected and by (C1)-(C3), we have (a) $G' - \{b, b'\}$ contains disjoint paths A, C from a, c to a', c', respectively, (b) $A \cup C$ is an induced subgraph in G', and (c) $G' - (A \cup C)$ is connected. By (C4) and (C5) and since $|V(G')| + |E(G')| < |V(G)| + |E(G)|$, either (1) $(G', (a, b, c), (a', b', c'), A, C)$ is a good ladder along a sequence $v_0 \dots v_m$, where $v_0 = b$ and $v_m = b'$ (and let $w_0 \dots w_n$ be the reduced sequence of $v_0 \dots v_m$, let $J' = \{w_0, \dots, w_n\}$, and let $L' = G'$), or (2) G' has a separation (J', L') such that $V(J' \cap L') = \{w_0, \dots, w_n\}$, $(J', w_0, \dots, w_n)$ is 3-planar, $(L', (a, b, c), (a', b', c'), A, C)$ is a good ladder along $v_0 \dots v_m$, where $v_0 = b$, $v_m = b'$, and $w_0 \dots w_n$ is the reduced sequence of $v_0 \dots v_m$. Let $J := J' \cup T$ and $L := L'$. Then (J, L) contradicts (C7). $\square$

By (3.2) and (3.4) and by symmetry, we may assume that $\{a, b\} \cap \{a', b'\} = \emptyset$. By (1) of (3.3), there is a collection $\mathcal{A}$ of pairwise disjoint subsets of $V(G - C) - \{a, b, a', b'\}$ such that $(G - C, \mathcal{A}, a, a', b', b)$ is 3-planar. We choose $\mathcal{A}$ to be minimal. Let $\mathcal{H} = \{S \cap V(H) : S \in \mathcal{A} \text{ and } S \cap V(H) \neq \emptyset\}$.

We will view $p(G - C, \mathcal{A})$ as a plane graph drawn in a closed disc in a plane such that a, a', b', b occur on the boundary of the disc in that clockwise order, and will view $p(H, \mathcal{H})$ as a plane subgraph of $p(G - C, \mathcal{A})$.

Since we will use (2.4) several times in the rest of the proof, we note the following: If we view $H, \mathcal{H}$ as $G, \mathcal{A}$ in (2.4), respectively, then conditions (ii) and (iii) of (2.4) are satisfied because H is 2-connected and because each $U \in \mathcal{H}$ with $|N_H(U)| = 3$ is also a member of $\mathcal{A}$ (and hence minimal). (See the remark after (2.4).)

(3.6) Lemma. *$\{b, b'\} \not\subseteq N_H(U)$ for any $U \in \mathcal{H}$.*

Proof. Suppose on the contrary $\{b, b'\} \subseteq N_H(U)$ for some $U \in \mathcal{H}$. Then $bb' \in E(p(H, \mathcal{H}))$. Let Y denote the union of those $X \in \mathcal{H}$ with $N_H(X) \subseteq N_H(U)$.

Claim 1. bb' is contained in the outer cycle of $p(H, \mathcal{H})$, $bb' \notin E(H)$, and for every $X \in \mathcal{H}$ with $X \not\subseteq Y$, $\{b, b'\} \not\subseteq N_H(X)$.

First, we show that bb' is contained in the outer cycle of $p(H, \mathcal{H})$. For otherwise, by planarity, $p(H, \mathcal{H})$ contains a separation (S', T') such that $V(S' \cap T') = \{b, b'\}$. Let $S = \{x \in V(H) : x \in S'$ or $x \in U_x$ for some $U_x \in \mathcal{H}$ with $N_H(U_x) \subseteq S'\}$, and let $T = \{x \in V(H) : x \in T'$ or $x \in U_x$ for some $U_x \in \mathcal{H}$ with $N_H(U_x) \subseteq T'\}$. Clearly, (S, T) is a separation of H and $V(S \cap T) = \{b, b'\}$, contradicting (3.5).

Clearly, $bb' \notin E(H)$; otherwise, H has a separation (S, T) such that $V(S \cap T) = \{b, b'\}$, contradicting (3.5).

Now assume that $\{b, b'\} \subseteq N_H(X)$ for some $X \in \mathcal{H}$ with $X \not\subseteq Y$. If $N_H(U) = \{b, b'\}$ or $N_H(X) = \{b, b'\}$, then H has a separation (S, T) such that $V(S \cap T) = \{b, b'\}$, contradicting (3.5). So $|N_H(U)| = 3 = |N_H(X)|$. Therefore, $N_H(U)$ and $N_H(X)$ both induce facial triangles in $p(H, \mathcal{H})$. Hence bb' is not contained in the outer cycle of $p(H, \mathcal{H})$, a contradiction.

Claim 2. For any $X \in \mathcal{H}$ with $X \subset Y$, $|N_H(X)| = |N_H(U)| = 3$.

For otherwise, let $X \in \mathcal{H}$ with $X \subset Y$ such that $N_H(X) = \{b, b'\}$. By (3.5), $V(H) = \{b, b'\} \cup X$. By (3) of (3.3), $\sigma(N(x)) \cap X \neq \emptyset$ for $x \in \{a, a'\}$. Let $X^* \in \mathcal{A}$ such that $X \subseteq X^*$. Since $a \neq a'$, $|N_{G-C}(X^*)| \geq 4$, a contradiction.

By Claim 2, $\sigma(N(A)) \cap Y = \emptyset$. By (C5), $\sigma(N(C)) \cap Y \neq \emptyset$. Let $N_H(U) = \{b, b', w\}$. By Claim 1 and because of (3) of (3.3), bb' is contained in the boundary of the infinite face of $p(G - C, \mathcal{A})$. Hence $(G - (C \cup Y), a, a', b', w, b)$ is 3-planar.

Claim 3. If $c \neq c'$ and $|\sigma(N(C))| \geq 2$, then $\sigma(N(C)) \subseteq Y$

Otherwise, there are distinct $u', v' \in V(C)$, $u \in \sigma(N(u') \cap V(H'))$, and $v \in \sigma(N(v') \cap V(H'))$ such that $u \in Y$ and $v \notin Y$.

Let $u^* = b$. If $v \notin X$ for any $X \in \mathcal{H}$ then let $v^* = v$; and if $v \in U_v$ for some $U_v \in \mathcal{H}$ then let $v^* \in N_H(U_v) - \{b, b'\}$ (by Claim 1). Since $p(H, \mathcal{H})$ is 2-connected, $p(H, \mathcal{H})$ has a $(\{u^*, b\}, \{v^*, b'\})$-linkage. Hence by (2.4) (with $H, \mathcal{H}$ as $G, \mathcal{A}$, respectively), H has a $(\{u, b\}, \{v, b'\})$-linkage.

Similarly, by defining $u^* = b'$, we can prove that H has a $(\{u, b'\}, \{v, b\})$-linkage.

Then $G - A$ has a $(\{b, c'\}, \{c, b'\})$-linkage, contradicting (1) of (3.3).

Claim 4. $G-((A-a)\cup(C-c'))$ has an $(\{a,b'\},\{b,c'\})$-linkage, and $G-((A-a')\cup(C-c))$ has a $(\{b,a'\},\{c,b'\})$-linkage.

Note that $\sigma(N(x)\cap V(H'))\cap Y\neq\emptyset$ for each $x\in\{c,c'\}$. This is obvious if $c=c'$ or $|\sigma(N(C))|=1$ (because $\sigma(N(C))\cap Y\neq\emptyset$). If $c\neq c'$ and $|\sigma(N(C))|\geq 2$, then this follows from Claim 3 and (3) of (3.3).

Let $u\in\sigma(N(c')\cap V(H'))\cap Y$. Define $u^*=b$. Let $v\in\sigma(N(a)\cap V(H'))$. Because $\sigma(N(A))\cap Y=\emptyset$ and by (3) of (3.3), $v\notin Y\cup\{b,b'\}$. If $v\notin X$ for any $X\in\mathcal{H}$ then let $v^*=v$; and if $v\in U_v$ for some $U_v\in\mathcal{H}$, then by Claim 1, let $v^*\in N_H(U_v)-\{b,b'\}$. Since $p(H,\mathcal{H})$ is 2-connected, $p(H,\mathcal{H})$ has a $(\{u^*,b\},\{v^*,b'\})$-linkage. Hence by (2.4), H has a $(\{u,b\},\{v,b'\})$-linkage. Therefore, $G-((A-a)\cup(C-c'))$ has an $(\{a,b'\},\{b,c'\})$-linkage.

Similarly, by choosing $u\in\sigma(N(c)\cap V(H'))\cap Y$ and $v\in\sigma(N(a')\cap V(H'))$ and by letting $u^*=b'$, we can prove that $G-((A-a')\cup(C-c))$ has a $(\{b,a'\},\{c,b'\})$-linkage.

Claim 5. $c\neq c'$ and $|\sigma(N(C))|\geq 2$.

If $c=c'$, then by (3.1) and by (1) of (3.3), $(G,(a,b,c),(a',b',c'))$ is a rung as in (2$'$) of (1.1). By Claim 1 and by (1) of (1.2), $(G,(a,b,c),(a',b',c'),A,C)$ is a good rung, contradicting (C6).

So $c\neq c'$. Suppose $|\sigma(N(C))|=1$. Then G has a 1-separation (G_1,G_2) such that $\{a,a',b',b\}\subseteq G_1$, $\{c,c'\}\subseteq G_2$. By (1) of (3.3), (G_1,a,a',b',b) is 3-planar (because $G_1\subseteq G-C$). Hence by (3.1), $(G,(a,b,c),(a',b',c'))$ is a rung as in (4) of (1.1). By Claim 4 and by (1) of (1.2), $(G,(a,b,c),(a',b',c'),A,C)$ is a good rung, contradicting (C6).

By Claim 3 and Claim 5, $\sigma(N(C))\subseteq Y$. Let H^* denote the subgraph of G induced by $V(C)\cup Y\cup N_H(U)\cup\{x\in V(H'):\sigma(x)\in Y\}$.

Claim 6. One of the following holds: (1) H^*-b has a $(\{c,b'\},\{c',w\})$-linkage or H^*-b' has a $(\{c,w\},\{c',b\})$-linkage; or (2) H^* has a separation (H_1,H_2) such that $V(H_1\cap H_2)=\{p,b,b'\}$, $\{c,c'\}\subseteq H_1$, $w\in H_2$, and (H_1,c',c,b,p,b') is 3-planar.

We wish to apply (2.7) to H^*,b,b',w,c,c' (as H,v,u,w,c,c' in (2.7)). Because U is minimal (since $\mathcal{A}$ is minimal) and because H is 2-connected, $H[U\cup N_H(U)]$ is 2-connected by (2.3). By Claim 5 and by (3) of (3.3), H^* is also 2-connected.

Condition (i) of (2.3) follows from (1) of (3.3) and (2.1). If (H^*,c',c,b,w,b') is 3-planar, then for some $X\in\mathcal{H}$ with $X\subseteq Y$, $\sigma(N(C))\cap X\neq\emptyset$, and so, $(H[X\cup N_H(X)],\mathcal{H}',b,w,b')$ is 3-planar for some $\mathcal{H}'\neq\{X\}$, contradicting the minimality of X. Hence (ii) of (2.7) holds. By (3.1), (iii) of (2.7) also holds. For any $X\in\mathcal{H}$ with $X\subseteq Y$, X is minimal and by (1) of (2.3), $H[X\cup N_H(X)]-\{b,b'\}$ is connected. Hence by (3) of (3.3), $H^*-\{b,b'\}$ is connected. By Claim 1, $bb'\notin E(H^*)$. Hence (iv) of (2.7) holds.

Now we have Claim 6 by applying (2.7) to H^*,b,b',w,c,c' (as H,v,u,w,c,c' in (2.7)).

By Claim 6, we have two cases.

Case 1. H^*-b has a $(\{c,b'\},\{c',w\})$-linkage or H^*-b' has a $(\{c,w\},\{c',b\})$-linkage.

By symmetry, we only consider the case when $H^* - b$ has a $(\{c, b'\}, \{c', w\})$-linkage. Then $H - (Y \cup \{b'\})$ contains no path from w to b; for otherwise $G - A$ would have a $(\{c, b'\}, \{b, c'\})$-linkage, contradicting (1) of (3.3). Hence, for any $x \in V(H) - (Y \cup \{w, b', b\})$, x and b belong to different components of $H - \{w, b'\}$. By (3.5), $\{b, b'\}$ is not a cut of H. Therefore, G has a separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{b', w\}$, $C \subset G_1$, and $A \subseteq G_2$. By (1) of (3.3), $(G - A, c, c', b', b)$ is 3-planar. Hence, (G_1, c, c', b', b) is 3-planar.

We claim that (G_2, a, a', b', w) is 3-planar. Otherwise, by (2.1), G_2 has an $(\{a, b'\}, \{w, a'\})$-linkage. By (1) of (2.3), $H[U \cup N_H(U)] - b'$ contains a path from w to b. Hence, $G - C$ contains an $(\{a, b'\}, \{a', b\})$-linkage, contradicting (1) of (3.3).

By (3.1), $(G, (a, b, c), (a', b', c'))$ is a rung as in ($5'$) of (1.1). By Claim 4 and by (1) of (1.2), $(G, (a, b, c), (a', b', c'), A, C)$ is a good rung, contradicting (C6).

Case 2. there exist a vertex p of H^* and a separation (H_1, H_2) in H^* such that $V(H_1 \cap H_2) = \{p, b, b'\}$, $\{c, c'\} \subseteq H_1$, $w \in H_2$, and (H_1, c', c, b, p, b') is 3-planar.

Let $G_c = H_1$, $M = H_2$, and $G_a = G - ((G_c \cup M) - \{b, b', w\})$. Then G_a, G_c, M are edge disjoint subgraphs of G, $G = G_a \cup G_c \cup M$, $V(G_a \cap M) = \{b, b', w\}$, $V(G_c \cap M) = \{b, b', p\}$, $V(G_a \cap G_c) = \{b, b'\}$, $\{a, a'\} \subseteq G_a$, $\{c, c'\} \subset G_c$, (G_a, a, a', b', w, b) is 3-planar, and (G_c, c', c, b, p, b') is 3-planar. Hence, by (3.1), $(G, (a, b, c), (a', b', c'))$ is a rung as in (7) of (1.1). By Claim 4 and by (1) of (1.2), $(G, (a, b, c), (a', b', c'), A, C)$ is a good rung, contradicting (C6). $\square$

By (3.5) and (3.6), $bb' \notin E(p(H, \mathcal{H}))$. Hence by (3) of (3.3), $p(H, \mathcal{H})$ has at least three vertices and, therefore, is a 2-connected plane graph. Let W denote the outer cycle of $p(H, \mathcal{H})$. By (3.5), $p(H, \mathcal{H})$ has a path from $bWb' - \{b, b'\}$ to $b'Wb - \{b, b'\}$ internally disjoint from W. In particular, $bWb' - \{b, b'\} \neq \emptyset$ and $b'Wb - \{b, b'\} \neq \emptyset$.

Next, we define subgraphs $H(b, b')$ and $H(b', b)$ of H, which will be heavily used in the rest of the proof.

Let $b_0 = b$, let $b_1, \ldots, b_{k-1}$ be the vertices on W in that clockwise order from b to b' such that (for each $i \in \{1, \ldots, k-1\}$) $p(H, \mathcal{H})$ contains a path from b_i to $b'Wb - \{b, b'\}$ internally disjoint from W, and let $b_k = b'$. By planarity of $p(H, \mathcal{H})$, either $b_{i-1}b_i \in E(bWb')$ or $\{b_{i-1}, b_i\}$ is a 2-cut of $p(H, \mathcal{H})$. Let $S_i \subseteq V(p(H, \mathcal{H}))$ such that $x \in S_i$ iff either $x \in \{b_{i-1}, b_i\}$ or x and $b'Wb - \{b, b'\}$ are contained in different components of $p(H, \mathcal{H}) - \{b_{i-1}, b_i\}$. Let H_i be the subgraph of H induced by $\{x \in V(H) : x \in S_i$ or $x \in U$ for some $U \in \mathcal{H}$ with $N_H(U) \subseteq S_i\}$. Let $H(b, b') = \bigcup_{i=1}^m H_i$. Note that $V(bWb') \subset H(b, b')$. Since H is 2-connected, H_i is connected or $V(H_i) = \{b_{i-1}, b_i\}$. By planarity, for any vertex $x \in p(H, \mathcal{H})$ with $x \notin H(b, b')$, $p(H, \mathcal{H}) - bWb'$ contains a path from x to $V(b'Wb) - \{b, b'\}$. Also, for any vertex $x \in H - H(b, b')$, $H - H(b, b')$ contains a path from x to $V(b'Wb) - \{b, b'\}$.

Similarly, let $b'_0 = b$, let $b'_1, \ldots, b'_{l-1}$ be the vertices on W in that counter clockwise order from b to b' such that $p(H, \mathcal{H})$ contains a path from b'_i to $bWb' - \{b, b'\}$ internally disjoint from W, and let $b'_l = b'$. Note that either $b'_{i-1}b'_i \in E(b'Wb)$ or $\{b'_{i-1}, b'_i\}$ is a 2-cut of $p(H,\mathcal{H})$. Let $S'_i \subseteq V(p(H,\mathcal{H}))$ such that $x \in S'_i$ iff either $x \in \{b'_{i-1}, b'_i\}$ or x and $bWb' - \{b,b'\}$ are contained in different components of $p(H,\mathcal{H}) - \{b'_{i-1}, b'_i\}$. Let H'_i be the subgraph of H induced by $\{x \in V(H) : x \in S'_i$ or $x \in U$ for some $U \in \mathcal{H}$ with $N_H(U) \subseteq S'_i\}$. Let $H(b',b) = \bigcup_{i=1}^{l} H'_i$. Note that $V(b'Wb) \subset H(b',b)$. Since H is 2-connected, H'_i is connected or $V(H'_i) = \{b'_{i-1}, b'_i\}$. By planarity, for any vertex $x \in p(H,\mathcal{H})$ with $x \notin H(b',b)$, $p(H,\mathcal{H}) - b'Wb$ contains a path from x to $V(bWb') - \{b,b'\}$. Also for any vertex $x \in H - H(b',b)$, $H - H(b',b)$ contains a path from x to $V(bWb') - \{b,b'\}$.

(3.7) Lemma. *For any $u \in H(b,b') - \{b,b'\}$ and $v \notin H(b,b')$ (respectively, $u \in H(b',b) - \{b',b\}$ and $v \notin H(b',b)$), H has both a $(\{u,b\},\{v,b'\})$-linkage and a $(\{u,b'\},\{v,b\})$-linkage.*

Proof. By symmetry, we only consider the case when $u \in H(b,b') - \{b,b'\}$ and $v \notin H(b,b')$.

Let $v^* = v$ if $v \notin U$ for any $U \in \mathcal{H}$. Suppose $v \in U_v$ for some $U_v \in \mathcal{H}$. Since $v \notin H(b,b')$, there exists $v^* \in N_H(U_v)$ such that $p(H,\mathcal{H})$ has a path Q from v^* to $b'Wb - \{b,b'\}$ and disjoint from bWb'.

Since $u \in H(b,b') - \{b,b'\}$, $u \in H_i$ for some i. Let $u^* = u$ if $u \notin U$ for any $U \in \mathcal{H}$. If $u \in U_u$ for some $U_u \in \mathcal{H}$, then let $u^* \in N_H(U_u)$ such that $u^* \neq b_i$. Then $p(H,\mathcal{H})$ has a path P from u^* to b such that $V(u^*Pb_{i-1}) \subseteq H_i - b'$ and $bPb_{i-1} = bWb_{i-1}$. Hence, $P \cup Q \cup b'Wb$ (and hence, $p(H,\mathcal{H})$) contains a $(\{u^*,b\},\{v^*,b'\})$-linkage. Therefore, by (2.4) (with $H,\mathcal{H}$ as $G,\mathcal{A}$, respectively), H contains a $(\{u,b\},\{v,b'\})$-linkage.

By selecting $u^* \in N_H(U_u)$ such that $u^* \neq b_{i-1}$ and by selecting P from u^* to b' such that $V(u^*Pb_i) \subseteq H_i - b$ and $b_iPb' = b_iWb'$, we can use (2.4) to show that H contains a $(\{u,b'\},\{v,b\})$-linkage. □

(3.8) Lemma. *$\sigma(N(A)) \subseteq H(b,b')$.*

Proof. Suppose this is false. Then there exists $x \in \sigma(N(A))$ such that $x \notin H(b,b')$. By symmetry between a and a', assume $x \in \sigma(N(A - a'))$. Let $x^* = x$ if $x \notin U$ for any $U \in \mathcal{H}$. If $x \in U_x$ for some $U_x \in \mathcal{H}$, then let $x^* \in N_H(U_x)$ such that $x^* \notin H(b,b')$. Hence, $p(H,\mathcal{H})$ contains a path P from x^* to some $x' \in V(b'Wb) - \{b,b'\}$ and disjoint from bWb'. That is, $p(H,\mathcal{H})$ has a $(\{b,b'\},\{x^*,x'\})$-linkage. By (2.4) (with $H,\mathcal{H},x,x^*,b,x'$ as $G,\mathcal{A},u,u^*,v=v^*,b$, respectively), H has an $(\{x,x'\},\{b,b'\})$-linkage. Hence, $G - C$ contains an $(\{a,x'\},\{b,b'\})$-linkage.

On the other hand, because $x' \in b'Wb$ and by (1) of (3.3), $(G - C, a, a', b', x', b)$ is 3-planar. Hence, by (2.1), $G - C$ contains no $(\{a,x'\},\{b,b'\})$-linkage, a contradiction. □

(3.9) Lemma. *$H(b,b') \neq H$ and $V(H(b,b') \cap H(b',b)) = \{b,b'\}$.*

Proof. Because $b'Wb - \{b, b'\} \neq \emptyset$, $H(b, b') \neq H$. Now assume that $V(H(b, b') \cap H(b', b)) \neq \{b, b'\}$. Let $r \in V(H(b, b') \cap H(b', b)) - \{b, b'\}$. Then $r \neq b_i$ and $r \neq b'_j$ for all $i \in \{0, \ldots, k\}$ and $j \in \{0, \ldots, l\}$. Hence, $r \in (H_i - \{b_{i-1}, b_i\}) \cap (H'_j - \{b'_{i-1}, b'_j\})$ for some i and j. This implies that $b_{i-1} = b = b'_{j-1}$ and $b_i = b' = b'_j$. Hence H has a separation (S, T) with $V(S \cap T) = \{b, b'\}$, contradicting (3.5). $\square$

(3.10) Lemma. $\sigma(N(C)) \not\subseteq H(b, b')$.

Proof. Suppose on the contrary that $\sigma(N(C)) \subseteq H(b, b')$. Then $\sigma(V(H' - H)) \subseteq H(b, b')$ by (3.8) and (C5).

Let $a_0 = a$. Let $a_1 \in V(A)$ with a_0Aa_1 maximal such that $a_1 = a_0$ or $\sigma(N(a_1) \cap V(H')) \cap V(H_1) \neq \emptyset$. Suppose that we have defined $a_0, \ldots, a_i$ for some integer $i \leq k - 2$. Let $a_{i+1} \in V(a_iAa')$ with a_iAa_{i+1} maximal such that $a_{i+1} = a_i$ or $\sigma(N(a_{i+1}) \cap V(H')) \cap V(H_{i+1}) \neq \emptyset$. Let $a_k = a'$.

Let $c_0 = c$. Let $c_1 \in V(C)$ with c_0Cc_1 maximal such that $c_1 = c_0$ or $\sigma(N(c_1) \cap V(H')) \cap V(H_1) \neq \emptyset$. Suppose that we have defined $c_0, \ldots, c_i$ for some integer $i \leq k - 2$. Let $c_{i+1} \in V(c_iCc')$ with c_iCc_{i+1} maximal such that $c_{i+1} = c_i$ or $\sigma(N(c_{i+1}) \cap V(H')) \cap V(H_{i+1}) \neq \emptyset$. Let $c_k = c'$.

For each $i \in \{1, \ldots, k\}$, let G_i denote the subgraph of G induced by $V(a_{i-1}Aa_i \cup c_{i-1}Cc_i \cup H_i) \cup \{x \in V(H') : \sigma(x) \in H_i - b_{i-1}\}$. Clearly, $G_i \subseteq G - (H - H(b, b'))$, $V(G_i \cap G_{i+1}) = \{a_i, b_i, c_i\}$. By (C2) and (C5), $E(G_i \cap G_{i+1}) \subseteq \{a_ib_i, c_ib_i\}$.

(a) We claim that $(G_i, \{a_{i-1}, c_{i-1}\}, \{a_i, c_i\}, (b_{i-1}, b_i))$ is an obstruction.

Otherwise, G_i contains three disjoint paths A', B' and C' between $\{a_{i-1}, b_{i-1}, c_{i-1}\}$ and $\{a_i, b_i, c_i\}$, none from b_{i-1} to b_i. Note that bWb_{i-1} and b_iWb' form a $(\{b, b_{i-1}\}, \{b_i, b'\})$-linkage in $p(H, \mathcal{H})$. By (2.4) (with $H, \mathcal{H}, b_{i-1}, b_i$ as $G, \mathcal{A}, u = u^*, v = v^*$, respectively), H has a $(\{b_{i-1}, b\}, \{b_i, b'\})$-linkage B, and it is disjoint from $H_i - \{b_{i-1}, b_i\}$. Now the union of such a linkage in H with A', B', C', A and C contains three disjoint paths between $\{a, b, c\}$ and $\{a', b', c'\}$ in G, none from b to b', contradicting the assumption that $(G, \{a, c\}, \{a', c'\}, (b, b'))$ is an obstruction.

(b) We also claim that $\bigcup_{i=1}^{k} G_i = G - (H - H(b, b'))$.

Clearly, each vertex of $G - (H - H(b, b'))$ is also a vertex in some G_i. So we only need to show that each edge of $G - (H - H(b, b'))$ is also an edge of some G_i. Let uv be an edge of $G - (H - H(b, b'))$. It is easy to check that uv is contained in some G_i if $uv \in H(b, b')$, or $uv \in E(H') - E(H)$, or $uv \in E(A \cup C)$. So we may assume by symmetry (between A and C and between u and v) that $u \in V(A)$ and $v \in V(H')$.

Because $\sigma(v) \in H(b, b') - \{b, b'\}$, $\sigma(v) \in H_i - b_{i-1}$ for some i. Then $u \in aAa_i$ by the definition of a_i. If $u \in a_{i-1}Aa_i$, then $uv \in G_i$. So assume that $u \in aAa_{i-1} - a_{i-1}$. Then $a_{i-1} \neq a$, and hence, by definition, a_{i-1} has a neighbor $w \in V(H')$ such that $\sigma(w) \in H_j - b_{j-1}$ for some $j < i$. Clearly, $p(H, \mathcal{H})$ has a $(\{b_{j-1}, b\}, \{b_i, b'\})$-linkage consisting of bWb_{j-1} and b_iWb'. By (2.4) (with $H, \mathcal{H}, b_{j-1}, b_i$ as $G, \mathcal{A}, u = u^*, v = v^*$, respectively), H has $(\{b_{j-1}, b\}, \{b_i, b'\})$-linkage which does not intersect $H_i - b_i$ and $H_j - b_{j-1}$. Hence H has a $(\{\sigma(w), b\}, \{\sigma(v), b'\})$-linkage. Now it is easy to see that $G - C$ contains an $(\{a, b'\}, \{b, a'\})$-linkage, contradicting (1) of (3.3). This proves (b).

Let $G_i^* = G_i$ if H_i is connected, and let $G_i^* = G_i + b_{i-1}b_i$ if H_i is not connected. Because $H(b, b') \neq H$ (by (3.9)), $|V(G_i^*)| + |E(G_i^*)| < |V(G)| + |E(G)|$. Let $A_i = A \cap G_i$ and $C_i = C \cap G_i$. Next, we describe J_i and L_i.

By (a), $(G_i^*, \{a_{i-1}, c_{i-1}\}, \{a_i, c_i\}, (b_{i-1}, b_i))$ is an obstruction. By (C1)-(C3), we have (i) $G_i^* - \{b_{i-1}, b_i\}$ contains disjoint paths A_i, C_i from a_{i-1}, c_{i-1} to a_i, c_i, respectively, (ii) $A_i \cup C_i$ is an induced subgraph of G_i^*, and (iii) $G_i^* - (A_i \cup C_i)$ is connected.

By (C4) and (C5) and by the choice of G, either (1) $(G_i^*, (a_{i-1}, b_{i-1}, c_{i-1}), (a_i, b_i, c_i), A_i, C_i)$ is a good ladder along a sequence $v_0^i \ldots v_{m_i}^i$, where $v_0^i = b_{i-1}$ and $v_{m_i}^i = b_i$ (and let $w_0^i \ldots w_{n_i}^i$ be the reduced sequence of $v_0^i \ldots v_{m_i}^i$, let $J_i = \{w_0^i, \ldots, w_{n_i}^i\}$, and let $L_i = G_i^*$), or (2) there is a separation (J_i, L_i) in G_i^* such that $V(J_i \cap L_i) = \{w_0^i, \ldots, w_{n_i}^i\}$, $(J_i, w_0^i, \ldots, w_{n_i}^i)$ is 3-planar, $(L_i, (a_{i-1}, b_{i-1}, c_{i-1}), (a_i, b_i, c_i), A_i, C_i)$ is a good ladder along $v_0^i \ldots v_{m_i}^i$, where $v_0^i = b_{i-1}$, $v_{m_i}^i = b_i$, and $w_0^i \ldots w_{n_i}^i$ is the reduced sequence of $v_0^i \ldots v_{m_i}^i$.

Let J be the graph obtained from $(H - (H(b, b') - \{b_0, \ldots, b_k\})) \cup (\bigcup_{i=1}^k J_i)$ by deleting all $b_{i-1}b_i$ with $b_{i-1}b_i \notin E(H)$, and let L be obtained from $\bigcup_{i=1}^k L_i$ by deleting all $b_{i-1}b_i$ with $b_{i-1}b_i \notin E(H)$. Let $w_0 \ldots w_n = w_0^1 \ldots w_{n_1}^1 w_1^2 \ldots w_{n_2}^2 \ldots w_1^k \ldots w_{n_k}^k$, and let $v_0 \ldots v_m = v_0^1 \ldots v_{m_1}^1 v_1^2 \ldots v_{m_2}^2 \ldots v_1^k \ldots v_{m_k}^k$. Because $\bigcup_{i=1}^k G_i = G - (H - H(b, b'))$, (J, L) is a separation of G. Moreover, $V(J \cap L) = \{w_0, \ldots, w_n\}$, $(J, w_0, \ldots, w_n)$ is 3-planar, $(L, (a, b, c), (a', b', c'), A, C)$ is a good ladder along $v_0 \ldots v_m$, where $v_0 = b$, $v_n = b'$, and $w_0 \ldots w_n$ is the reduced sequence of $v_0 \ldots v_m$. If $E(J) = \emptyset$ then we have a contradiction to (C6), and otherwise, (J, L) contradicts (C7). □

(3.11) Lemma. $\sigma(N(C)) \not\subseteq H(b', b)$.

Proof. Suppose on the contrary that $\sigma(N(C)) \subseteq H(b', b)$. Recall the definition of H_i' and $H(b', b)$. Let $c_0' = c$. Let $c_1' \in V(C)$ with $c_0'Cc_1'$ maximal such that $c_1' = c_0'$ or $\sigma(N(c_1') \cap V(H')) \cap V(H_1') \neq \emptyset$. Suppose that we have defined $c_0', c_1', \ldots, c_i'$ for some integer $i \leq l - 2$. Let $c_{i+1}' \in V(c_i'Cc')$ with $c_i'Cc_{i+1}'$ maximal such that $c_{i+1}' = c_i'$ or $\sigma(N(c_{i+1}') \cap V(H')) \cap V(H_{i+1}') \neq \emptyset$. Let $c_l' = c'$.

Let G_i' be the subgraph of G induced by $V(H_i' \cup c_{i-1}Cc_i) \cup \{x \in V(H') : \sigma(x) \in H_i - b_{i-1}'\}$. Then, for each $i \in \{1, \ldots, l - 1\}$, $V(G_i' \cap G_{i+1}') = \{b_i', c_i'\}$ (by definition) and $E(G_i' \cap G_{i+1}') \subset \{b_i'c_i'\}$ (by (C5)). Because $(G - C, c, c', b', b)$ is 3-planar, by the same argument as for (b) in the proof of (3.10), we can show that $\bigcup_{i=1}^l G_i'$ is the subgraph of G induced by $V(C) \cup \{z \in V(H') : \sigma(z) \in H(b', b)\}$.

Moreover, for each $i \in \{1, \ldots, l\}$, $(G_i', b_{i-1}', b_i', c_i', c_{i-1}')$ is 3-planar. For otherwise, by (2.1), G_i' contains a $(\{b_{i-1}', c_i'\}, \{c_{i-1}', b_i'\})$-linkage. Since $p(H, \mathcal{H}) - (H_i' - \{b_{i-1}', b_i'\})$ has a $(\{b'_{i-1}, b\}, \{b'_i, b'\})$-linkage and by (2.4) (with $H, \mathcal{H}, b'_{i-1}, b'_i$ as $G, \mathcal{A}, u = u^*, v = v^*$, respectively), H has a $(\{b'_{i-1}, b\}, \{b'_i, b'\})$-linkage, which is disjoint from $H'_i - \{b'_{i-1}, b'_i\}$. Hence, $G - A$ contains a $(\{b, c'\}, \{c, b'\})$-linkage, contradicting (1) of (3.3).

Therefore, (G, a', b', c', c, b, a) is 3-planar. By (3.1), $(G, (a, b, c), (a', b', c'))$ is a rung as in (3) of (1.1).

Let $x \in \sigma(N(a) \cap V(H'))$ and $y \in \sigma(N(c') \cap V(H'))$. Then $x \in H(b, b') - \{b, b'\}$ (by (3.8) and (3) of (3.3)) and $y \in H(b', b) - \{b, b'\}$ (by the assumption $\sigma(N(C)) \subseteq H(b', b)$ and by (3) of (3.3)). By (3.7), H contains an $(\{x, b'\}, \{y, b\})$-linkage. Thus, $G - ((A - a) \cup (C - c'))$ contains an $(\{a, b'\}, \{b, c'\})$-linkage.

Similarly, by choosing $x \in \sigma(N(a') \cap V(H'))$ and $y \in \sigma(N(c) \cap V(H'))$, we can prove that $G - ((A - a') \cup (C - c))$ contains a $(\{b, a'\}, \{c, b'\})$-linkage.

Therefore by (1) of (1.2), $(G, (a, b, c), (a', b', c'), A, C)$ is a good rung, contradicting (C6). □

(3.12) Lemma. *$c \neq c'$ and $\sigma(N(C)) \not\subseteq H(b, b') \cup H(b', b)$.*

Proof. First, suppose $c = c'$. By (3.1) and by (1) of (3.3), $(G, (a, b, c), (a', b', c'))$ is a rung as in ($2'$) of (1.1). By (3.10), there is a vertex $x \in \sigma(N(c) \cap V(H'))$ such that $x \notin H(b, b')$. Let $y \in \sigma(N(a) \cap V(H'))$. Then by (3.8) and (3) of (3.3), $y \in H(b, b') - \{b, b'\}$. By (3.7), H contains an $(\{x, b\}, \{y, b'\})$-linkage. Therefore, $G - ((A - a) \cup (C - c'))$ contains an $(\{a, b'\}, \{b, c'\})$-linkage. Similarly, we can pick $y \in \sigma(N(a') \cap V(H'))$ to prove that $G - ((A - a') \cup (C - c))$ contains a $(\{b, a'\}, \{c, b'\})$-linkage. By (1) of (1.2), $(G, (a, b, c), (a', b', c'), A, C)$ is a good rung, contradicting (C6).

So $c \neq c'$. Now assume that $\sigma(N(C)) \subseteq H(b, b') \cup H(b', b)$. Because $c \neq c'$, by (3) of (3.3) and by (3.10) and (3.11), there are distinct $u, v \in V(C)$, $x \in \sigma(N(u) \cap V(H'))$, and $y \in \sigma(N(v) \cap V(H'))$ such that $x \in H(b, b') - H(b', b)$ and $y \in H(b', b) - H(b, b')$. By (3.7), H contains both an $(\{x, b'\}, \{y, b\})$-linkage and an $(\{x, b\}, \{y, b'\})$-linkage. Then $G - A$ contains a $(\{c, b'\}, \{b, c'\})$-linkage, contradicting (1) of (3.3). □

(3.13) Lemma. *$|\sigma(N(C))| \geq 2$.*

Proof. Otherwise, let x be the only vertex in $\sigma(N(C))$. Then G has a separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{x\}$, $A \cup H \subset G_1$, and $C \subseteq G_2$. Because $G_1 \subseteq G - C$ and by (1) of (3.3), (G_1, a, a', b', b) is 3-planar. By (3.1), $(G, (a, b, c), (a', b', c'))$ is a rung as in (4) of (1.1).

By (3.12), $x \notin H(b, b') \cup H(b', b)$. Let $y \in \sigma(N(a) \cap V(H'))$. By (3) of (3.3) and by (3.8), $y \in H(b, b') - \{b, b'\}$. By (3.7), H has an $(\{x, b\}, \{y, b'\})$-linkage. Hence, $G - ((A - a) \cup (C - c'))$ contains an $(\{a, b'\}, \{b, c'\})$-linkage.

Similarly, we can pick $y \in \sigma(N(a') \cap V(H'))$ to prove that $G - ((A - a') \cup (C - c))$ contains a $(\{b, a'\}, \{c, b'\})$-linkage. By (1) of (1.2), $(G, (a, b, c), (a', b', c'), A, C)$ is a good rung, contradicting (C6). □

(3.14) Lemma. *There is no $z \in V(H) - \{b, b'\}$ such that $\sigma(N(C)) \subseteq V_z \cup V'_z$, where $V_z \subseteq V(H)$ and $V'_z \subseteq V(H)$ are defined as follows: for each $r \in V(H)$, $r \in V_z$ iff either $r \in \{z, b\}$ or r and b' belong to different components of $H - \{z, b\}$, and $r \in V'_z$ iff either $r \in \{z, b'\}$ or r and b belong to different components of $H - \{z, b'\}$.*

Proof. Suppose $\sigma(N(C)) \subseteq V_z \cup V'_z$ for some vertex z.

By (3.12), $z \notin H(b, b') \cup H(b', b)$. If $V_z \neq \{b, z\}$ and $V'_z \neq \{b', z\}$, then the component of $G - \{z, b, b'\}$ containing $H(b', b) - \{b, b'\}$ contains no element of $V(A \cup C) \cup \{b, b'\}$, contradicting (C5). So either $V_z = \{b, z\}$ or $V'_z = \{b', z\}$. By symmetry, assume that $V'_z = \{b', z\}$.

Then by (2) of (3.3), $\sigma(N(C)) \subseteq V_z$. Hence, G has a separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{z, b\}$, $V(A) \cup \{b, b'\} \subseteq G_1$, and $V(C) \cup V_z \subseteq G_2$. Note that G_2 contains no $(\{b, c'\}, \{c, z\})$-linkage; for otherwise, such a linkage and a path in $H - b$ from z to b' form a $(\{b, c'\}, \{c, b'\})$-linkage in $G - A$, contradicting (1) of (3.3). By (2.1), (G_2, c', c, b, z) is 3-planar. Since $G_1 \subseteq G - C$ and by (1) of (3.3), (G_1, a, a', b', b) is 3-planar. By (3.1), $(G, (a, b, c), (a', b', c'))$ is a rung as in (5) of (1.1).

Let $y \in \sigma(N(a) \cap V(H'))$ and $x \in \sigma(N(c') \cap V(H'))$. Then $y \in H(b, b') - \{b, b'\}$ (by (3.8) and by (3) of (3.3)), and $x \notin H(b, b') \cup H(b', b)$ (because $x \in V_z - \{b\}$ and $z \notin H(b, b') \cup H(b', b)$). By (3.7), H has an $(\{x, b\}, \{y, b'\})$-linkage. Hence, $G - ((A - a) \cup (C - c'))$ contains an $(\{a, b'\}, \{b, c'\})$-linkage.

Similarly, we can pick $y \in \sigma(N(a') \cap V(H'))$ and $x \in \sigma(N(c) \cap V(H'))$ to prove that $G - ((A - a') \cup (C - c))$ contains a $(\{b, a'\}, \{c, b'\})$-linkage.

By (1) of (1.2), $(G, (a, b, c), (a', b', c'), A, C)$ is a good rung, contradicting (C6). □

(3.15) Lemma. *$\sigma(N(C)) \subseteq U$ for some $U \in \mathcal{H}$.*

Proof. Suppose this is not true. Because $c \neq c'$ (by (3.12) and by (3) of (3.3), there are distinct $x', y' \in V(C)$, $x \in \sigma(N(x') \cap V(H'))$, $y \in \sigma(N(y') \cap V(H'))$ such that $\{x, y\} \not\subseteq U$ for any $U \in \mathcal{H}$. By (3.12) and without loss of generality, we may assume $x \notin H(b, b') \cup H(b', b)$.

Then $y \notin H(b, b') \cup H(b', b)$. Otherwise, by (3.7), H has a both an $(\{x, b\}, \{y, b'\})$-linkage and an $(\{x, b'\}, \{y, b\})$-linkage. Hence, $G - A$ contains a $(\{b, c'\}, \{c, b'\})$-linkage, contradicting (1) of (3.3).

Define $x^* = x$ if $x \notin U$ for any $U \in \mathcal{H}$; and otherwise, $x \in U_x$ for some $U_x \in \mathcal{H}$, and let $x^* \in N_H(U_x) - \{b, b'\}$. Let $y^* = y$ if $y \notin U$ for any $U \in \mathcal{H}$; and otherwise, $y \in U_y$ for some $U_y \in \mathcal{H}$, and let $y^* \in N_H(U_y) - \{b, b'\}$. Note that x^*, y^* are well defined by (3.6).

(a) We may assume that $x^* \notin H(b, b') \cup H(b', b)$ or $y^* \notin H(b, b') \cup H(b', b)$.

For otherwise, $\{x^*, y^*\} \subseteq H(b, b') \cup H(b', b)$ for all choices of x^* and y^*. Then $x^* \neq x$ and $y^* \neq y$ (because $x, y \notin H(b, b') \cup H(b', b)$), $N_H(U_x) \subseteq W$, and $N_H(U_y) \subseteq W$. Also because $x, y \notin H(b, b') \cup H(b', b)$, $N_H(U_x) \not\subseteq bWb'$, $N_H(U_x) \not\subseteq b'Wb$, $N_H(U_y) \not\subseteq bWb'$, and $N_H(U_y) \not\subseteq b'Wb$. Hence, we may assume $x^* \in bWb' - \{b, b'\}$ and $y^* \in b'Wb - \{b, b'\}$. By planarity, $p(H, \mathcal{H})$ contains both an $(\{x^*, b\}, \{y^*, b'\})$-linkage and an $(\{x^*, b'\}, \{y^*, b\})$-linkage. By (2.4) (with $H, \mathcal{H}, \{x, y\}, \{x^*, y^*\}$ as $G, \mathcal{A}, \{u, v\}, \{u^*, v^*\}$, respectively), H has both an $(\{x, b\}, \{y, b'\})$-linkage and an $(\{x, b'\}, \{y, b\})$-linkage. Hence, $G - A$ contains a $(\{b, c'\}, \{c, b'\})$-linkage, contradicting (1) of (3.3).

(b) Then $\{x^*, y^*\} \subseteq D_z \cup D'_z$ for some $z \in V(p(H, \mathcal{H})$, where $D_z \subseteq V(p(H, \mathcal{H}))$ and $D'_z \subseteq V(p(H, \mathcal{H}))$ are defined as follows: for every $r \in V(p(H, \mathcal{H}))$, $r \in D_z$ iff either $r \in \{z, b\}$ or r and b' belong to different components of $p(H, \mathcal{H}) - \{z, b\}$, and $r \in D'_z$ iff either $r \in \{z, b'\}$ or r and b belong to different components of $p(H, \mathcal{H}) - \{z, b'\}$.

We wish to apply (2.5) to $p(H, \mathcal{H}), W, x^*, y^*$ (as G, C, x, y in (2.5)). By (a), (i) of (2.5) holds. If (b) does not hold, then (1) and (2) of (2.5) do not hold, and so, (ii) of (2.5) must fail. Hence, if (b) does not hold, then $p(H, \mathcal{H})$ contains both an $(\{x^*, b\}, \{y^*, b'\})$-linkage and an $(\{x^*, b'\}, \{y^*, b\})$-linkage. Then by (2.4) (as in the proof of (a)), we derive a contradiction to (1) of (3.3).

(c) There exists $t \in \sigma(N(C))$, $t \notin D_z \cup D'_z$ and $t \in U$ for any $U \in \mathcal{H}$ with $N_H(U) \subseteq D_z \cup D'_z$.

For otherwise, let $V_z \subseteq V(H)$ and $V'_z \subseteq V(H)$ such that for every $r \in V(H)$, $r \in V_z$ iff $r \in D_z$ or $r \in U$ for some $U \in \mathcal{H}$ with $N_H(U) \subseteq D_z$, and $r \in V'_z$ iff $r \in D'_z$ or $r \in U$ for some $U \in \mathcal{H}$ with $N_H(U) \subset D'_z$. But then $\sigma(N(C)) \subseteq V_z \cup V'_z$, contradicting (3.14).

By (a) and (b), $x^*, y^*, z \notin H(b, b') \cup H(b', b)$. By (c), there is some $t \in \sigma(N(C))$ such that $t \notin D_z \cup D'_z$ if $t \notin U$ for any $U \in \mathcal{H}$, or $N_H(U_t) \not\subset D_z \cup D'_z$ if $t \in U_t$ for some $U_t \in \mathcal{H}$. Let $t^* = t$ if $t \notin U$ for any $U \in \mathcal{H}$, and otherwise, let $t^* \in N_H(U_t) - (D_z \cup D'_z)$. Let $t' \in V(C)$ such that $t \in \sigma(N(t') \cap V(H'))$. Then $t' \neq x'$ or $t' \neq y'$. By symmetry, assume that $t' \neq x'$. We view $p(H, \mathcal{H}), W, x^*, t^*$ as G, C, x, y in (2.5), respectively. Then (i) of (2.5) holds (because $x^* \notin H(b, b') \cup H(b', b)$) and the conclusion of (2.5) does not hold (by (b) and (c)). Hence (ii) of (2.5) must fail. Thus, $p(H, \mathcal{H})$ contains both an $(\{x^*, b\}, \{t^*, b'\})$-linkage and an $(\{x^*, b'\}, \{t^*, b\})$-linkage. By (2.4) (with $H, \mathcal{H}, \{x, t\}, \{x^*, t^*\}$ as $G, \mathcal{A}, \{u, v\}, \{u^*, v^*\}$, respectively), H has both an $(\{x, b\}, \{y, b'\})$-linkage and an $(\{x, b'\}, \{y, b\})$-linkage. Hence, $G - A$ contains a $(\{b, c'\}, \{c, b'\})$-linkage, contradicting (1) of (3.3). □

4 Proof of the main result

Assume on the contrary that (1.3) is false, and let $(G, \{a, c\}, \{a', c'\}, (b, b'))$ be a counter example to (1.3) such that $|V(G)| + |E(G)|$ is minimum. Then we have (C1)-(C7) as in Section 3.

For convenience, let $H' := G - (A \cup C)$. We will use $N(X)$ and $N(x)$ instead of $N_G(X)$ and $N_G(x)$ for any $X \subseteq G$ and $x \in V(G)$. By (3.2) and (3.4) and by symmetry, we may assume that $\{a,b\} \cap \{a',b'\} = \emptyset$ and $\{b,b'\}$ is contained in a block H of H'. By (1) of (3.3), there is a collection $\mathcal{A}$ of pairwise disjoint subsets of $V(G-C) - \{a,b,a',b'\}$ such that $(G-C, \mathcal{A}, a, a', b', b)$ is 3-planar. We choose $\mathcal{A}$ to be minimal. Let $\mathcal{H} = \{S \cap V(H) : S \in \mathcal{A}$ and $S \cap V(H) \neq \emptyset\}$. We will view $p(G-C, \mathcal{A})$ as a plane graph drawn in a closed disc in a plane such that a, a', b', b occur on the boundary of the disc in that clockwise order, and will view $p(H, \mathcal{H})$ as a plane subgraph of $p(G-C, \mathcal{A})$.

By (3.15), $\sigma(N(C)) \subseteq U$ for some $U \in \mathcal{H}$. Let H^* be the subgraph of G induced by $U \cup N_H(U) \cup V(C) \cup \{x \in V(H') : \sigma(x') \in U\}$. We note that if $|N_H(U)| = 3$ then $U \in \mathcal{A}$, and so, U is minimal. Hence if $|N_H(U)| = 3$, then $H[U \cup N_H(U)]$ is 2-connected (by (1) and (3) of (2.3)). So H^* is 2-connected; because $|\sigma(N(C))| \geq 2$ (by (3.13)), and $\sigma(N(c)) \cap U \neq \emptyset$ and $\sigma(N(c')) \cap U \neq \emptyset$(by (3) of (3.3)).

Let $K := H - U$, let $\mathcal{K} := \mathcal{H} - \{U\}$, and let S denote the set of 2-element subsets of $N_H(U)$. Then $p(H, \mathcal{H}) = p(K, \mathcal{K}) + S$. We view $p(K, \mathcal{K})$ as a plane subgraph of $p(H, \mathcal{H})$. Recall that W is the outer cycle of $p(H, \mathcal{H})$, and recall the definition of $H(b, b')$ and $H(b', b)$ (before (3.7)).

We complete the proof of Theorem (1.3) by distinguishing two cases.

Case 1. W is also the outer cycle of $p(K, \mathcal{K})$.

In this case, let B denote the block of $p(K, \mathcal{K})$ containing W. For each $x \in N_H(U)$, define $x_B \in V(B)$ as follows: let $x_B = x$ if $x \in V(B)$, and otherwise, let x_B be the cut vertex of $p(K, \mathcal{K})$ which is contained in all paths in $p(K, \mathcal{K})$ from x to B. Let $U_B = \{x_B : x \in N_H(U)\}$. By (3.5), $U_B \neq \{b, b'\}$. Therefore, by (3.14), $|U_B - \{b, b'\}| \geq 2$.

(a) For any $z \in V(p(H, \mathcal{H})) - \{b, b'\}$ and for any choice of $\{u_B, v_B\} \subseteq U_B - \{b, b'\}$, $p(H, \mathcal{H}) - \{z, b\}$ has a path from $\{u_B, v_B\}$ to b' and $p(H, \mathcal{H}) - \{z, b'\}$ has a path from $\{u_B, v_B\}$ to b.

Suppose (a) does not hold. Then there exists $z \in V(p(H, \mathcal{H})) - \{b, b'\}$ such that $N_H(U) \subseteq D_z \cup D'_z$, where $D_z \subseteq V(p(H, \mathcal{H}))$ and $D'_z \subseteq V(p(H, \mathcal{H}))$ are defined as follows: for every $r \in V(p(H, \mathcal{H}))$, $r \in D_z$ iff either $r \in \{z, b\}$ or r and b' belong to different components of $p(H, \mathcal{H}) - \{z, b\}$, and $r \in D'_z$ iff either $r \in \{z, b'\}$ or r and b belong to different components of $p(H, \mathcal{H}) - \{z, b'\}$.

Let $V_z \subseteq V(H)$ and $V'_z \subseteq V(H)$ such that for every $r \in V(H)$, $r \in V_z$ iff $r \in D_z$ or $r \in U_r$ for some $U_r \in \mathcal{H}$ with $N_H(U_r) \subseteq D_z$, and $r \in V'_z$ iff $r \in D'_z$ or $r \in U_r$ for some $U_r \in \mathcal{H}$ with $N_H(U_r) \subset D'_z$. But then $\sigma(N(C)) \subseteq V_z \cup V'_z$, contradicting (3.14). So (a) holds.

(b) We may select $u, v \in N_H(U)$ and $u_B, v_B \in U_B - \{b, b'\}$ such that (i) either $u_B \notin H(b, b') \cup H(b', b)$, or $u_B \in H(b, b')$ and $v_B \in H(b', b)$, and (ii) $H^* - (N_H(U) - \{u, v\})$ has disjoint paths from $\{u, v\}$ to $\{c, c'\}$.

Since $\sigma(N(C)) \not\subseteq H(b, b') \cup H(b', b)$ (by (3.12)), $U_B \not\subseteq H(b, b')$ and $U_B \not\subseteq H(b', b)$. Hence, there are distinct $u_B, v_B \in U_B - \{b, b'\}$ such that (i) holds.

Clearly, (ii) holds if $|N_H(U)| = 2$ because $|\sigma(N(C))| \geq 2$ (by (3.13)) and because H is 2-connected. So let $N_H(U) - \{u, v\} = \{w\}$. Then $U \in \mathcal{A}$ and U is minimal. Suppose that $H^* - w$ does not contain disjoint paths from $\{u, v\}$ to $\{c, c'\}$. Then H^* has a 2-separation (H_u, H_c) such that $V(H_u \cap H_c) = \{w, z\}$, $\{u, v\} \subseteq H_u$, and $\{c, c'\} \subseteq H_c$. It is clear that $C \subseteq H_c$ because $w \notin C$. We may assume that $z \in H$; otherwise, $\{w, \sigma(z)\}$ is a 2-cut of H^*, and we could choose (H_u, H_c) so that $V(H_u \cap H_c) = \{w, \sigma(z)\}$ (instead of $\{w, z\}$). Then $w \notin \{b, b'\}$; otherwise, $\sigma(N(C)) \subseteq V_z \cup V_z'$, contradicting (3.14). By (2) of (2.3) (with U, z as A, x, respectively), $H[U \cup \{u, v, w\}]$ contains a $(\{z, u\}, \{v, w\})$-linkage and a $(\{z, v\}, \{u, w\})$-linkage. Hence $H_u - \{w, v\}$ has a path from z to u, and $H_u - \{u, w\}$ has a path from z to v. Because H^* is 2-connected, H_c has disjoint paths from $\{z, w\}$ to $\{c, c'\}$. Therefore, $H^* - v$ has disjoint paths from $\{u, w\}$ to $\{c, c'\}$, and $H^* - u$ has disjoint paths from $\{v, w\}$ to $\{c, c'\}$. If $u_B \notin H(b, b') \cup H(b', b)$, then it is easy to see that (i) and (ii) hold for $\{u_B, w_B\}$ (instead of $\{u_B, v_B\}$). So by (i), we may assume that $u_B \in H(b, b')$ and $v_B \in H(b', b)$. If $w_B \notin H(b, b') \cup H(b', b)$ or $w_B \in H(b, b')$, then (i) and (ii) hold for $\{w_B, v_B\}$ (instead of $\{u_B, v_B\}$). If $w_B \in H(b', b)$, then (i) and (ii) hold for $\{w_B, u_B\}$ (instead of $\{u_B, v_B\}$). This prove (b).

(c) B contains both a $(\{u_B, b\}, \{v_B, b'\})$-linkage and a $(\{u_B, b'\}, \{v_B, b\})$-linkage.

This is obvious if $u_B \in H(b, b')$ and $v_B \in H(b', b)$. So by (i) of (b), we may assume that $u_B \notin H(b, b') \cup H(b', b)$. We view B, W, u_B, v_B as G, C, x, y in (2.5), respectively. Then (i) of (2.5) holds and the conclusion of (2.5) does not hold (by (a)). Hence (ii) of (2.5) must fail. Therefore, B contains both a $(\{u_B, b\}, \{v_B, b'\})$-linkage and a $(\{u_B, b'\}, \{v_B, b\})$-linkage.

By (c) and (2.4) (with $K, \mathcal{K}, \{u_B, v_B\}$ as $G, \mathcal{A}, \{u = u^*, v = v^*\}$, respectively), K contains both a $(\{u_B, b\}, \{v_B, b'\})$-linkage and a $(\{u_B, b'\}, \{v_B, b\})$-linkage. By (ii) of (b), H^*, and hence $G - A$, contains a $(\{b, c'\}, \{c, b'\})$-linkage, contradicting (1) of (3.3).

Case 2. W is not the outer cycle of $p(K, \mathcal{K})$.

In this case, there are vertices $u, v \in N_H(U)$ such that $uv \in W$, $uv \notin E(G)$, and $\{u, v\} \not\subseteq N_H(X)$ for any $X \in \mathcal{K}$. Because $N_H(U) \not\subseteq H(b, b') \cup H(b', b)$ (by (3.12)), $|N_H(U)| = 3$. Hence $U \in \mathcal{A}$ and U is minimal. Let $N_H(U) = \{u, v, w\}$ with $w \notin H(b, b')$.

We will only consider the case $uv \in E(bWb')$; the case $uv \in E(b'Wb)$ is symmetric. Without loss of generality, we may assume that b, v, u, b' occur on W in that clockwise order. By (3.6), $v \neq b$ or $u \neq b'$. By symmetry, we may assume that $u \neq b'$. Recall that H^* is 2-connected. We wish to apply (2.6).

(a) We claim that $(H^*, c', c, a_1, a_2, a_3)$ is not 3-planar for any sequence $a_1a_2a_3 \in \{wvu, vwu, vuw\}$

Suppose that there is a collection $\mathcal{X}$ of pairwise disjoint subsets of $V(H^*) - \{c, c', u, v, w\}$ such that $(H^*, \mathcal{X}, c', c, a_1, a_2, a_3)$ is 3-planar for some $a_1a_2a_3 \in \{wvu, vwu, vuw\}$. Then $(H[U \cup N_H(U)], \mathcal{U}, a_1, a_2, a_3)$ is 3-planar, where $\mathcal{U} = \{X \cap U :$

$X \in \mathcal{X}\}$ and $\mathcal{U} \neq \{U\}$ (because $\sigma(N(C)) \cap U \neq \emptyset$). This contradicts the minimality of $U \in \mathcal{A}$.

(b) We claim that H^* contains no $(\{c, u\}, \{c', v\})$-linkage.

Note that bWv and uWb' form a $(\{v, b\}, \{u, b'\})$-linkage in $p(K, \mathcal{K})$. By (2.4) (with $K, \mathcal{K}, v, u$ as $G, \mathcal{A}, u = u^*, v = v^*$, respectively), K contains a $(\{v, b\}, \{u, b'\})$-linkage, disjoint from $H^* - \{u, v, w\}$. If H^* has a $(\{c, u\}, \{c', v\})$-linkage, then $G - A$ has a $(\{b, c'\}, \{c, b'\})$-linkage, contradicting (1) of (3.3).

We view H^* as H in (2.6). Then (ii) of (2.6) holds by (a). Hence, (i) of (2.6) is not satisfied, or (1) and (2) of (2.6) hold. Thus, H^* has a $(\{c, w\}, \{c', v\})$-linkage L_v (by (b)), or H^* has a $(\{c, u\}, \{c', w\})$-linkage L_u .

(c) If (1) and (2) of (2.6) hold, then we can choose L_u such that if $v = b$ then v is not used by L_u.

Otherwise, by (1) and (2) of (2.6), H^* has a separation (H_1, H_2) such that $V(H_1 \cap H_2) = \{b, q\}$, $\{c, c', b\} \subseteq H_1$, and $\{u, w\} \subset H_2$. Then $q = c'$; otherwise, we can choose q to be in $V(H)$, and so, $\sigma(N(C)) \subseteq V_q \cup V_q'$ (defined as in (3.14) with q as z), contradicting (3.14). Therefore, G has a separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{a, b, c'\}$, $\{a, b, c\} \subseteq G_1$, $\{a', b', c'\} \subseteq G_2$, contradicting (3.1).

(d) We claim that $v = b$, $wb \in E(b'Wb)$, and $\{w, b\} \not\subseteq N_H(X)$ for any $X \in \mathcal{K}$.

Suppose $v \neq b$, or $v = b$ and $wb \notin E(b'Wb)$, or $v = b$ and $wb \in E(b'Wb)$ but $\{w, b\} \subseteq N_H(X)$ for some $X \in \mathcal{K}$. Then $p(K, \mathcal{K})$ has a $(\{w, b\}, \{u, b'\})$-linkage consisting of the following: uWb', a path in $p(H, \mathcal{H}) - bWb'$ from w to some $x \in V(b'Wb) - \{b, b'\}$ (because $w \notin H(b, b')$), and xWb. Clearly, if $v \neq b$, then v is not used by this linkage. Also $p(K, \mathcal{K})$ has a $(\{w, b'\}, \{v, b\})$-linkage consisting of the following: bWv, a path in $p(H, \mathcal{H}) - bWb'$ from w to some $x \in V(b'Wb) - \{b, b'\}$, and $b'Wx$. Clearly, u is not used by this linkage. By (2.4) (with $K, \mathcal{K}, \{w, u\}$ as $G, \mathcal{A}, \{u, v\}$, respectively), K contains a $(\{w, b\}, \{u, b'\})$-linkage R_u and a $(\{w, b'\}, \{v, b\})$-linkage R_v. Thus L_u and R_u or L_v and R_v form a $(\{b, c'\}, \{c, b'\})$-linkage in $G - A$, contradicting (1) of (3.3).

(e) We claim that $(G, (a, b, c), (a', b', c'))$ is a rung.

Let $G_a = G - (H^* - \{u, w\})$. By (d), (H^*, G_a) is a 2-separation in G, $V(H^* \cap G_a) = \{u, w\}$, and (G_a, a, a', b', w, u) is 3-planar.

We view H^* as H in (2.6). Then (ii) of (2.6) holds by (a). Note that H^* has no $(\{c, w\}, \{c', v\})$-linkage; otherwise, it is easy to see that $G - A$ contains a $(\{b, c'\}, \{c, b'\})$-linkage, contradicting (1) of (3.3). Hence (i) of (2.6) holds (by (b)). By (1) of (2.6), H^* has a 2-separation (G_c, M) (as (H_1, H_2)) such that $V(G_c \cap M) = \{p, q\}$, $\{c, c', b\} \subseteq G_c$, $\{u, w\} \subseteq M$, and (G_c, c', c, b, p, q) is 3-planar.

If $\{p, q\} \cap \{u, w\} \neq \emptyset$, then it is easy to see that (G, a', b', c', c, b, a) is 3-planar, and by (3.1), $(G, (a, b, c), (a', b', c'))$ is a rung as in (3) of (1.1). If $\{p, q\} \cap \{u, w\} = \emptyset$, then $G_a \cap G_c = \emptyset$, and by (3.1), $(G, (a, b, c), (a', b', c'))$ is a rung as in (6) of (1.1).

Because U is minimal, $H[U \cup N_H(U)] - \{u, w\}$ is connected (by (1) of (2.1)). Because $\sigma(N(C)) \subseteq U$ and by (3) of (3.3), $H^* - ((C - c') \cup \{u, w\})$ has a path from c' to b. By (3) of (3.3), $G_a - (A - a)$ has a path from a to b'. Hence, $G - ((A - a) \cup (C - c'))$ has an $(\{a, b'\}, \{b, c'\})$-linkage.

Let $x \in \sigma(N(c) \cap V(H'))$. Then $x \in U$ (because $\sigma(N(C)) \subset U$). By (2) of (2.3), $H[U \cup N_H(U)]$ has an $(\{x, w\}, \{u, v\})$-linkage. Hence $H^* - (C - c)$ has a $(\{c, w\}, \{b, u\})$-linkage. Let $y \in \sigma(N(a') \cap V(H))$. Then $y \in H(b, b') - \{b, b'\}$ by (3) of (3.3). Therefore, H has a $(\{u, y\}, \{b', w\})$-linkage. Hence, $G_a - (A - a')$ has a $(\{u, a'\}, \{w, b'\})$-linkage. So $G - ((A - a') \cup (C - c))$ has an $(\{b, a'\}, \{c, b'\})$-linkage.

Hence, by (1) of (1.2), $(G, (a, b, c), (a', b', c'), A, C)$ is a good rung, contradicting (C6). $\square$

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