Notes on complex analysis

John McCuan

April 17, 2018

Introduction

These are some notes for a graduate course in complex analysis given at Georgia Tech in the Spring semester 2018. The primary text for the course is that of Ahlfors *Complex Analysis* third edition (McGraw-Hill 1979).

Much of the following consists of solutions and notes on the exercises. Ahlfors sections and subsections each chapter essentially independently. Thus, one finds § 1.1 and § 1.2 in both chapter 1 and chapter 2. When distinctions between different section numbers with respect to the chapter seem necessary, I will use roman numerals. Thus, in these notes chapter 2 is denoted by chapter II, and section 1.2 of chapter 2 is denoted by § 1.2 (in the understood context of chapter II) or by § II 1.2 if necessary.

We may also refer to a set of exercise solutions for the first three chapters of Ahlfors published electronically Dustin Smith and a similar collection of solutions for chapters 4-7 by Matt Rosenzweig.

Main theorems/topics of complex analysis

Here are what I consider the main theorems and topics for this course. I will list theorems in bold face and give references in parentheses.

- complex arithmetic (chapter I)
- differentiability/Cauchy-Riemann equations (chapter II)
- harmonic functions, Schwarz reflection (chapter IV § 6)
- elementary functions (chapter II § 3)
    1. complex powers/roots
    2. complex exponential/logarithm

3. complex trigonometric functions
4. Riemann surfaces

- conformal mapping (chapter III and chapter VI)
    1. **Riemann mapping theorem** (chapter VI § 1)
    2. stereographic projection (chapter I § 2.4)
    3. Riemann sphere (chapter I § 2.4)
    4. LFTs (linear fractional transformations) (chapter II § 1.4 and chapter III § 3)
- **Cauchy integral formula**/complex integration (chapter IV)
- Laurent series (chapter V) and residue calculus (chapter IV § 5)

It will be noted that there are only two main theorems and a lot of "main" topics.

Here are what I consider some additional topics:

- entire functions (chapter V § 3)
- product expansions and the Riemann zeta function (chapter V § 2 and § 4)
- elliptic functions (chapter VII)

Contents

Chapter 1

Chapter I

1.1 Lecture 1: The complex number field

In the first part of the lecture I will cover § 1.1 and § 1.3. Then I will come back to § 1.2 on complex square roots.

The complex numbers are the smallest field containing $\mathbb{R}$ and a root of the polynomial equation

$$x^2 + 1 = 0.$$

The root is denoted by i, and the complex numbers are denoted by $\mathbb{C}$.

What does it mean to be a **field**?

$\mathbb{F}$ is a field if ($\mathbb{F}$ is a set and) there are operations (addition and multiplication) on $\mathbb{F}$ satisfying the following:

1. Addition and multiplication are associative and commutative:

$$(a + b) + c = a + (b + c); \quad a + b = b + a; \quad (ab)c = a(bc); \quad ab = ba.$$

2. Multiplication is distributive with respect to addition:

$$a(b + c) = ab + ac.$$

3. There are additive and multiplicative identities, 0 and 1:

$$a + 0 = a; \qquad a \cdot 1 = a.$$

4. every element has an additive inverse and every nonzero element has a multiplicative inverse:

$$a + (-a) = 0; \quad bb^{-1} = 1,\ b \neq 0.$$

5. There are no zero divisors:

$$ab = 0 \quad \text{implies} \quad a = 0 \text{ or } b = 0.$$

1.1.1 The real numbers $\mathbb{R}$ as a field

The real numbers $\mathbb{R}$ have two (or three) important additional properties:

($\mathbb{R}$1) The real numbers are *ordered*:

Given $a, b \in \mathbb{R}$, then exactly one of the following holds:

$$a < b,\ a = b,\ \text{or } a > b.$$

($\mathbb{R}$2) The real numbers contain *least upper bounds*:

If S is a nonempty subset of $\mathbb{R}$ and there is some $b \in \mathbb{R}$ such that $x \leq b$ for every $x \in S$, i.e., b is an upper bound for S, then there is some $b_0 \in \mathbb{R}$ such that

(a) $x \leq b_0$ for every $x \in X$, i.e., b_0 is an upper bound for S, and

(b) If $x \leq c$ for every $x \in S$, then $b_0 \leq c$, i.e., b_0 is the least upper bound.

There is also a way to measure distance between real numbers (a metric) given by the absolute value of the difference:

$$d(a, b) = |b - a|.$$

It can be shown that the least upper bound property **($\mathbb{R}$2)** implies the following:

Every sequence $x_1, x_2, x_3, \ldots$ of real numbers satisfying the following condition:

Cauchy's Condition *For any $\epsilon > 0$, there is some $N > 0$ such that*

$$j, k > N \implies d(x_j, x_k) < \epsilon.$$

must also converge, i.e., there is some limit L (a real number in this case) such that for any $\epsilon > 0$, there is some $N > 0$ with

$$j > N \implies d(x_j, L) < \epsilon.$$

In short, every Cauchy sequence converges.

1.1.2 Comparison of $\mathbb{C}$ to $\mathbb{R}$

$$\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}$$

is a field:

$$\begin{aligned} \text{addition:} \quad & (a + bi) + (c + di) = (a + c) + (b + d)i \\ \text{multiplication:} \quad & (a + bi)(c + di) = (ac - bd) + (ad + bc)i. \end{aligned}$$

See Nexercise 2.

$$\mathbb{R} = \{a + 0i : a \in \mathbb{R}\} \quad \text{and} \quad i\mathbb{R} = \{0 + bi : b \in \mathbb{R}\}.$$

See Nexercise 1.

The complex numbers are not ordered. (The least upper bound property does not make sense in $\mathbb{C}$.)

However, $\mathbb{C}$ is a metric space:

$$d(a + bi, c + di) = \sqrt{(a - c)^2 + (b - d)^2}. \tag{1.1}$$

See Nexercise 3.

We'll often write a single letter to represent an element of $\mathbb{C}$. For example, it is typical to write $z = x + iy$, $w = a + ib$, and/or $a = \alpha + i\beta$. If we write $w = a + ib$, then it usually means a and b are real, but sometimes it doesn't, so we have to keep track.

Finally, $\mathbb{C}$ is **complete** in the sense that every Cauchy sequence converges in $\mathbb{C}$. Notice that this is a property $\mathbb{C}$ has in common with $\mathbb{R}$. See Nexercise 4.

Nexercises

Nexercise 1 *Show that*

$$\mathbb{Q} = \left\{ \frac{p}{q} : p, q \in \mathbb{N},\ q \neq 0 \right\}$$

is a field, but $i\mathbb{R}$ is not.

Nexercise 2 *Verify the field axioms for* $\mathbb{C}$.

Nexercise 3 *Show* $d : \mathbb{C} \times \mathbb{C} \to \mathbb{R}$ *by the formula given in (1.1) satisfies the metric axioms:*

(symmetric) $d(z, w) = d(w, z)$

(positive definite) $d(z, w) \geq 0$ *with equality if and only if* $z = w$.

(triangle inequality) $d(z, w) \leq d(z, \zeta) + d(\zeta, w)$.

Nexercise 4 *Assume every Cauchy sequence of real numbers converges in* $\mathbb{R}$, *i.e., assume* $\mathbb{R}$ *is complete, and show* $\mathbb{C}$ *is complete.*

1.1.3 §1.1 exercises

Exercise 1

$$(1 + 2i)^3 = (-3 + 4i)(1 + 2i) = -11 - 2i.$$

$$\frac{5}{-3 + 4i} = \frac{-3 - 4i}{5}.$$

$$\left(\frac{2 + i}{3 - 2i}\right)^2 = \left(\frac{4 + 7i}{13}\right)^2 = \frac{-33 + 56i}{169}.$$

Let's check by doing this a different way:

$$\left(\frac{2 + i}{3 - 2i}\right)^2 = \frac{3 + 4i}{5 - 12i} = \frac{-33 + 56i}{169}.$$

Finally, we consdier

$$(1 + i)^n + (1 - i)^n.$$

There are various ways to simplify this expression (and even to express it). Let's start by considering $(1 + i)^n$ and $(1 - i)^n$ separately.

$$(1 + i)^1 = 1 + i; \quad (1 + i)^2 = 2i; \quad (1 + i)^3 = 2(-1 + i); \quad (1 + i)^4 = -4.$$

Notice, first of all, that the fourth power is interesting; it is purely real. And after the fourth power, there is a recursion: $(1 + i)^5 = -4(1 + i)$;

$$(1 + i)^{4+j} = -4(1 + i)^j.$$

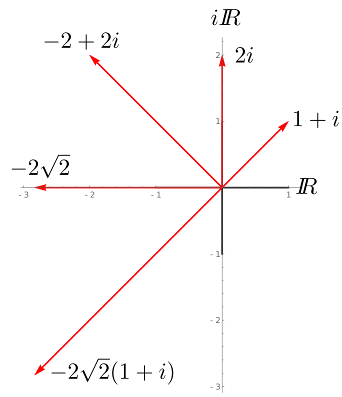

Figure 1.1: Here are the first five powers of $1+i$. One can make a similar picture for the powers of $1-i$. This picture suggests the use of trigonometric functions/Euclidean polor coordinates to represent the powers. One can also see the sums of the powers using Euclidean vector addition.

This means there is a kind of cylcle of length 4, and after the fourth power, one picks up another factor of -4. That is, every n may be written as $n = 4j + k$ for some unique nonnegative j and $k = 0, 1, 2$, or 3, and we have

$$(1+i)^n = \begin{cases} (-4)^j & \text{if } k = 0 \\ (-4)^j(1+i) & \text{if } k = 1 \\ (-4)^j(2i) & \text{if } k = 2 \\ (-4)^j[2(-1+i)] & \text{if } k = 3. \end{cases}$$

As we have checked the base case above, this assertion follows easily from induction on j.

Plotting the first couple cycles geometrically and thinking in terms of polar coordinates, it becomes evident that the expression for $(1+i)^n$ may be expressed by the single formula

$$(1+i)^n = \sqrt{2^n}\left[\cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4}\right].$$

Similar considerations apply to $(1-i)^n$:

$$(1-i)^1 = 1-i; \quad (1-i)^2 = -2i; \quad (1-i)^3 = 2(-1-i); \quad (1-i)^4 = -4;$$

$$(1-i)^{4+j} = -4(1-i)^j.$$

$$(1-i)^{4j+k} = \begin{cases} (-4)^j & \text{if } k=0 \\ (-4)^j(1-i) & \text{if } k=1 \\ (-4)^j(-2i) & \text{if } k=2 \\ (-4)^j[2(-1-i)] & \text{if } k=3. \end{cases}$$

$$(1-i)^n = \sqrt{2^n}\left[\cos\frac{n\pi}{4} - i\sin\frac{n\pi}{4}\right].$$

Combining the appropriate expressions above, we get

$$(1+i)^{4j+k} + (1-i)^{4j+k} = \begin{cases} 2(-4)^j = (-1)^j 2^{2j+1} & \text{if } k=0 \\ 2(-4)^j = (-1)^j 2^{2j+1} & \text{if } k=1 \\ 0 & \text{if } k=2 \\ (-4)^{j+1} & \text{if } k=3. \end{cases}$$

or

$$(1+i)^n + (1-i)^n = 2\sqrt{2^n}\cos\frac{n\pi}{4}.$$

Exercise 2

$z = x + iy$.

$$\begin{aligned} z^4 &= (x^2 - y^2 + 2xyi)^2 \\ &= (x^2-y^2)^2 - 4x^2y^2 + 4xy(x^2+y^2)i \\ &= x^4 - 6x^2y^2 + y^4 + 4xy(x^2+y^2)i. \end{aligned}$$

$$\frac{1}{z} = \frac{x-iy}{x^2+y^2}.$$

$$\frac{z-1}{z+1} = \frac{(x-1+i)(x+1-i)}{(x+1)^2+1} = \frac{x^2+2i}{x^2+2x+2}.$$

$$\frac{1}{z^2} = \frac{x^2-y^2-2xyi}{(x^2+y^2)^2}.$$

Exercise 3

$$\left(\frac{-1 \pm i\sqrt{3}}{2}\right)^3 = \frac{(-2 \mp 2i\sqrt{3})(-1 \pm i\sqrt{3})}{8} = 1.$$

In our expressions, the signs are coordinated.

$$\left(\frac{\pm 1 \pm i\sqrt{3}}{2}\right)^6 = (\mp 1)^6 \left[\left(\frac{-1 - i\sqrt{3}}{2}\right)^3\right]^2 = 1.$$

With the other choice of sign:

$$\left(\frac{\pm 1 \mp i\sqrt{3}}{2}\right)^6 = (\mp 1)^6 \left[\left(\frac{-1 + i\sqrt{3}}{2}\right)^3\right]^2 = 1.$$

1.1.4 Lecture 1 (continued) complex square roots

In § 1.2 Ahlfors talks about complex square roots. It seems, more or less, that he wishes the student to consider the topic from the point of view of algebraic and arithmetic manipulations without the benefit of geometric interpretation, which he introduces in § I 2.1. I will talk about some of the geometric material at the very beginning of § I 2.1, so this part of the lecture can also be considered a start at that section on complex geometry.

Let us start by looking for a complex number (i.e., all complex numbers) $x + iy$ such that

$$\sqrt{i} = x + iy.$$

If this condition is to hold, then we should have

$$x^2 - y^2 = 0 \quad \text{and} \quad 2xy = 1.$$

Squaring and adding these equations, we get

$$x^4 - 2x^2y^2 + y^4 + 4x^2y^2 = (x^2 + y^2)^2 = 1.$$

Sinces $x^2 + y^2$ is real and positive, this means

$$x^2 + y^2 = 1.$$

Looking at the first equation $x^2 - y^2 = 0$ above, we get

$$x = \pm 1/\sqrt{2}.$$

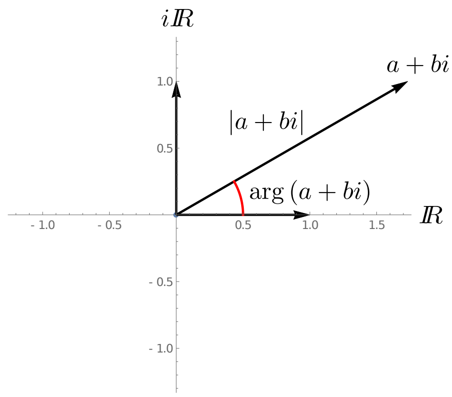

Figure 1.2: The complex plane.

The equation for the imaginary parts $2xy = 1$ then gives

$$y = \frac{1}{2x} = \pm\frac{\sqrt{2}}{2}.$$

Thus, we get two square roots of i:

$$\sqrt{i} = \pm\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = \pm\frac{1}{\sqrt{2}}(1 + i).$$

This approach works in general to find the square root of a complex number; see the exercises.

1.1.5 Geometry of the complex square root

Each complex number corresponds to a point in the (complex) plane; the complex plane is isomorphic to $\mathbb{R}^2$:

$$a + bi \sim (a, b).$$

The absolute value of a complex number (covered again in the next section § I 1.4) is the distance from $a + bi$ to the origin:

$$|a + bi| = \sqrt{a^2 + b^2}.$$

This is also called the **magnitude** or **modulus** of the complex number. The **argument** is the angle the vector (a, b) makes with the positive real axis:

$$\arg(a+bi) = \begin{cases} \tan^{-1}(b/a), & a > 0 \\ \text{sign}(b)\pi/2, & a = 0 \text{ (and } b \neq 0) \\ \pi + \tan^{-1}(b/a), & a < 0. \end{cases}$$

The argument is not really rigorously defined here as pointed out by Ahlfors. In any case, the argument is only defined up to an additive multiple of 2π.

As will be seen in the exercises, it seems Ahlfors intends for the student to struggle with the algebraic manipulations without the geometry. Some of the exercises, however, seem exceedingly difficult without the rudimentary identification of $\mathbb{C}$ with the plane given above.

Let us apply the geometric interpretation to the complex square root.

If we reconsider $\sqrt{i}$ geometrically, we see $\sqrt{i}$ has the two values indicated in Figure 1.3. Each has unit length (or mudulus) and one has argument $\pi/4$.

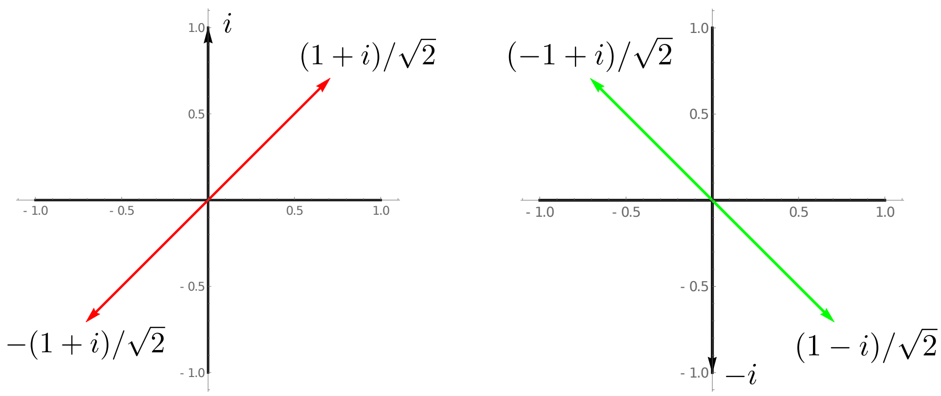

Figure 1.3: the square roots of i and $-i$

We consider this one the **principal square root**. The other square root has argument $\pi/4 + \pi = 5\pi/4$.

A similar discussion applies to $\sqrt{-i}$ or to the square root of any complex number. Notice the argument of (at least one of) the square roots of a complex number z is half the argument of the number z. This observation is justified more fully in § I 2.1. See also Nexercise 6 below.

We will not repeat Ahlfors' derivation of the square root formula here,[1]

[1]See the solution to Exercise 4 of this section below.

but let us briefly consider his conclusions from it with respect to the geometric interpretation:

- The square root of any complex number always exists and has (up to) two opposite values. (Apart from $z = 0$, there are exactly two opposite values.) Taking the complex number with modulus $\sqrt{|z|}$ and argument $\arg(z)/2$, we obtain a complex number $|z|[\cos(\arg(z)/2) + i\sin(\arg(z)/2)]$ whose square is z. We will return to this observation again in § I 2.1. The "opposite" value of this square root will have argument $\arg(z)/2 + \pi$. Thus, when the "opposite" value is squared, the argument will be $\arg(z) + 2\pi$ giving again z.

- The two "opposite" square roots of $z \in \mathbb{C}$ are the same only when $z = 0$.

- Setting $z = 0$ aside, only positive numbers have real square roots: $\sqrt{a + 0i} = \pm\sqrt{a}$ when $a > 0$.

- Setting $z = 0$ aside, only negative numbers have purely imaginary square roots: $\sqrt{a + 0i} = \pm\sqrt{-a}$ when $a < 0$.

What we have said above about the prinipal square root is something of a contradiction to Ahlfors' assertion at the end of the section that it is not "correct" to distinguish between the two complex square roots. In fact, the argument provides a natural way to do so. There is always exactly one square root of a nonzero complex number with argument in a certain range, for example between $-\pi/2$ and $\pi/2$ (or equal to $\pi/2$). We can call this one the **principal square root**. The interval we have suggested is, more or less, standard though occasionally the principal square root it taken with argument in $[0, \pi)$.

Nexercise 5 *Show the complex metric given in (1.1) is also given by*

$$d(z, w) = |z - w|.$$

Nexercise 6 *Show the argument of one of the square roots of* $z \in \mathbb{C}$ *is* $\arg(z)/2$*, and the other square root has argument* $\arg(z)/2 + \pi$*.*

1.1.6 §1.2 exercises

Exercise 1

$\sqrt{i} = \pm(1+i)/\sqrt{2}$ (calculated above).

$\sqrt{-i}$:

$$x^2 - y^2 = 0 \quad \text{and} \quad 2xy = -1.$$

$$x^2 + y^2 = 1 \quad \text{so} \quad x = \pm\frac{1}{\sqrt{2}}.$$

$y = \mp\sqrt{2}/2$.

$$\sqrt{-i} = \pm\left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right).$$

$\sqrt{1+i}$:

$$x^2 - y^2 = 1 \quad \text{and} \quad 2xy = 1.$$

$$x^2 + y^2 = \sqrt{2} \quad \text{so} \quad x = \pm\sqrt{\frac{1+\sqrt{2}}{2}} = \pm\frac{\sqrt{2(1+\sqrt{2})}}{2}.$$

$$y = \pm\frac{\sqrt{2(\sqrt{2}-1)}}{2}.$$

$$\sqrt{1+i} = \pm\left(\frac{\sqrt{2(1+\sqrt{2})}}{2} + i\,\frac{\sqrt{2(\sqrt{2}-1)}}{2}\right).$$

$\sqrt{(1-i\sqrt{3})/2}$:

$$x^2 - y^2 = \frac{1}{2} \quad \text{and} \quad 2xy = -\frac{\sqrt{3}}{2}.$$

$$x^2 + y^2 = 1 \quad \text{so} \quad x = \pm\frac{\sqrt{3}}{2}.$$

$$y = \mp\frac{1}{2}.$$

$$\sqrt{\frac{1-\sqrt{3}}{2}} = \pm\left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right).$$

Exercise 2

If $z = \sqrt[4]{-1}$, then $z^2 = \sqrt{-1} = \pm i$. There are two roots of $-i$ and two roots of i as calculated in Exercise 1 above. Any of these (and only these) give fourth roots of -1:

$$\sqrt[4]{-1} = \pm(1+i)/\sqrt{2},\ \pm(1-i)/\sqrt{2}.$$

Exercise 3

If $z = \sqrt[4]{i}$, then $z^2 = \sqrt{i} = \pm(1+i)/\sqrt{2}$. From the point of view of the argument, the principal square root of $(1+i)/\sqrt{2} = \cos(\pi/4) + i\sin(\pi/4)$ is $\cos(\pi/8) + i\sin(\pi/8)$, and all the other roots may be found geometrically:

$$\sqrt[4]{i} = \pm[\cos(\pi/8) + i\sin(\pi/8)], \quad \pm[-\sin(\pi/8) + i\cos(\pi/8)]$$

with $\pm[-\sin(\pi/8) + i\cos(\pi/8)]$ being the two square roots of $-(1+i)/\sqrt{2} = \cos(5\pi/4) + i\sin(5\pi/4)$. From the arithmetical point of view, it would be nice to have some values for $\sin(\pi/8)$ and $\cos(\pi/8)$. Thus, let us consider $\cos(\pi/8) + i\sin(\pi/8) = x + iy$. Then

$$x^2 - y^2 = \frac{1}{\sqrt{2}} = 2xy,$$

whence $x^2 + y^2 = 1$ (which is already obvious geometrically). It follows that

$$2x^2 = 1 + \frac{1}{\sqrt{2}} = \frac{2+\sqrt{2}}{2} \quad \text{and} \quad x = \pm\frac{\sqrt{2+\sqrt{2}}}{2},$$

and we take the positive squre root here to get the principal root $x + iy$. That is,

$$\cos(\pi/8) = \frac{\sqrt{2+\sqrt{2}}}{2} \quad \text{and} \quad \sin(\pi/8) = y = \frac{\sqrt{2}}{4x} = \frac{\sqrt{2-\sqrt{2}}}{2}$$

owing to the fact that $x = 1/\sqrt{2(2-\sqrt{2})}$. In view of this calculation, our roots become

$$\sqrt[4]{i} = \pm\left[\frac{\sqrt{2+\sqrt{2}}}{2} + i\frac{\sqrt{2-\sqrt{2}}}{2}\right], \quad \pm\left[-\frac{\sqrt{2-\sqrt{2}}}{2} + i\frac{\sqrt{2+\sqrt{2}}}{2}\right].$$

For the fourth roots of $-i$, we can simply add $\pi/4$ (i.e., multiply by $\sqrt{i}$) to the argument of the roots we just found:

$$\begin{aligned}\sqrt[4]{-i} &= \pm[\sin(\pi/8) + i\cos(\pi/8)], \quad \pm[-\cos(\pi/8) + i\sin(\pi/8)] \\ &= \pm\left[\frac{\sqrt{2-\sqrt{2}}}{2} + i\frac{\sqrt{2+\sqrt{2}}}{2}\right], \quad \pm\left[-\frac{\sqrt{2+\sqrt{2}}}{2} + i\frac{\sqrt{2-\sqrt{2}}}{2}\right].\end{aligned}$$

Figure 1.4: the fourth roots of i and $-i$

Exercise 4

The quadratic formula, derived from completing the square, is also valid over $\mathbb{C}$. In fact, if we write $\zeta = \alpha + i\beta$ and $\eta = \gamma + i\delta$, and z satisfies $z^2 + \zeta z + \eta = 0$, then

$$(z - \zeta/2)^2 = \zeta^2/4 - \eta,$$

and

$$z = \zeta/2 \pm \sqrt{\zeta^2/4 - \eta} = \frac{\zeta \pm \sqrt{\zeta^2 - 4\eta}}{2}. \tag{1.2}$$

In this formula, $\pm\sqrt{\zeta^2 - 4\eta}$ represents the two square roots of the complex number $\zeta^2 - 4\eta$. In order to proceed further, we need the formula Ahlfors derives for square roots of complex numbers. Let us give a slight veriation of that here:

The complex square root (computation)

$$\sqrt{\alpha + i\beta} = x + iy.$$

$$x^2 - y^2 = \alpha \quad \text{and} \quad 2xy = \beta. \tag{1.3}$$

$$x^2 + y^2 = \sqrt{\alpha^2 + \beta^2}, \quad \text{so} \quad x = \pm\sqrt{\frac{\alpha + \sqrt{\alpha^2+\beta^2}}{2}}. \tag{1.4}$$

Because $2xy = \beta$, we know

$$y = \frac{\beta}{2x} = \pm\frac{\sqrt{2}}{2}\,\frac{\beta}{\sqrt{\alpha + \sqrt{\alpha^2+\beta^2}}} \tag{1.5}$$

unless, of course, $x = 0$. The value of x can only vanish if $\beta = 0$ and $\alpha < 0$. In this case, $y = \pm\sqrt{-\alpha}$ and $\sqrt{\alpha + 0i} = \pm i\sqrt{-\alpha}$.

Let us assume $x \neq 0$ and come back to the exceptional case $\alpha + i\beta < 0$ later. We have then

$$\sqrt{\alpha + i\beta} = \pm\left(\sqrt{\frac{\alpha + \sqrt{\alpha^2+\beta^2}}{2}} + i\frac{\sqrt{2}}{2}\,\frac{\beta}{\sqrt{\alpha + \sqrt{\alpha^2+\beta^2}}}\right). \tag{1.6}$$

If $\beta = 0$ and $\alpha \geq 0$, then this formula simplifies to

$$\sqrt{\alpha + 0i} = \sqrt{\alpha}.$$

If $\beta \neq 0$, then $\sqrt{\pm\alpha + \sqrt{\alpha^2+\beta^2}} > 0$, so from (1.5)

$$y = \pm\frac{\beta}{\sqrt{\beta^2}}\,\frac{\sqrt{-\alpha + \sqrt{\alpha^2+\beta^2}}}{2},$$

and (1.6) becomes

$$\sqrt{\alpha + i\beta} = \pm\left(\sqrt{\frac{\alpha + \sqrt{\alpha^2+\beta^2}}{2}} + i\frac{\beta}{|\beta|}\,\frac{\sqrt{-\alpha + \sqrt{\alpha^2+\beta^2}}}{2}\right).$$

This is Ahlfors' formula.

Notice we can write $\beta/|\beta| = \mathrm{sign}(\beta)$ with

$$\mathrm{sign}(\beta) = \begin{cases} \beta/|\beta| & \text{if } \beta \neq 0 \\ 0 & \text{if } \beta = 0. \end{cases}$$

In this way, the singularity when $\beta = 0$ and $\alpha > 0$ is effectively removed and the cases with $\beta \neq 0$ and $\alpha + \beta i > 0$ may be unified:

$$\sqrt{\alpha + i\beta} = \pm \left(\sqrt{\frac{\alpha + \sqrt{\alpha^2 + \beta^2}}{2}} + i\,\mathrm{sign}(\beta)\, \frac{\sqrt{-\alpha + \sqrt{\alpha^2 + \beta^2}}}{2} \right). \qquad (1.7)$$

In the special case $\alpha + \beta i > 0$, this formula simplifies immediately to the correct expression

$$\sqrt{\alpha + 0i} = \pm\sqrt{\alpha} \quad \text{when } \alpha > 0.$$

As pointed out by Ahlfors, this case is worth remembering separately. We recall also the exceptional case $\alpha + 0i < 0$ which was set aside above. The formula (1.7) does not give the correct expression in this case, and we state it separately:

$$\sqrt{\alpha + 0i} = \pm i\sqrt{-\alpha} \quad \text{when } \alpha < 0.$$

Let us now apply these formulas to the square root appearing in the quadratic formula.

Exercise 4 (continued)

In our discussion above, we were led to consider the square root

$$\sqrt{\zeta^2 - 4\eta}$$

where $\zeta = \alpha + i\beta$ and $\eta = \gamma + i\delta$. The complex number inside the square root is

$$\alpha^2 - \beta^2 - 4\gamma + (2\alpha\beta - 4\delta)i.$$

If $\alpha\beta - 2\delta \neq 0$ or $\alpha\beta = 2\delta$ and $\alpha^2 - \beta^2 - 4\gamma \geq 0$, we can apply (1.7) to (1.2) and obtain an expression for the roots of the quadratic equation. Before we make that substitution, we note for reference

$$(\alpha^2 - \beta^2 - 4\gamma)^2 + (2\alpha\beta - 4\delta)^2 = (\alpha^2 + \beta^2)^2 - 8\gamma(\alpha^2 - \beta^2) + 16\alpha\beta\delta + 16\delta^2.$$

Here is the complex quadratic formula for z satisfying

$$z^2 + (\alpha + i\beta)z + \gamma + i\delta = 0$$

in terms of real and imaginary parts when $\alpha\beta \neq 2\delta$:

$$\begin{aligned}
z &= \frac{\alpha + i\beta \pm \sqrt{\zeta^2 - 4\eta}}{2} \\
&= \frac{1}{2}\left[\alpha + i\beta \pm \sqrt{\frac{\alpha^2 - \beta^2 - 4\gamma + \sqrt{(\alpha^2 - \beta^2 - 4\gamma)^2 + (2\alpha\beta - 4\delta)^2}}{2}}\right. \\
&\qquad \left. \pm i\,\mathrm{sign}(\alpha\beta - 2\delta)\sqrt{\frac{\beta^2 - \alpha^2 + 4\gamma + \sqrt{(\alpha^2 - \beta^2 - 4\gamma)^2 + (2\alpha\beta - 4\delta)^2}}{2}}\right] \\
&= \frac{1}{2}\left(\alpha \pm \sqrt{\frac{\alpha^2 - \beta^2 - 4\gamma + \sqrt{(\alpha^2 - \beta^2 - 4\gamma)^2 + (2\alpha\beta - 4\delta)^2}}{2}}\right) \\
&+ \frac{i}{2}\left(\beta \pm \mathrm{sign}(\alpha\beta - 2\delta)\sqrt{\frac{\beta^2 - \alpha^2 + 4\gamma + \sqrt{(\alpha^2 - \beta^2 - 4\gamma)^2 + (2\alpha\beta - 4\delta)^2}}{2}}\right).
\end{aligned}$$

When $\alpha\beta = 2\delta$ and $\alpha^2 - \beta^2 \geq 4\gamma$, the formula simplifies to

$$z = \frac{1}{2}\left(\alpha \pm \sqrt{\alpha^2 - \beta^2 - 4\gamma} + i\beta\right).$$

Lastly, we have the exceptional case $\alpha\beta = 2\delta$ and $\alpha^2 - \beta^2 < 4\gamma$. In this case,

$$z = \frac{1}{2}\left[\alpha + i\left(\beta \pm \sqrt{\beta^2 - \alpha^2 + 4\gamma}\right)\right].$$

As can be seen clearly from this exercise, the quadratic formula (1.2) in the complex case, looks superficially the same as the formula for the real case, but when the complex quadratic formula is expressed in terms of real and imaginary parts, it is rather more complicated.

1.1.7 Lecture 2: Conjugation and inequalities

This lecture covers § I 1.4,5.

When $z = a + bi$ with $a, b \in \mathbb{R}$, we write

$$\begin{aligned} a &= \operatorname{Re}(z) \qquad \text{(the "real part" of } z\text{)} \\ b &= \operatorname{Im}(z) \qquad \text{(the "imaginary part" of } z\text{).} \end{aligned}$$

The real and imaginary parts of z are both real numbers. We have then

$$\bar{z} = \operatorname{Re}(z) + i \operatorname{Im}(z),$$

and we also write

$$\bar{z} = \operatorname{Re}(z) - i \operatorname{Im}(z).$$

$\bar{z}$ is called the **conjugate** of z.

Figure 1.5: complex conjugate

Useful facts about complex conjugation:

$z \in \mathbb{R}$ if and only if $\bar{z} = z$.

$$\begin{aligned} z + \bar{z} &= 2 \operatorname{Re}(z) \\ z - \bar{z} &= 2i \operatorname{Im}(z) \in i\mathbb{R}. \end{aligned}$$

$$\bar{\bar{z}} = z \qquad \text{(Conjugation is an \textit{involution} of } \mathbb{C}.)$$

$$\overline{z + w} = \bar{z} + \bar{w} \qquad \text{(Conjugation is "linear.")}$$

$$\overline{(a + bi)(c + di)} = ac - bd - (ad + bc)i = (a - bi)(c - di) \qquad \checkmark$$

This is really useful; conjugation is multiplicative:

$$\overline{zw} = \bar{z}\bar{w} \quad \text{for } z, w \in \mathbb{C}.$$

As a result of these observations, conjugation "propogates" through all kinds of algebraic expressions:

$$\begin{aligned} P(z) &= a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 \\ \overline{P(z)} &= \bar{a}_n \bar{z}^n + \bar{a}_{n-1} \bar{z}^{n-1} + \cdots + \bar{a}_1 \bar{z} + \bar{a}_0. \end{aligned}$$

Proposition 1 *If z is a root of the polynomial equation*

$$P(z) = \sum_{j=0}^{n} a_j z^j = 0$$

and $a_0, \ldots, a_n \in \mathbb{R}$, i.e., the coefficients are real, then $\bar{z}$ is also a root.

$$\overline{\left(\frac{z^2 + w}{z + w^2}\right)} = \frac{\bar{z}^2 + \bar{w}}{\bar{z} + \bar{w}^2}.$$

We already talked about the modulus:

$$|z| = \sqrt{\operatorname{Re}(z)^2 + \operatorname{Im}(z)^2} = \sqrt{z\bar{z}} \qquad \big(\ (a+bi)(a-bi) = a^2 + b^2 + 0i\ \big)\,.$$

Important consequence:

$$|zw| = \sqrt{zw\bar{z}\bar{w}} = |z||w|.$$

Similarly,

$$\left|\frac{z}{w}\right| = \sqrt{\frac{z}{w}\frac{\bar{z}}{\bar{w}}} = \frac{|z|}{|w|}, \qquad w \neq 0.$$

$$\begin{aligned} |z+w|^2 &= (z+w)(\bar{z}+\bar{w}) \\ &= |z|^2 + z\bar{w} + w\bar{z} + |w|^2 \\ &= |z|^2 + 2\operatorname{Re}(z\bar{w}) + |w|^2. \end{aligned}$$

Therefore, $|z+w| \neq |z| + |w|$, but

$$\begin{aligned} |z+w| &= \sqrt{|z|^2 + 2\operatorname{Re}(z\bar{w}) + |w|^2} \\ &\leq \sqrt{|z|^2 + 2|z\bar{w}| + |w|^2} \\ &= |z| + |w| \qquad \text{since } |\bar{w}| = |w|. \end{aligned}$$

This is the proof of the triangle inequality for the modulus (norm) on $\mathbb{C}$. We will repeat and extend this discussion below.

Some other useful inequalities

$$-|z| \le \mathrm{Re}(z) \le |z| = \sqrt{\mathrm{Re}(z)^2 + \mathrm{Im}(z)^2}$$
$$-|z| \le \mathrm{Im}(z) \le |z|.$$
$$\left|\sum_{j=1}^{n} z_j\right| \le \left|\sum_{j=1}^{n-1} z_j\right| + |z_n| \le \cdots \le \sum_{j=1}^{n} |z_j|.$$
$$|z-w| \ge |z| - |w| \qquad (Why?) \tag{1.8}$$

Answer: $|z-w| + |w| \ge |z-w+w|$.

$$|z-w| = |w-z| \ge |w| - |z|. \tag{1.9}$$

Maybe the right sides of the inequalities in (1.8) and (1.9) are negative, but...

$$|z-w| \ge \big|\,|z| - |w|\,\big|.$$

Nexercise 7 *Show* $|z+w| \ge \big|\,|z| - |w|\,\big|$.

Equality in the triangle inequality

The triangle inequality is one of the metric axioms for $\mathbb{C}$ as described in Nexercise 3. One sees from Nexercise 5 that the complex absolute value (or modulus) gives a more primitive form of the metric. That is, the inequality

$$|z-w| \le |z-\zeta| + |\zeta - w|$$

for $z, w, \zeta \in \mathbb{C}$ follows from

$$|z+w| \le |z| + |w|, \tag{1.10}$$

and this latter inequality is also called the triangle inequality. It is one of the abstract axioms which defines what it means to be a *normed space*. In fact, $\mathbb{C}$ is also a normed space with norm given by the absolute value. Also, every normed vector space is a metric space with a distance between two elements defined to be the norm of the difference of those two elements. In any case, we can derive the triangle inequality for the modulus as follows:

$$\begin{aligned} |z+w|^2 &= |z|^2 + 2\,\mathrm{Re}(z\bar{w}) + |w|^2 \\ &\le |z|^2 + 2|z\bar{w}| + |w|^2 \\ &= |z|^2 + 2|z||w| + |w|^2 \\ &= (|z| + |w|)^2. \end{aligned}$$

Looking at the derivation, we see equality holds if and only if

$$\mathrm{Re}(z\bar{w}) = |z\bar{w}| = \sqrt{[\mathrm{Re}(z\bar{w})]^2 + [\mathrm{Im}(z\bar{w})]^2}.$$

That is, if and only if $\mathrm{Im}(z\bar{w}) = 0$ and $z\bar{w} \geq 0$. In fact, only the latter condition characterizes equality:

> *Equality holds in the triangle inequality $|z + w| \leq |z| + |w|$ if and only if $z\bar{w} \geq 0$.*

The Cauchy-Schwarz inequality

We've mentioned that $\mathbb{C}$ is a metric space with distance

$$d(z, w) = |z - w|.$$

The distance comes from the modulus which (we've mentioned) is a **norm** on $\mathbb{C}$

$$|z| = \sqrt{\mathrm{Re}(z)^2 + \mathrm{Im}(z)^2}.$$

The concepts of distance (or **metric**) and **norm** are familiar from

$$\mathbb{R}^n = \{\mathbf{x} = (x_1, \ldots, x_n) : x_1, \ldots, x_n \in \mathbb{R}\}$$

where

$$|\mathbf{x}| = \sqrt{\sum_{j=1}^{n} x_j^2} \quad \text{and} \quad d(\mathbf{x}, \mathbf{y}) = \sqrt{\sum_{j=1}^{n} (x_j - y_j)^2}.$$

It will be recalled that there is also a *dot product* on $\mathbb{R}^n$:

$$\mathbf{x} \cdot \mathbf{y} = \sum_{j=1}^{n} x_j y_j.$$

Obviously, the norm can be derived from the dot product:

$$|\mathbf{x}| = \sqrt{\mathbf{x} \cdot \mathbf{x}}.$$

The dot product is a special case of an **inner product**. There are abstract axioms defining each of these structures (metric, norm, inner product) on a vector space. Also, once you have an inner product

$$(\mathbf{v}, \mathbf{w}) \mapsto \langle \mathbf{v}, \mathbf{w} \rangle,$$

then you *always* get a norm

$$\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$$

and a metric/distance

$$d(\mathbf{v}, \mathbf{w}) = \|\mathbf{v} - \mathbf{w}\|.$$

Finally, when you have an inner product, you always get a fundamental inequality relating the inner product and the norm called the Cauchy-Schwarz inequality:

$$|\langle \mathbf{v}, \mathbf{v} \rangle| \leq \|\mathbf{v}\|\|\mathbf{w}\|. \tag{1.11}$$

The (absolute value of the) inner product is less than (or equal to) the product of the norms.

The discussion above applies to the vector space

$$\mathbb{C}^n = \{(z_1, \ldots, z_n) : z_1, \ldots, z_n \in \mathbb{C}\}$$

which is the proto-typical *complex* vector space. So far, we have introduced the norm (modulus) and distance for $\mathbb{C}^1$. Due to the structural form/arithmetic of complex numbers, the inner product on $\mathbb{C}^n$ and especially the Cauchy-Schwarz inequality are important directly in $\mathbb{C}$. The inner product on $\mathbb{C}^n$ is

$$\langle (z_1, \ldots, z_n), (w_1, \ldots, w_n) \rangle = \sum_{j=1}^{n} z_j \bar{w}_j. \tag{1.12}$$

Let's prove the Cauchy-Schwarz inequality in $\mathbb{C}^n$:

$$\begin{aligned} 0 &\leq \sum_{j=1}^{n} |z_j + \lambda w_j| \qquad (1.13)\\ &= \left(\sum_{j=1}^{n} |w_j|^2\right) |\lambda|^2 + 2\,\mathrm{Re}\left[\lambda \sum_{j=1}^{n} w_j \bar{z}_j\right] + \sum_{j=1}^{n} |z_j|^2 \\ &\leq \left(\sum_{j=1}^{n} |w_j|^2\right) |\lambda|^2 + 2\left|\sum_{j=1}^{n} w_j \bar{z}_j\right| |\lambda| + \sum_{j=1}^{n} |z_j|^2. \end{aligned}$$

The last expression is quadratic in $|\lambda|$ with a minimum occuring when the argument of the quatratic polynomial is

$$-\frac{\left|\sum_{j=1}^{n} w_j \bar{z}_j\right|}{\sum_{j=1}^{n} |w_j|^2}$$

assuming $\sum_{j=1}^n |w_j|^2 > 0$. Of course, $|\lambda|$ can not be expected to take this value. If we estimate in the opposite direction, however, we find

$$\left(\sum_{j=1}^n |w_j|^2\right)|\lambda|^2 + 2\,\mathrm{Re}\left[\lambda\sum_{j=1}^n w_j\bar{z}_j\right] + \sum_{j=1}^n |z_j|^2 \\ \geq \left(\sum_{j=1}^n |w_j|^2\right)|\lambda|^2 - 2\left|\sum_{j=1}^n w_j\bar{z}_j\right||\lambda| + \sum_{j=1}^n |z_j|^2. \tag{1.14}$$

Thus, optimizing the inequality (1.13) by choosing

$$|\lambda| = \frac{\left|\sum_{j=1}^n w_j\bar{z}_j\right|}{\sum_{j=1}^n |w_j|^2} \qquad \text{and} \qquad \lambda = -\frac{\sum_{j=1}^n z_j\bar{w}_j}{\sum_{j=1}^n |w_j|^2}$$

we obtain equality in (1.14) and

$$0 \leq \frac{\left|\sum_{j=1}^n w_j\bar{z}_j\right|^2}{\sum_{j=1}^n |w_j|^2} - 2\frac{\left|\sum_{j=1}^n w_j\bar{z}_j\right|^2}{\sum_{j=1}^n |w_j|^2} + \sum_{j=1}^n |z_j|^2.$$

This immediately simplifies to (1.11):

$$\left|\sum_{j=1}^n z_j\bar{w}_j\right| \leq \sqrt{\sum_{j=1}^n |z_j|^2}\sqrt{\sum_{j=1}^n |w_j|^2}.$$

This completes the proof of the Cauchy-Schwarz inequality for $\mathbb{C}$. Replacing w_j with $\bar{w}_j$, we obtain what Ahlfors calls the Cauchy inequality:

$$\left|\sum_{j=1}^n z_j w_j\right|^2 \leq \left(\sum_{j=1}^n |z_j|^2\right)\left(\sum_{j=1}^n |w_j|^2\right).$$

Nexercise 8 *What do the triangle inequality for the norm and the metric on $\mathbb{C}^n$ say about complex numbers?*

1.1.8 §1.4 exercises

Exercise 3

We are asked to show that when $|z| = 1$ or $|w| = 1$, then

$$\left|\frac{w - z}{1 - \bar{w}z}\right| = 1.$$

We are also asked to find an exceptional case when $|z| = |w| = 1$. The basic assertion is easy since

$$\left|\frac{w-z}{1-\bar{w}z}\right| = \frac{w-z}{1-\bar{w}z} \cdot \overline{\frac{w-z}{1-\bar{w}z}} = \frac{w-z}{1-\bar{w}z} \cdot \frac{\bar{w}-\bar{z}}{1-\bar{z}w} = \frac{|w|^2 - z\bar{w} - w\bar{z} + |z|^2}{1 - z\bar{w} - w\bar{z} + |z|^2|w|^2}.$$

This fraction evidently reduces to unity if $|z| = 1$ or $|w| = 1$.

Presumably, the exceptional case would be when the denominator in the original expression $1 - \bar{w}z$ vanishes. The condition $1 - \bar{w}z = 0$ is the same as $w - |w|^2 z = 0$ or

$$z = \frac{w}{|w|^2}. \tag{1.15}$$

Notice the original exceptional condition can never hold when $w = 0$. In the situation where $|w| = 1$, this becomes $z = w$ which indeed implies $|z| = |w| = 1$. We conclude then that the exceptional case when $|z| = |w| = 1$ is the case $z = w$ for which the original assertion doesn't make sense.

It is interesting to consider the collection of all pairs $(z, w) \in \mathbb{C}^2$ for which the condition (1.15) holds. A first observation is that if $|w| = r > 0$, then $|z| = 1/r$. This gives a kind of product structure to the singular set

$$\Sigma = \{(z, w) : w \neq 0 \text{ and } z = w/|w|^2\}.$$

In fact, if $w = r(\cos\theta + i\sin\theta)$ is in the circle of radius $r > 0$ centered at 0, then $z = (\cos\theta + i\sin\theta)/r$ is the corresponding point in the circle of radius $1/r$. This means Σ is a (two-dimensional) surface of some sort in $\mathbb{C}^2$ which is (geo)metrically isomorphic to $\mathbb{R}^4$. This surface, in fact, is parameterized by

$$X(r,\theta) = ((\cos\theta + i\sin\theta)/r, r(\cos\theta + i\sin\theta)) \sim (\cos\theta/r, \sin\theta/r, r\cos\theta, r\sin\theta)$$

where $(r, \theta) \in (0, \infty) \times \mathbb{R}$ and the parameterization is 2π-periodic in θ.

A further observation is that Σ projects simply onto the three-dimensional sphere

$$\{(z, w) \in \mathbb{C}^2 : |z|^2 + |w|^2 = 1\} \sim \{(x, y, z, w) \in \mathbb{R}^4 : x^2 + y^2 + z^2 + w^2 = 1\}$$

with the projection given by

$$\rho \circ X = \frac{1}{\sqrt{r^2 + 1/r^2}} (\cos\theta/r, \sin\theta/r, r\cos\theta, r\sin\theta).$$

To see this, note that $r\cos\theta = \tilde{r}\cos\tilde{\theta}$ and $r\sin\theta = \tilde{r}\sin\tilde{\theta}$ implies first that $r^2 = \tilde{r}^2$ and, hence, that $r = \tilde{r}$. Then the angles θ and $\tilde{\theta}$ must agree up to an additive multiple of 2π as well.

Finally, the three-sphere projects stereographically onto $\mathbb{R}^3$ by

$$\sigma : (x, y, z, w) \mapsto \frac{(x, y, z)}{1 - w}$$

with the exception of the north pole $(0, 0, 0, 1)$. Since the non-vanishing of the fourth coordinate in $\rho \circ X$ implies the non-vanishing of the second coordinate, we know the north pole corresponds to no point in Σ. We conclude that the image of $\sigma \circ \rho \circ X$ represents a nonsingular embedded surface in $\mathbb{R}^3$, and we may be able to get some reasonable idea of what the surface Σ looks like in $\mathbb{C}^2$ by plotting this double projection.

In fact, the double projection is a very interesting surface. The portion indicated in Figure 1.6 appears to be a double twisted strip. The $r = 0$

Figure 1.6: the projection of an annular piece of Σ corresponding to $0 \le r < 1$

boundary is nonsingular with the double projection of the circle $r = 0$ coinciding with the unit circle in the x, y-plane. In particular, one can conclude that Σ is *orientable.*

Exercise 4

Here we consider the equation $az + b\bar{z} + c = 0$ for $z \in \mathbb{C}$ and are asked to determine conditions under which there is a unique solution (and to find that solution).

Writing $a = a_1 + a_2 i$, $b = b_1 + b_2 i$, $c = c_1 + c_2 i$, and $z = x + iy$ the equation becomes

$$(a_1 + a_2 i)(x + iy) + (b_1 + b_2 i)(x - iy) + c_1 + c_2 i = 0.$$

This is equivalent to the two real linear equations

$$\begin{cases} (a_1 + b_1)x - (a_2 - b_2)y = -c_1 \\ (a_2 + b_2)x + (a_1 - b_1)y = -c_2. \end{cases}$$

Such a solution has a unique solution precisely when the coefficient matrix has nonzero determiant. That is,

$$a_1^2 - b_1^2 + a_2^2 - b_2^2 \neq 0 \quad \text{or} \quad |a| \neq |b|.$$

When this condition holds, the solution is given by Cramer's rule:

$$x = \frac{c_1(a_1 - b_1) + c_2(a_2 - b_2)}{|b|^2 - |a|^2}, \quad y = \frac{c_2(a_1 + b_1) + c_1(a_2 + b_2)}{|b|^2 - |a|^2}.$$

Thus, we can write

$$z = \frac{c_1(a_1 - b_1) + c_2(a_2 - b_2) + [c_2(a_1 + b_1) + c_1(a_2 + b_2)]i}{|b|^2 - |a|^2}.$$

This is a somewhat unsatisfying solution due to all the subscripts. These could be removed by writing $c_1 = \mathrm{Re}(c)$ etc., but that wouldn't be much better, and we can do better.

$$\begin{aligned} c_1(a_1 - b_1) + c_2(a_2 - b_2) &+ [c_2(a_1 + b_1) + c_1(a_2 + b_2)]i \\ &= \mathrm{Re}[(a - b)\bar{c}] + i\,\mathrm{Im}[(\bar{a} + \bar{b})c] \\ &= \frac{1}{2}\left[(a - b)\bar{c} + (\bar{a} - \bar{b})c + (\bar{a} + \bar{b})c - (a + b)\bar{c}\right] \\ &= \bar{a}c - b\bar{c}. \end{aligned}$$

In fact, it is easily checked that

$$z = \frac{\bar{a}c - b\bar{c}}{|b|^2 - |a|^2} \tag{1.16}$$

represents the same solution we found above and, of course, solves the equation. This brings up another question. Is there a cleaner way to treat/solve the equation from the beginning without the use of subscripts? I'm not exactly sure there is. With hindsight from our answer, however, we can do the following.

Let us assume a solution of the form $z = \alpha c + \beta \bar{c}$. (It is not clear why such an assumption is justified.) The equation may then be written as

$$a\alpha c + a\beta\bar{c} + b\bar{\alpha}\bar{c} + b\bar{\beta}c + c = 0.$$

That is,

$$(a\alpha + b\bar{\beta} + 1)c + (b\bar{\alpha} + a\beta)\bar{c} = 0.$$

Making another (apparently unjustified) assumption that the coefficients of c and $\bar{c}$ should vanish, we obtain the system

$$\begin{cases} a\alpha + b\bar{\beta} & = -1 \\ \bar{b}\alpha + \bar{a}\bar{\beta} & = 0. \end{cases}$$

This is a system of two linear equations with complex coefficients and, presumably, a solution $(\alpha, \bar{\beta}) \in \mathbb{C}^2$. Notice the determinant of the coefficient matrix is $|a|^2 - |b|^2$. In fact, Cramer's rule applies here as well; see Ahlfors' discussion in § 1.1, and we find

$$\alpha = \frac{\bar{a}}{|b|^2 - |a|^2} \quad \text{and} \quad \beta = -\frac{b}{|b|^2 - |a|^2}$$

in agreement with the solution (1.16). □

Is there a straightforward way to treat the original equation with no unjustified assumptions?

Okay, I've got it. Incidentally, there's a short discussion of Cramer's rule in the 2×2 complex case below.

A better solution for exercise 1.4.4:

Let's start with the assumption that a and b are both nonzero. Then we have for any z with $az + b\bar{z} + c = 0$ that

$$\bar{z} = -\frac{az + c}{b}.$$

Taking the conjugate of the original equation, we have $\bar{a}\bar{z} + \bar{b}z + \bar{c} = 0$, and this implies

$$\bar{z} = -\frac{\bar{b}z + \bar{c}}{\bar{a}}.$$

Equating the two expressions for $\bar{z}$ we obtain

$$|b|^2 z + b\bar{c} = |a|^2 z + \bar{a}c.$$

That is,

$$(|b|^2 - |a|^2|)z = \bar{a}c - b\bar{c}.$$

Clearly, we get a unique solution given by (1.16) as long as $|b| \neq |a|$. Now all we need to do is check that this same condition is necessary and sufficient and leads to the same solution when one of a or b is zero. For example, if $b = 0$, then the condition to have a unique solution is $a \neq 0$, which is equivalent to $|a| \neq |b|$ in this case. Furthermore, if $b = 0 \neq a$, then $z = -c/a$ which is exactly the same as (1.16) when $b = 0$. The situation when $a = 0$ is very similar. $\square$

Cramer's Rule: If $a_{11}a_{22} - a_{12}a_{21} \neq 0$, then

$$\begin{cases} a_{11}z + a_{12}w &= c_1 \\ a_{21}z + a_{22}w &= c_2 \end{cases}$$

has the unique solution

$$z = \frac{c_1 a_{22} - c_2 a_{12}}{a_{11}a_{22} - a_{12}a_{21}} \quad \text{and} \quad w = \frac{a_{11}c_2 - a_{21}c_1}{a_{11}a_{22} - a_{12}a_{21}}.$$

Proof: It is easily checked that the values of z and w provide a solution. Uniqueness either follows from substitution and checking various cases or from a dimension argument (the set of solutions is a complex affine vector space). $\square$

Exercise 5: Lagrange's Identity

$$\left|\sum_i^n z_i w_i\right|^2 = \left(\sum_i^n |z_i|^2\right)\left(\sum_i^n |w_i|^2\right) - \sum_{1\leq i<j\leq n} |z_i\bar{w}_j - z_j\bar{w}_i|^2.$$

Perhaps the easiest way to see this is by induction. The base case $n = 1$ is okay, however, the last sum in the identity is vacuous in this case, so we will verify the cases $n = 1$ and $n = 2$. Then we will carry out the inductive step for $n \geq 3$. For $n = 1$, we write $z = z_1 = x + iy$ and $w = w_1 = \xi + i\eta$. Then

$$\begin{aligned}
|zw|^2 &= |(x+iy)(\xi+i\eta)|^2 \\
&= (x\xi - y\eta)^2 + (x\eta + y\xi)^2 \\
&= x^2\xi^2 + y^2\eta^2 + x^2\eta^2 + y^2\xi^2 \\
&= (x^2+y^2)(\xi^2+\eta^2) \\
&= |z|^2|w|^2.
\end{aligned}$$

For $n = 2$,

$$\begin{aligned}
|z_1w_1 + z_2w_2|^2 &= (z_1w_1 + z_2w_2)(\bar{z}_1\bar{w}_1 + \bar{z}_2\bar{w}_2) \\
&= |z_1|^2|w_1|^2 + |z_2|^2|w_2|^2 + z_1w_1\bar{z}_2\bar{w}_2 + z_2w_2\bar{z}_1\bar{w}_1 \\
&= (|z_1|^2 + |w_1|^2)(|z_2|^2 + |w_2|^2) - |z_1\bar{w}_2 - z_2\bar{w}_1|^2
\end{aligned}$$

as

$$\begin{aligned}
|z_1\bar{w}_2 - z_2\bar{w}_1|^2 &= (z_1\bar{w}_2 - z_2\bar{w}_1)(\bar{z}_1w_2 - \bar{z}_2w_1) \\
&= |z_1|^2|w_2|^2 - z_1\bar{w}_2\bar{z}_2w_1 - z_2\bar{w}_1\bar{z}_1w_2 + |z_2|^2|w_1|^2.
\end{aligned}$$

Finally, for $n \geq 3$ we proceed by induction:

$$\begin{aligned}
\left|\sum_{i=1}^{n} z_iw_i\right|^2 &= \left|\sum_{i=1}^{n-1} z_iw_i + z_nw_n\right|^2 \\
&= \left(\sum_{i=1}^{n-1} z_iw_i + z_nw_n\right)\left(\sum_{i=1}^{n-1} \bar{z}_i\bar{w}_i + \bar{z}_n\bar{w}_n\right) \\
&= \left(\sum_{i=1}^{n-1} |z_i|^2\right)\left(\sum_{i=1}^{n-1} |w_i|^2\right) - \sum_{1\leq i<j\leq n-1} |z_i\bar{w}_j - z_j\bar{w}_i|^2 \\
&\qquad + z_nw_n\sum_{i=1}^{n-1} \bar{z}_i\bar{w}_i + \bar{z}_n\bar{w}_n\sum_{i=1}^{n-1} z_iw_i + |z_n|^2|w_n^2| \\
&= \left(\sum_{i=1}^{n-1} |z_i|^2 + |z_n|^2\right)\left(\sum_{i=1}^{n-1} |w_i|^2 + |w_n|^2\right) \\
&\qquad - \left(\sum_{1\leq i<j\leq n-1} |z_i\bar{w}_j - z_j\bar{w}_i|^2 + \sum_{i=1}^{n-1} |z_i\bar{w}_n - z_n\bar{w}_i|^2\right)
\end{aligned}$$

since

$$\sum_{i=1}^{n-1} |z_i \bar{w}_n - z_n \bar{w}_i|^2 = \sum_{i=1}^{n-1} \left(|w_n|^2 |z_i|^2 - \bar{w}_n \bar{z}_n z_i w_i - z_n w_n \bar{z}_i \bar{w}_i + |z_n|^2 |w_i|^2 \right).$$

Notice Lagrange's identity gives the Cauchy-Schwarz inequality as pointed out by Ahlfors in §1.5. This kind of inequality asserts generally that "the dot product is less than the product of the norms," or more precisely, the absolute value of the inner product is less than or equal to the product of the norms. In this case, we are working in the space $\mathbb{C}^n$ where n-tuples of complex numbers $\mathbf{z} = (z_1, \ldots, z_n)$ and $\mathbf{w} = (w_1, \ldots, w_n)$ have inner product

$$\langle \mathbf{z}, \mathbf{w} \rangle = \sum_i^n z_i \bar{w}_i$$

and norm

$$\|\mathbf{z}\| = \left(\sum_i^n |z_i|^2 \right)^{1/2}.$$

Also, the case of equality in the Cauchy-Schwarz inequality holds when the residual term

$$\sum_{1 \le i < j \le n} |z_i \bar{w}_j - z_j \bar{w}_i|^2$$

vanishes. This means $z_i \bar{w}_j - z_j \bar{w}_i = 0$ for $1 \le i < j \le n$. Depending on one's interpretation of the phrase "is proportional to," Ahlfors' assertion concerning the case of inequality is not quite correct. One case of equality is when $\mathbf{w} = 0 \in \mathbb{C}^n$. In this case, there need not be a constant λ for which $\mathbf{z} = \lambda \bar{\mathbf{w}}$. Of course, it would be true in this case that there is some λ for which $\mathbf{w} = \lambda \mathbf{z}$, namely, $\lambda = 0$.

Alternatively, there is some $w_{i_0} \neq 0$. In this latter case, we have

$$z_i \bar{w}_{i_0} = z_{i_0} \bar{w}_i \quad \text{and} \quad z_{i_0} \bar{w}_j = z_j \bar{w}_{i_0} \quad \text{for all } 1 \le i < i_0 < j \le n.$$

If there is any $z_i \neq 0$ with $i < i_0$ or $z_j \neq 0$ with $j > i_0$, then it follows that $w_i \neq 0$, $w_j \neq 0$, and the associated quotients $z_i/\bar{w}_i$ and $z_j/\bar{w}_j$ have the common value

$$\frac{z_i}{\bar{w}_i} = \frac{z_j}{\bar{w}_j} = \lambda = \frac{z_{i_0}}{\bar{w}_{i_0}}.$$

Furthermore, in this case $\mathbf{z} = \lambda \bar{\mathbf{w}}$ as Ahlfors asserts.

1.1.9 §1.5 exercises

Exercise 1

$$\left|\frac{a-b}{1-\bar{a}b}\right| < 1 \quad \text{if} \quad |a|, |b| < 1.$$

The square of the left side is

$$\frac{|a|^2 - \bar{a}b - a\bar{b} + |b|^2}{1 - \bar{a}b - a\bar{b} + |ab|^2}.$$

Since $|ab|^2 = |a|^2|b|^2$, the assertion will clearly follow if we can show

$$|a|^2 + |b|^2 < 1 + |a|^2|b|^2.$$

Our derivation of this fact relies on the simple fact that the sum $|a| + |b|$ and the product $|a||b|$ satisfy

$$|a|+|b| = \max\{|a|, |b|\}+\min\{|a|, |b|\} \quad \text{and} \quad |a||b| = \max\{|a|, |b|\} \min\{|a|, |b|\}.$$

$$\begin{aligned}
|a|^2 + |b|^2 &\leq \max\{|a|, |b|\}(|a| + |b|) \\
&= \max\{|a|, |b|\} \left(\max\{|a|, |b|\} + \min\{|a|, |b|\}\right) \\
&= \left(\max\{|a|, |b|\}\right)^2 + \max\{|a|, |b|\} \min\{|a|, |b|\} \\
&< 1 + |a||b|.
\end{aligned}$$

1.1.10 Exercise 3

If $|z_j| \leq 1$ and $\lambda_j > 0$ for $j = 1, 2, \ldots, n$ and $\lambda_1 + \lambda_2 + \cdots + \lambda_n = 1$, then by the triangle inequality

$$\left|\sum \lambda_j z_j\right| \leq \sum |\lambda_j z_j| = \sum \lambda_j |z_j| \leq \sum \lambda_j = 1.$$

Exercise 4

I will restate this exercise for somewhat subtle pedagogical reasons:

Show there are complex numbers z satisfying

$$|z - a| + |z + a| = 2|c|$$

if and only if $|a| \le |c|$. If this condition is satisfied, find the smallest and largest values of $|z|$.

By the triangle inequality one always has

$$|z - a| + |z + a| \ge \max\{2|a|, 2|z|\}. \tag{1.17}$$

Taking $2|a|$ on the right, we see that if (any) $z \in \mathbb{C}$ satisfies the equation, we must have $2|c| \ge 2|a|$. Thus, the condition $|a| \le |c|$ is necessary for solutions to exist.

Taking $2|z|$ on the right in (1.17), we obtain an upper bound

$$|z| \le |c| \tag{1.18}$$

for any solutions. We still have not shown the existence of any solutions.

Let us look for solutions of the form $z = \lambda a$ with $\lambda \in \mathbb{R}$. The equation then becomes

$$(|\lambda - 1| + |\lambda + 1|)|a| = 2|c|. \tag{1.19}$$

If $|\lambda| \le 1$, then the scalar λ vanishes from the equation, and we get the condition $|a| = |c|$. Thus, if $|a| = |c|$, then any $z = \lambda a$ with $-1 \le \lambda \le 1$ is a solution of the equation. On the other hand, by (1.17) with $2|a| = 2|c|$ on the right, we see that every solution z of the equation implies an equality in the triangle inequality. This means

$$(a - z)(\bar{a} + \bar{z}) = |a|^2 + a\bar{z} - \bar{a}z - |z|^2 \ge 0.$$

Since $a\bar{z} - \bar{a}z$ is purely imaginary, this means $a\bar{z} - \bar{a}z = 0$ and $|z| \le |a| = |c|$. From the first condition, $\bar{a}z$ must also be real. Thus, we can write $\bar{a}z = \mu \in \mathbb{R}$ so that either $a = 0$ or $z = \mu a/|a|^2 = \lambda a$ where $\lambda = \mu/|a|^2$.

Let us consider these two cases separately. If $a = c = 0$, then the equation clearly has exactly one solution $z = 0$. There are not "complex numbers" z satisfying the equation as Ahlfors asserts, but only one complex number satisfying the equation.

If $|a| = |c| > 0$, then z must have the form $z = \lambda a$, and we know the solution set is precisely

$$\{\lambda a : -1 \le \lambda \le 1\}.$$

In each of these cases, the question of minimum and maximum modulus is also clear. If $a = c = 0$, then the single solution $z = 0$ gives both the minimum and maximum modulus of $|z| = 0$.

If $|a| = |c| > 0$, then the minimum modulus is again attained by a solution $z = 0$ with $|z| = 0$. The maximum modulus is $|c|$ and is attained by the two points in the solution set corresponding to $\lambda = -1$ and $\lambda = 1$, namely $z = \pm a$.

Returning to (1.19), we have also the case $|\lambda| > 1$. In this case, we find

$$2|\lambda||a| = 2|c|.$$

The cases $a = c = 0$ and $|a| = |c| > 0$ have already been settled, so we may assume $|a| < |c|$. Again, we have some cases:

If $|a| = 0$, we cannot find any solution of the form $z = \lambda a$, but the equation simplifies to $|z| = |c|$ with $|c| > 0$. Evidently, the solution set is

$$\{z = x + iy \in \mathbb{C} : x^2 + y^2 = |c|^2\}$$

is nonempty (for example $z = \pm|c|$ and $z = \pm|c|i$ are solutions) and is in one-to-one correspondence with a circle. The minimum and maximum modulus values are the same with $|z| = |c|$ and are attained at each solution.

The last case is $0 < |a| < |c|$. We have then

$$|\lambda| = \frac{|c|}{|a|} > 1.$$

It follows that there are solutions $z = \pm|c|a/|a|$ with modulus $|z| = |c|$. In view of the uppper bound (1.18), these solutions yield the maximum value of the modulus.

We have shown:

There are "complex numbers" $z \in \mathbb{C}$ with

$$|z - a| + |z + a| = 2|c|$$

if and only if $|a| \leq |c|$. There are at least two such complex numbers unless $|a| = |c| = 0$. In this case there is only one, namely $z = 0$, and we should say there is "a complex number" etc.. Furthermore, we have noted

$$|z| \leq |c|$$

for all solutions, and in all cases we have found solutions with $|z| = |c|$, that is, we have found solutions with the largest possible modulus.

It would seem that we have completed most of the exercise. It remains to find the complex numbers z satisfying $|z-a|+|z+a| = 2|c|$ with the smallest modulus. In fact, we have settled the question of minimum modulus as well in all cases except under the assumption $0 < |a| < |c|$.

In this last case, it is algebraically less obvious where to look for additional solutions and those of minimum modulus in particular. It would be helpful, perhaps, to have an inequality of the form $|z| \geq b$ giving a bound from below on the modulus of solutions in this case, if there is such an estimate. Squaring the equation, we find

$$|z|^2 + |a|^2 + |z^2 - a^2| = 2|c|^2.$$

Thus, solutions must satisfy

$$|z^2 - a^2| = 2|c|^2 - |a|^2 - |z|^2.$$

Squaring again, we have

$$|a|^4 - (a^2\bar{z}^2 + \bar{a}^2z^2) = 4|c|^4 - 4|c|^2|a|^2 + |a|^4 - 2(2|c|^2 - |a|^2)|z|^2.$$

That is,

$$2(2|c|^2 - |a|^2)|z|^2 = 4|c|^4 - 4|c|^2|a|^2 + a^2\bar{z}^2 + \bar{a}^2z^2.$$

Note $a^2\bar{z}^2 + \bar{a}^2z^2 = 2\operatorname{Re}(a^2\bar{z}^2) \geq -2|a|^2|z|^2$. Therefore,

$$4|c|^2|z|^2 \geq 4|c|^4 - 4|c|^2|a|^2,$$

or

$$|z|^2 \geq |c|^2 - |a|^2.$$

Thus, we have an estimate of the desired form

$$|z| \geq \sqrt{|c|^2 - |a|^2}. \tag{1.20}$$

Still it is perhaps not so obvious where to find complex numbers z for which the lower bound $|z| = \sqrt{|c|^2 - |a|^2}$ is attained. It turns out that $z = \lambda i a$ with $\lambda \in \mathbb{R}$ is the correct choice:

$$|\lambda i a - a| + |\lambda i a + a| = (|\lambda i - 1| + |\lambda i + 1|)|a| = 2\sqrt{1+\lambda^2}|a|.$$

Setting this expression equal to $2|c|$ yields two values

$$\lambda = \pm\frac{\sqrt{|c|^2 - |a|^2}}{|a|}$$

and two complex numbers

$$z = \pm i \, \frac{\sqrt{|c|^2 - |a|^2}}{|a|} \, a,$$

which are solutions and yield the minimum (possible) modulus. This completes the exercise from the algebraic/analytic point of view suggested by Ahlfors.

There were, in some sense, two rather tricky points arising in the solution, both in connection to the minimum modulus. These, for me, were the form of the minimum modulus inequality (1.20) and the form of the minimum modulus "numbers" $z = \lambda i a$. I must confess that, in each case, I was guided by geometry, visualizing the solutions as "points" in the complex plane rather than as "numbers" in the abstract algebraic space $\mathbb{C}$. It seems worthwhile to describe the geometry and carry out the associated computations a bit more fully. If a is a nonzero point in the complex plane, then the equation

$$|z - a| + |z + a| = 2|c|$$

prescribes the locus of points z the sum of whose distances to the points a and $-a$ is a constant $2|c|$. This is, of course, the definition of an **ellipse** with focal points at $\pm a$ and major semi-axis of length $|c|$. The conditions $|a| > |c|$ and $|c| = 0$ correspond to degenerate cases in which there are no solutions and one solution respectively, as we have verified analytically above. If $0 < |a| = |c|$ one has the degenerate case of a straight line segment connecting $-a$ to a; the special case $|a| = 0 < |c|$ gives a circle. These cases were also easy to handle. The remaining case $0 < |a| < |c|$ gives a nondegenerate ellipse. In analytic geometry, one usually treats ellipses in standard position with equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \tag{1.21}$$

where the larger of a and b (assumed positive) is the length of the major (i.e., longer) semi-axis. In this case, our ellipse (in the appropriate coordinates) should correspond to

$$\frac{x^2}{|c|^2} + \frac{y^2}{b^2} = 1,$$

with b satisfying $b^2 = |c|^2 - |a|^2$ where $|a|$ is half the distance between the focal points. Sometimes rotations of an ellipse, as we have here, are considered, but the computations are somewhat complicated and the resulting

Figure 1.7: ellipse with focal points at $\pm a \in \mathbb{C}$

equation involves products xy whose relation to the standard form is also correspondingly complicated. It is easy to check, however, in the standard form (1.21) that a parameterization giving all points on the ellipse is given by

$$x = a\cos\theta \quad \text{and} \quad y = b\sin\theta$$

which is quite simple. Notice this gives not only the points of minimum and maximum modulus, but all points on the ellipse. In our case, we should have points corresponding to

$$x = |c|\cos\theta \quad \text{and} \quad y = b\sin\theta.$$

Let us see if we can translate the corresponding geometry into complex analytic terminology and verify that this is the case. The general point on the ellipse should be

$$z = |c|\cos\theta\,\frac{a}{|a|} + b\sin\theta\,\frac{ia}{|a|} \quad \text{where} \quad b = \sqrt{|c|^2 - |a|^2}. \tag{1.22}$$

Plugging this point into the equation we get

$$\begin{aligned}|z-a|+|z+a| &= ||c|\cos\theta - |a| + ib\sin\theta| + ||c|\cos\theta + |a| + ib\sin\theta| \\ &= \sqrt{|a|^2 - 2|a||c|\cos\theta + |c|^2 - |a|^2\sin^2\theta} \\ &\qquad + \sqrt{|a|^2 + 2|a||c|\cos\theta + |c|^2 - |a|^2\sin^2\theta} \\ &= \sqrt{|a|^2\cos^2\theta - 2|a||c|\cos\theta + |c|^2} \\ &\qquad + \sqrt{|a|^2\cos^2\theta + 2|a||c|\cos\theta + |c|^2} \\ &= |\,|a|\cos\theta - |c|\,| + |\,|a|\cos\theta + |c|\,| \\ &= 2|c|.\end{aligned}$$

Conversely, it is not difficult to see that any point z in the plane can be expressed in the form

$$z = \alpha\frac{a}{|a|} + \beta\, i\,\frac{a}{|a|} = \frac{a}{|a|}(\alpha + i\beta) \tag{1.23}$$

with $\alpha, \beta \in \mathbb{R}$. This is essentially equivalent to the vectors (a_1, a_2) and $(-a_2, a_1)$ constitute a **basis** for vectors in $\mathbb{R}^2$. Thus, if the arbitrary point of the form (1.23) satisfies the equation $|z-a|+|z+a| = 2|c|$, then we have

$$\begin{aligned}2|c| &= |\alpha + i\beta - |a|| + |\alpha + i\beta + |a|| \\ &= \sqrt{(\alpha - |a|)^2 + \beta^2} + \sqrt{(\alpha + |a|)^2 + \beta^2}.\end{aligned}$$

Squaring both sides, we find

$$\alpha^2 + \beta^2 + |a|^2 + \sqrt{(\alpha^2 + \beta^2 + |a|^2)^2 - 4|a|^2\alpha^2} = 2|c|^2.$$

Isolating the remaining square root, and squaring again, we find

$$(\alpha^2 + \beta^2 + |a|^2)^2 - 4|a|^2\alpha^2 = [2|c|^2 - (\alpha^2 + \beta^2 + |a|^2)]^2.$$

Therefore,

$$(|c|^2 - |a|^2)\alpha^2 + |c|^2\beta^2 = |c|^2(|c|^2 - |a|^2),$$

or

$$\left(\frac{\alpha}{|c|}\right)^2 + \left(\frac{\beta}{\sqrt{|c|^2 - |a|^2}}\right)^2 = 1.$$

This means there is some angle θ for which

$$\frac{\alpha}{|c|} = \cos\theta \quad \text{and} \quad \frac{\beta}{b} = \sin\theta.$$

Thus, z has the form (1.22), and such points represent the entire locus.

1.2 Lecture 3 Gometric interpretation

This lecture covers § 2 of Chapter 1.

We start in § 2.1 by filling in some details of the geometry of complex numbers. § 2.2 extends our observations about squares and square roots to arbitrary integral powers and roots. § 2.3 is about expressing concepts of plane analytic geometry and linear algebra in the language of the complex plane. Finally, § 2.4 introduces stereographic projection and the Riemann sphere.

1.2.1 Geometry of basic complex arithmetic

Complex addition may be interpreted in terms of the usual geometry of vector addition involving parallelograms in $\mathbb{R}^2$.

Figure 1.8: addition and subtraction

$$|z + w| \leq |z| + |w|.$$

In complex notation, it is very easy to see that the "sum of the squares of the lengths of the diagonals of a parallelogram is equal to the sum of the squares of the lengths of the sides:"

$$|v - w|^2 + |v + w|^2 = 2|v|^2 + 2|w|^2.$$

The introduction of the geometric modulus and argument leads to the following important calculation:

$$
\begin{aligned}
z &= |z|(\cos\theta + i\sin\theta), \quad \theta = \arg(z) \\
z &= |w|(\cos\phi + i\sin\phi), \quad \phi = \arg(w).
\end{aligned}
$$

$$
\begin{aligned}
zw &= |z||w|[\cos\theta\cos\phi - \sin\theta\sin\phi + i(\cos\theta\sin\phi + \cos\phi\sin\theta) \\
&= |z||w|[\cos(\theta+\phi) + i\sin(\theta+\phi)].
\end{aligned}
$$

$$|zw| = |z||w| \qquad \text{and} \qquad \arg(zw) = \arg(z) + \arg(w).$$

In particular,

$$|z^2| = |z|^2 \qquad \text{and} \qquad \arg(z^2) = 2\arg(z).$$

This justifies our previous discussion of the principal square root. The geometric interpretation concerning "adding arguments" in regard to multiplication is fairly clear. Ahlfors also gives a nice geometric interpretation of the modulus relation in terms of similar triangles. If one considers the triangle determined by 0, 1 and z and constructs a similar triangle with w corresponding to 1 as indicated in Figure 1.9, then the side corresponding to z in the new triangle is the product. The ratios of side lengths then read

Figure 1.9: complex multiplication and similar triangles

$$\frac{|w|}{1} = \frac{L}{|z|} \qquad \text{or} \qquad L = |z||w|.$$

I cannot say I have ever used this "similar triangles" construction for complex multiplication, but (of course) the more you know, the more you know. There's also a version for division as indicated in Figure 1.10:

Figure 1.10: complex division and similar triangles

$$\text{For } \frac{w}{z}: \quad \frac{L}{1} = \frac{|w|}{|z|}; \qquad \text{for } \frac{z}{w}: \quad \frac{L}{1} = \frac{|z|}{|w|}.$$

1.2.2 §2.1 exercises

Exercise 1

If w is a complex number, the reflection of z across the line $\mathrm{Re}(z) = \mathrm{Im}(z)$ is $i\bar{w}$. A nice way to see this is by first rotating clockwise by $\pi/4$ so that the line of reflection coincides with the real axis. Then reflection corresponds to conjugation:

$$w \mapsto e^{-i\pi/4}w \mapsto e^{i\pi/4}\bar{w}.$$

Then just rotate back:

$$e^{i\pi/4}\bar{w} \mapsto e^{i\pi/4}e^{i\pi/4}\bar{w} = e^{i\pi/2}\bar{w} = i\bar{w}.$$

One can also just figure out what happens to the coordinates. The reflection of w across the line $\mathrm{Re}(z) = -\mathrm{Im}(z)$ is $-i\bar{w}$.

Figure 1.11: symmetric points

Exercise 2

We are to show that for any three complex numbers z_1, z_2, and z_3 the conditions

$$|z_1 - z_2| = |z_2 - z_3| = |z_3 - z_1| \tag{1.24}$$

and

$$z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1 \tag{1.25}$$

are equivalent. The first condition is interpreted to mean the three points z_1, z_2, and z_3 are the vertices of an equilateral triangle in the complex plane $\mathbb{C}$.

Let's start with a simple special case and see if we can establish the equivalence. Namely, if

$$z_3 = 0, \quad z_2 = a \geq 0, \quad \text{and} \quad z_1 = z \text{ is arbitrary,}$$

then conditions (1.24) and (1.25) become

$$|z - a| = a = |z| \tag{1.26}$$

and

$$z^2 + a^2 = az \tag{1.27}$$

respectively. If $z^2 - az + a^2 = 0$, then by the quadratic formula

$$z = \frac{a \pm \sqrt{a^2 - 4a^2}}{2} = \left(\frac{1}{2} \pm i\frac{\sqrt{3}}{2}\right) a.$$

Therefore,

$$|z - a| = \left|\frac{1}{2} \pm i\frac{\sqrt{3}}{2} - 1\right| a = a = |z|.$$

Conversely, if (1.26) holds, then since $a \geq 0$, we know

$$|z|^2 - \bar{z}a - za + a^2 = a^2 = |z|^2.$$

This means

$$|z|^2 - a(z + \bar{z}) = 0 \qquad \text{and} \qquad a^2 - a(z + \bar{z}) = 0.$$

If $a = 0$, then $|z| = 0$ and we certainly get (1.27). If $a > 0$, then the second equation gives

$$z + \bar{z} = 2\,\mathrm{Re}(z) = a.$$

The first equation then says $|z|^2 = a^2$ or

$$(\mathrm{Im}\, z)^2 = a^2 - \frac{a^2}{4} = \frac{3a^2}{4}.$$

This means

$$z = \frac{a}{2} \pm i\frac{\sqrt{3}}{2}a,$$

and we already know these values both satisfy (1.27).

We have established the equivalence in the special case; see Figure 1.12.

Figure 1.12: In the special case where one point is the origin and another lies along the positive real axis, there are precisely two choices for the remaining point z; the condition (1.25) becomes in this speical case a quadratic equation of which these two points are the roots.

Now, say $z_3 = 0$ (still) but z_2 and z_1 are arbitrary. Then the two conditions read

$$|z_1 - z_2| = |z_2| = |z_3| \tag{1.28}$$

and

$$z_1^2 + z_2^2 = z_1 z_2. \tag{1.29}$$

If (1.28) holds, then $a = z_2\bar{z}_2 \geq 0$ and $z = z_1\bar{z}_2$ satisfy

$$\begin{aligned} |z - a| &= |z_1\bar{z}_2 - z_2\bar{z}_2| = |z_1 - z_2||z_2| \\ &= |z_2\bar{z}_2| = a \\ &= |z_3\bar{z}_2| = |z|. \end{aligned}$$

This means (1.26) holds for $a = z_2\bar{z}_2$ and $z = z_1\bar{z}_2$. That is,

$$z_1^2\bar{z}_2^2 + z_2^2\bar{z}_2^2 = z_1 z_2 \bar{z}_2^2. \tag{1.30}$$

If $z_2 = 0$, then $z_1 = z_2 = 0$, and (1.29) clearly holds. If $z_2 \neq 0$, then (1.30) implies

$$z_1^2 + z_2^2 = z_1 z_2$$

which is (1.29).

Conversely, if (1.29) holds, then we can multiply the relation (1.29) by $\bar{z}_2^2$ to conclude

$$z^2 + a^2 = az$$

where $a = z_2\bar{z}_2$ and $z = z_1\bar{z}_2$. Since we know (1.29) is equivalent to (1.28), we know

$$|z - a| = a = |z| \qquad \text{or} \qquad |z_1\bar{z}_2 - z_2\bar{z}_2| = z_2\bar{z}_2 = |z_1\bar{z}_2|.$$

Again, if $z_2 = 0$, we can go back to (1.29) directly and see $z_1 = z_2 = 0$, so (1.28) holds. Otherwise, we get

$$|z_1 - z_2| = |z_2| = |z_1|$$

which is (1.28).

We have established the equivalence if one of the points is $0 \in \mathbb{C}$.

Finally, let's consider the general case: If (1.24) holds, then

$$\tilde{z}_1 = z_1 - z_3 \qquad \text{and} \qquad \tilde{z}_2 = z_2 - z_3 \tag{1.31}$$

satisfy

$$\begin{aligned}|\tilde{z}_1 - \tilde{z}_2| &= |z_1 - z_2| = |z_2 - z_3| = |z_3 - z_1| \\ &= |\tilde{z}_2| = |\tilde{z}_1|.\end{aligned}$$

Since (1.28) and (1.29) are equivalent, we know

$$\tilde{z}_1^2 + \tilde{z}_2^2 = \tilde{z}_1\tilde{z}_2. \tag{1.32}$$

Substituting from (1.31) this means

$$(z_1 - z_3)^2 + (z_2 - z_3)^2 = (z_1 - z_3)(z_2 - z_3)$$

or

$$z_1^2 - 2z_1z_3 + z_3^2 + z_2^2 - 2z_2z_3 + z_3^2 = z_1z_2 - z_1z_3 - z_2z_3 + z_3^2$$

or

$$z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_1z_3 + z_2z_3$$

which is (1.25).

We are almost done, and it looks like we will be successful. If (1.25) holds, we can reverse the last calculation to conclude

$$(z_1 - z_3)^2 + (z_2 - z_3)^2 = (z_1 - z_3)(z_2 - z_3).$$

This means, $\tilde{z}_1 = z_1 - z_3$ and $\tilde{z}_2 = z_2 - z_3$ satisfy (1.32) which looks suspiciously like (1.29)—which is equivalent to (1.28). Precisely, we can conclude

$$|\tilde{z}_1 - \tilde{z}_2| = |\tilde{z}_2| = |\tilde{z}_1|,$$

and

$$|z_1 - z_2| = |z_2 - z_3| = |z_3 - z_1|$$

which, of course, is (1.24). □

Exercise 3

If z and w are two vertices of a square, what are the possibilities for the other two vertices?

Here, as in the last problem, we proceed from a special case. If $w = 0$ and $z = a > 0$, then there are two possibilities: The additional vertices may be ai and $a + ai$ or $-ai$ and $a - ai$. That is, the two additional vertices are

$$\pm ai \qquad \text{and} \qquad a \pm ai$$

in general (for this special case).

Now, if w is still the origin, but z is arbitrary (and nonzero), then denoting the additional vertices by ζ and $\tilde{\zeta}$, we know 0, $a = |z|$, $\zeta z/|z|$ and $\tilde{\zeta} z/|z|$ are the vertices of a square, that is we may take (without loss of generality)

$$\zeta z/|z| = \pm|z|i \qquad \text{and} \qquad \tilde{\zeta} z/|z| = |z| \pm |z|i.$$

That is, we obtain in this case additional vertices

$$|z|i\bar{z} \qquad \text{and} \qquad \bar{z}|z|(1 \pm i).$$

Finally, in the general case, if z, w, ζ, and $\tilde{\zeta}$ are the vertices of a square, then so are $z - w$, 0, $\zeta - w$, and $\tilde{\zeta} - w$. It follows that (up to renaming ζ and $\tilde{\zeta}$) we must have

$$\zeta - w = |z - w|i\overline{z - w} \qquad \text{and} \qquad \tilde{\zeta} - w = \overline{z - w}|z - w|(1 \pm i).$$

The two additional vertices are given by

$$w + |z - w|i\overline{z - w} \qquad \text{and} \qquad w + \overline{z - w}|z - w|(1 \pm i).$$

Exercise 4

Given z_1, z_2 and z_3 any three noncolinear points in $\mathbb{C}$, find the center and radius of the circle that circumscribes the triangle with vertices z_1, z_2 and z_3. Put the expressions in symmetric form.

Let us try the strategy of the previous two problems. If one of the points, say z_3, were the origin and we had $z_2 = a > 0$ with $z_1 = z$ arbitrary, then the center should be the intersection of the perpendicular bisectors of the segments connecting the origin to a and z. The first of these lines is $\mathrm{Re}(\zeta) = a/2$. Thus, we look for a point $z/2 + tiz/2$ subject to the condition $\mathrm{Re}(z/2 + tiz/2) = a/2$. That is, $\mathrm{Re}(z) - t\,\mathrm{Im}(z) = a$. This means

$$z + \bar{z} + it(z - \bar{z}) = 2a \qquad \text{or} \qquad t = i\frac{z + \bar{z} - 2a}{z - \bar{z}}.$$

In this case, the center is

$$z_0 = \frac{z}{2} - \frac{z + \bar{z} - 2a}{z - \bar{z}} \cdot \frac{z}{2} = \frac{z}{2}\left(\frac{z - \bar{z} - (z + \bar{z} - 2a)}{z - \bar{z}}\right) = \frac{a - \bar{z}}{z - \bar{z}}\, z.$$

The radius is the modulus of this point:

$$r = \frac{|a-\bar{z}|}{|z-\bar{z}|}|z| = \frac{|a-z|}{|2\,\mathrm{Im}(z)|}|z| = \frac{|z^2-az|}{|2\,\mathrm{Im}(z)|}.$$

It's not entirely clear what Ahlfors means by "symmetric form." Presumably, he means the expressions should be symmetric in z_1, z_2 and z_3. He also talks about expressions which are invariant under switching z and $\bar{z}$. The expression we have here does not satisfy this kind of symmetry condition. The center may also be written the following forms

$$\begin{aligned} z_0 &= \frac{a\bar{z} - az - \bar{z}^2 + |z|^2}{2|z|^2} z = \frac{a|z|^2 - az^2 - \bar{z}|z|^2 + z|z|^2}{2|z|^2} \\ &= \frac{(a+z-\bar{z})|z|^2 - az^2}{2|z|^2} = \frac{1}{2}\left(a + z - \bar{z} - a\frac{z^2}{|z|^2}\right). \\ &= \frac{a(|z|^2 - z^2)}{2|z|^2} + i\,\mathrm{Im}\, z. \end{aligned}$$

We proceed to the case in which two vertices z_1 and z_2 are arbitrary, while the third $z_3 = 0$. By rotation we reduce to the previous case with

$$a = \frac{z_2\bar{z}_2}{|z_2|} = |z_2| \qquad \text{and} \qquad z = \frac{z_1\bar{z}_2}{|z_2|}.$$

The center is then given by

$$\begin{aligned} z_0 &= \frac{a-\bar{z}}{z-\bar{z}}\, z = \frac{|z_2|^2 - \bar{z}_1 z_2}{z_1\bar{z}_2 - \bar{z}_1 z_2} \cdot \frac{z_1\bar{z}_2}{|z_2|} \\ &= \frac{z_2\bar{z}_2 - \bar{z}_1 z_2}{z_1\bar{z}_2 - \bar{z}_1 z_2} \cdot \frac{z_1}{z_2} = \frac{\bar{z}_2 - \bar{z}_1}{z_1\bar{z}_2 - \bar{z}_1 z_2}\, z_1 \end{aligned}$$

(This needs more work.)

1.2.3 §2.2 n-th roots of a complex number (binomial equation)

We have seen that the complex square $\zeta(z) = z^2$ covers the complex plane with a right half plane so that any complex number ζ has a unique square root $z = \sqrt{\zeta}$ (a principal square root) with

$$-\frac{\pi}{2} < \arg(z) \leq \frac{\pi}{2}.$$

Figure 1.13: squaring the right half plane

Alternatively, we can use the upper half plane to get a principal square root $z = \sqrt{\zeta}$ with

$$0 \leq \arg(z) < \pi.$$

Of course, we could use any half plane, but these are the standard choices, and each has its advantages and disadvantages. The second choice, indexed from the positive real axis, is somewhat more standard for other integer powers where

Figure 1.14: squaring the upper half plane

$$z^n = |z|^n[\cos(n\theta) + i\sin(n\theta)] \qquad \text{for } \theta = \arg z \text{ and } n = 3, 4, 5, \ldots$$

The principal n-th root is

Figure 1.15: complex cube function

$$\zeta^{1/n} = |\zeta|^{1/n}\left(\cos\frac{\phi}{n} + i\,\sin\frac{\phi}{n}\right) \quad \text{where } \arg(\zeta) = \phi \text{ and } 0 \le \phi < 2\pi.$$

The other n-th roots of ζ are

$$\begin{aligned}
&|\zeta|^{1/n}\left[\cos\left(\frac{\phi}{n} + \frac{2\pi}{n}\right) + i\sin\left(\frac{\phi}{n} + \frac{2\pi}{n}\right)\right],\\
&|\zeta|^{1/n}\left[\cos\left(\frac{\phi}{n} + 2\cdot\frac{2\pi}{n}\right) + i\sin\left(\frac{\phi}{n} + 2\cdot\frac{2\pi}{n}\right)\right],\\
&\vdots\\
&|\zeta|^{1/n}\left[\cos\left(\frac{\phi}{n} + (n-1)\frac{2\pi}{n}\right) + i\sin\left(\frac{\phi}{n} + (n-1)\frac{2\pi}{n}\right)\right].
\end{aligned}$$

Taking $\zeta = 1$, we get the n-th roots of unity:

$$1,\ \omega = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n},\ \omega^2, \ldots \omega^{n-1}.$$

The first root ω is called the *principal n-th root of unity* or the *primitive n-th root of unity.*

Figure 1.16: third, fourth, and fifth roots of unity

1.2.4 §2.2 exercises

Exercise 1

Ahlfors mentions that expressions for the trigonometric functions of multiple angles, $\cos n\theta$ and $\sin n\theta$, may be found easily (in an "extremely simple way") in terms of $\cos\theta$ and $\sin\theta$. This exercise asks for some of those expressions. The basis for the calculation is the binomial formula:

$$(x+iy)^n = \sum_{j=0}^{n} \left(\begin{array}{c} n \\ j \end{array} \right) x^j (iy)^{n-j}$$

where

$$\left(\begin{array}{c} n \\ j \end{array} \right) = \frac{n!}{j!(n-j)!}$$

is the number of distinct subsets having j elements that are contained in a set with n (distinct) elements. The ordering of the j elements in a subset is irrelevant, and this number is called the *number of combinations of j elements taken from among n elements,* or *the number of combinations of j taken from n,* or simply the "combination of n taken j." The number is also called the *binomial coefficient.*

Alternatively, switching the order of the terms,

$$(x+iy)^n = \sum_{j=0}^{n} i^j \left(\begin{array}{c} n \\ j \end{array} \right) x^{n-j} y^j.$$

If n is even the sum has $n+1$ terms, and there will be one more term in which y has an even power than an odd one:

$$(x+iy)^n = \\ \sum_{k=1}^{n/2}(-1)^k \begin{pmatrix} n \\ 2k \end{pmatrix} x^{n-2k}y^{2k} + i\sum_{k=1}^{n/2-1}(-1)^k \begin{pmatrix} n \\ 2k+1 \end{pmatrix} x^{n-2k-1}y^{2k+1}.$$

If n is odd the terms split evenly:

$$(x+iy)^n = \\ \sum_{k=1}^{(n-1)/2}(-1)^k \begin{pmatrix} n \\ 2k \end{pmatrix} x^{n-2k}y^{2k} + i\sum_{k=1}^{(n-1)/2}(-1)^k \begin{pmatrix} n \\ 2k+1 \end{pmatrix} x^{n-2k-1}y^{2k+1}.$$

These expressions may be unified by using the "floor" function

$$\left\lfloor \frac{n}{2} \right\rfloor = \begin{cases} \frac{n}{2}, & n \text{ even} \\ \frac{n-1}{2}, & n \text{ odd.} \end{cases}$$

One finds in all cases

$$(x+iy)^n = \\ \sum_{k=1}^{\lfloor n/2 \rfloor}(-1)^k \begin{pmatrix} n \\ 2k \end{pmatrix} x^{n-2k}y^{2k} + i\sum_{k=1}^{\lfloor (n-1)/2 \rfloor}(-1)^k \begin{pmatrix} n \\ 2k+1 \end{pmatrix} x^{n-2k-1}y^{2k+1}.$$

Owing to the addition of arguments which results from multiplication of complex numbers, the special case where $x = \cos\theta$ and $y = \sin\theta$ leads to the relation

$$\cos n\theta + i\sin n\theta = (\cos\theta + i\sin\theta)^n = \sum_{j=0}^{n} \cos^j\theta(i\sin\theta)^{n-j}.$$

That is, we obtain general multiple angle formulas:

$$\cos n\theta = \sum_{k=0}^{\lfloor n/2 \rfloor}(-1)^k \begin{pmatrix} n \\ 2k \end{pmatrix} \cos^{n-2k}\theta \sin^{2k}\theta$$

and

$$\sin n\theta = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor}(-1)^k \begin{pmatrix} n \\ 2k+1 \end{pmatrix} \cos^{n-2k-1}\theta \sin^{2k+1}\theta.$$

Using these formulas we find

$$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta,$$

$$\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta,$$

and

$$\sin 5\theta = 5\cos^4\theta\sin\theta - 20\cos^2\theta\sin^3\theta + \sin^5\theta$$

as requested by Ahlfors. There are a number of interesting observations about the general formulas which may be made. Some of those may be relevant to the next problem.

Exercise 2

Here we are requested to "simplify" the expressions

$$1 + \cos\phi + \cos 2\phi + \cdots + \cos n\phi = \sum_{j-0}^{n} \cos j\phi$$

and

$$\sin\phi + \sin 2\phi + \cdots + \sin n\phi = \sum_{j-0}^{n} \sin j\phi$$

One obvious thing to do is write

$$\sum_{j-0}^{n} \cos j\phi + i\sum_{j-0}^{n} \sin j\phi = \sum_{j-0}^{n} (\cos\phi + i\sin\phi)^j.$$

This is a partial sum for a geometric series with ratio $r = \cos\phi + i\sin\phi$. The usual telescoping sum for such a series yields

$$\begin{aligned}\sum_{j-0}^{n} (\cos\phi + i\sin\phi)^j &= \frac{1 - r^{n+1}}{1 - r} \\ &= \frac{1 - \cos(n+1)\phi - i\sin(n+1)\phi}{1 - \cos\phi - i\sin\phi}.\end{aligned}$$

The last expression may be written as

$$\begin{aligned}\frac{1}{(1-\cos\phi)^2 + \sin^2\phi}[1 - \cos\phi - \cos(n+1)\phi + \cos(n+1)\phi\cos\phi + \sin(n+1)\phi\sin\phi \\ + i(\sin\phi - \sin(n+1)\phi - \sin\phi\cos(n+1)\phi + \sin(n+1)\phi\cos\phi)]\end{aligned}$$

or

$$\frac{1-\cos\phi+\cos n\phi-\cos(n+1)\phi+i(\sin\phi+\sin n\phi-\sin(n+1)\phi)}{2(1-\cos\phi)}.$$

Equating real and imaginary parts, we obtain the formulas

$$\sum_{j-0}^{n}\cos j\phi=\frac{1-\cos\phi+\cos n\phi-\cos(n+1)\phi}{2(1-\cos\phi)}$$

and

$$\sum_{j-0}^{n}\cos j\phi=\frac{\sin\phi+\sin n\theta-\sin(n+1)\phi}{2(1-\cos\phi)}.$$

These formulas are not so satisfying in several ways.

Exercise 4

We are to show

$$1+\omega^k+\omega^{2k}+\cdots+\omega^{(n-1)k}=\sum_{j=0}^{n-1}\omega^{jk}=0$$

where ω is the principal n-th root of unity

$$\omega=\cos\frac{2\pi}{n}+i\sin\frac{2\pi}{n}$$

and k is not a multiple of n. We can use here the formula for the sum of the first n terms of a geometric series:

$$\sum_{j=0}^{n-1}r^j=\frac{1-r^n}{1-r}. \tag{1.33}$$

The formula is valid if $r\neq 1$ and follows from the telescoping expansion

$$(1-r)\sum_{j=0}^{n-1}r^j=\sum_{j=0}^{n-1}r^j-\sum_{j=0}^{n-1}r^{j+1}=1-r^n.$$

If $r=1$, then the sum is just n.

In this case we take $r = \omega^k$, and

$$\sum_{j=0}^{n-1} \omega^{jk} = \frac{1-\omega^{kn}}{1-\omega^k} = \frac{1-(\omega^n)^k}{1-\omega^k} = \frac{1-(1)^k}{1-\omega^k} = 0$$

as long as $\omega^k \neq 1$. If $\omega^k = 1$, then k must be a multiple of n and the sum takes the value n. Perhaps this point deserves a little explanation. Evidently Ahlfors intends for k to be a non-negative (and probably positive) integer. If $0 < k < n$, we know $\omega^k \neq 1$; see Figure 1.16. If $k \geq n$, then we can use the division algorithm to write $k = \alpha n + \beta$ where α and β are (unique positive) integers with $0 \leq \beta < n$. Therefore,

$$\omega^k = \omega^{\alpha n + \beta} = \omega^\beta.$$

If $\omega^k = 1$, therefore, then $\beta = 0$ and $k = \alpha n$ is a multiple of n.

Exercise 5

Here we are presented with the alternating version of the sum in the last problem:

$$1 - \omega^k + \omega^{2k} - \cdots + (-1)^{n-1}\omega^{(n-1)k} = \sum_{j=0}^{n-1}(-1)^j\omega^{jk} = \sum_{j=0}^{n-1}(-\omega^k)^j.$$

Here, obviously, we have $r = -\omega^k$ and

$$\sum_{j=0}^{n-1}(-1)^j\omega^{jk} = \frac{1-(-\omega^k)^n}{1+\omega^k}$$

as long as $\omega^k \neq -1$. To complete the exercise, we should simplify the answer and examine the exceptional case(s) when the sum becomes n.

First of all, it is possible to have $\omega^k = -1$ only when n is even; see Figure 1.16. In the case where n is even, the discussion in the previous problem leads to the conclusion that $\omega^k = 0$ if and only if k is a multiple of $n/2$. This takes care of the exceptional cases.

If k is not an integer multiple of $n/2$, then

$$\sum_{j=0}^{n-1}(-1)^j\omega^{jk} = \frac{1-(-1)^n(\omega^n)^k}{1+\omega^k} = \frac{1-(-1)^n}{1+\omega^k} = \begin{cases} 0 & \text{if } n \text{ is even} \\ 2/(1+\omega^k) & \text{if } n \text{ is odd.} \end{cases}$$

Finally, when n is odd we get

$$\frac{2}{1+\omega^k} = \frac{2}{1+\cos\frac{2\pi k}{n}+\sin\frac{2\pi k}{n}} = \frac{1+\cos\frac{2\pi k}{n} - i\sin\frac{2\pi k}{n}}{1-\cos\frac{2\pi k}{n}}.$$

1.2.5 §2.2a The complex dot product and orthonormal bases

Most really good textbooks[2] are *really good* as much for what the author has left unsaid as for what he has written. The art of making the student work toward that which is unsaid seems to be the key. At the risk of ruining that experience for someone, I have decided to add many extra details, pictures, and relatively long-winded solutions to Ahlfors' problems. This is a short "additional" section of this nature.

In $\mathbb{R}^2$ one has for vectors $\mathbf{v} = (v_1, v_2)$ and $\mathbf{w} = (w_1, w_2)$ the dot product

$$\mathbf{v}\cdot\mathbf{w} = v_1w_1 + v_2w_2 = |\mathbf{v}||\mathbf{w}|\cos\theta$$

where θ is the angle between $\mathbf{v}$ and $\mathbf{w}$. We also have the orhonormality generalizing the standard basis $\{\mathbf{e}_1 = (1,0), \mathbf{e}_1 = (0,1)\}$ so that, whenever $\{\mathbf{u}_1, \mathbf{u}_2\}$ satisfies $|\mathbf{u}_1| = |\mathbf{u}_2| = 1$ and $\mathbf{u}_1\cdot\mathbf{u}_2 = 0$, any vector $\mathbf{v}$ may be written immediately as

$$\mathbf{v} = (\mathbf{v}\cdot\mathbf{u}_1)\mathbf{u}_1 + (\mathbf{v}\cdot\mathbf{u}_2)\mathbf{u}_2.$$

In the complex plane, naturally, these structures are still operative but look somewhat different. Let us take two complex numbers v and w which, as we know, are identified with two real vectors $\mathbf{v}$ and $\mathbf{w}$ respectively. Computing the products

$$vw = (x+iy)(\xi+i\eta) = x\xi - y\eta + i(x\eta + y\xi)$$

and

$$v\bar{w} = (x+iy)(\xi-i\eta) = x\xi + y\eta + i(y\xi - x\eta)$$

we see among the coefficients (real and imaginary parts) the dot product:

$$\mathbf{v}\cdot\mathbf{w} = \mathrm{Re}(v\bar{w}).$$

[2]Some examples of really good textbooks are Ahlfors' *Complex Analysis*, Royden's (or Rudin's) *Real* (*and Complex*) *Analysis*, and Lang's *Algebra*, though it is also Lang's peculiar charm of expressing things poorly and in a somewhat disorganized way that makes his book surpass, say, Hungerford's *Algebra*.

This is, perhaps, the main observation of this section. We can go on to observe

$$\mathbf{v} \cdot \mathbf{w} = \mathrm{Re}(v\bar{w}) = \mathrm{Re}(\bar{v}w) = \frac{1}{2}(v\bar{w} + \bar{v}w).$$

Also, given any nonzero $\zeta \in \mathbb{C}$, we can write $u = \zeta/|\zeta|$ and form what amounts to a right handed orthonormal basis $\{u, iu\}$. Given any complex number z, we should have

$$\begin{aligned} z &= \mathrm{Re}(z\bar{u})\, u + \mathrm{Re}(z\overline{iu})\, (iu) \\ &= [\mathrm{Re}(z\bar{u}) - i\,\mathrm{Re}(iz\bar{u})]u \\ &= [\mathrm{Re}(z\bar{u}) + i\,\mathrm{Im}(z\bar{u})]u. \end{aligned}$$

Indeed,

$$\mathrm{Re}(z\bar{u}) + i\,\mathrm{Im}(z\bar{u}) = \left[\frac{z\bar{u} + \bar{z}u}{2} + i\frac{z\bar{u} - \bar{z}u}{2i}\right] = z\bar{u}$$

so that $[\mathrm{Re}(z\bar{u}) + i\,\mathrm{Im}(z\bar{u})]u = z|u|^2 = z$. We will use the following summary of our discussion several times below:

Lemma 1 *If u is any complex number with unit modulus and z is any complex number, then there are unique real numbers α and β (representing the real coordinates of z in the "basis" $\{u, iu\}$) such that*

$$z = (\alpha + i\beta)u.$$

1.2.6 §2.3 Analytic geometry (and linear algebra)

We have already considered, in Exercise 4 of § 1.5, the equation of an ellipse. Also, Exercise 1 in the section currently under consideration involves the equation of a line and amounts to the reexpressing some concepts familiar from the linear algebra of $\mathbb{R}^2$ in complex notation. Conic sections are central to the study of analytic geometry in the plane, and it seems appropriate to focus on Exercise 2 of this section where we are asked to consider the conic sections. The most general ellipse (taking the Euclidean perspective) has a translated center (h, k) and a rotated major axis parallel to $(\cos\psi, \sin\psi)$ for some angle ψ. As we know, any vector $(x - h, y - k) \in \mathbb{R}^2$ may be written as

$$\begin{aligned} &[(x-h)\cos\psi + (y-k)\sin\psi](\cos\psi, \sin\psi) \\ &\qquad + [-(x-h)\sin\psi + (y-k)\cos\psi](-\sin\psi, \cos\psi) \end{aligned}$$

so that the equation of an ellipse becomes

$$\frac{[(x-h)\cos\psi + (y-k)\sin\psi]^2}{a^2} + \frac{[-(x-h)\sin\psi + (y-k)\cos\psi]^2}{b^2} = 1 \qquad (1.34)$$

where $a > b > 0$ and $c = \sqrt{a^2 - b^2}$ is the focal length. It may be verified that this equation is expressing the geometric relation

The sum of the distances from the point (x, y) to the focal points

$$(h, k) \pm c(\cos\psi, \sin\psi)$$

is $2a$.

That is,

$$|(x-h, y-k) + c(\cos\psi, \sin\psi)| + |(x-h, y-k) - c(\cos\psi, \sin\psi)| = 2a. \quad (1.35)$$

Nexercise 9 *Show the equation (1.34) for an ellipse can be written in the form (1.35).*

Figure 1.17: ellipses

As noted previously, the direction vector $(\cos\psi, \sin\psi) \sim u = \cos\psi + i\sin\psi$ has complementary basis vector iu in complex notation. More generally, any nonzero $u = \zeta/|\zeta| \in \mathbb{C}$ may be completed to a right handed "basis" so that every $z \in \mathbb{C}$ may be expressed as

$$z = \alpha u + \beta(iu).$$

In fact, $\alpha = \operatorname{Re}(z\bar{u})$ and $\beta = \operatorname{Im}(z\bar{u})$ with

$$z = \frac{z\bar{u} + \bar{z}u}{2}u + \frac{z\bar{u} - \bar{z}u}{2i}(iu).$$

Notably, the "dot product" of any two vectors z and ζ is $\operatorname{Re}(z\bar{\zeta}) = \operatorname{Re}(\bar{z}\zeta)$ and

$$\operatorname{Re}(\bar{z}iu) = -\operatorname{Im}(\bar{z}u) = \operatorname{Im}(z\bar{u}).$$

This observation allows us to express (1.35) in complex notation and write the $\mathbb{R}^2$ norms appearing in (1.35), in particular, as complex absolute values. It is, furthermore, natural to specify the focal points w_1 and w_2 directly and independently in the complex setting:

$$|z - w_1| + |z - w_2| = 2a \tag{1.36}$$

where

$$\frac{w_1 + w_2}{2} = h + ik \tag{1.37}$$

is the center, and w_1 and w_2 are the focal points with (say)

$$w_j = h+ik+(-1)^j c(\cos\psi + i\sin\psi) = h+ik+(-1)^j cu \qquad \text{for } j = 1, 2. \tag{1.38}$$

Starting directly with (1.36) we may write

$$\left| z - \frac{w_1 + w_2}{2} + \frac{w_2 - w_1}{2} \right| + \left| z - \frac{w_1 + w_2}{2} - \frac{w_2 - w_1}{2} \right| = 2a. \tag{1.39}$$

The relations (1.37) and (1.38) may then be recovered with

$$c = \frac{|w_2 - w_1|}{2} \qquad \text{and} \qquad \psi = \arg(w_2 - w_1).$$

For a nondegenerate ellipse it is required that $2a > |w_2 - w_1| > 0$. That

$$\{z \in \mathbb{C} : |z - w_1| + |z - w_2| = 2a\}$$

is a locus with the properties we associate to an ellipse follows by setting

$$u = \frac{w_2 - w_1}{|w_2 - w_1|} \quad \text{and writing} \quad z - (h + ik) = \alpha u + \beta(iu) = (\alpha + i\beta)u$$

as in Exercise 4 of § 1.5: Squaring (1.39)

$$\begin{aligned} 4a^2 &= (|(\alpha + i\beta)u + cu| + |(\alpha + i\beta)u - cu|)^2 \\ &= (\alpha + c)^2 + \beta^2 + 2|\alpha^2 - c^2 - \beta^2 + 2\alpha\beta\, i| + (\alpha - c)^2 + \beta^2 \\ &= 2(\alpha^2 + c^2 + \beta^2) + 2|\alpha^2 - c^2 - \beta^2 + 2\alpha\beta\, i|. \end{aligned}$$

That is,

$$\sqrt{(\alpha^2 - c^2 - \beta^2)^2 + 4\alpha^2\beta^2} = 2a^2 - (\alpha^2 + c^2 + \beta^2).$$

Squaring once again,

$$(\alpha^2 + \beta^2)^2 - 2c^2(\alpha^2 - \beta^2) + c^4 = (\alpha^2 + \beta^2 + c^2)^2 - 4a^2(\alpha^2 + \beta^2 + c^2) + 4a^4.$$

That is

$$-2c^2(\alpha^2 - \beta^2) = 2c^2(\alpha^2 + \beta^2) - 4a^2(\alpha^2 + \beta^2 + c^2) + 4a^4.$$

This becomes

$$(a^2 - c^2)\alpha^2 + a^2\beta^2 = a^2(a^2 - c^2).$$

That is,

$$\frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} = 1$$

where $b^2 = a^2 - c^2$ is the length of the minor semi-axis and we see the usual/standard form of the ellipse.

The parabola

The general parabola in $\mathbb{R}^2$ with vertex at (h, k) is given by

$$-(x - h)\sin\psi + (y - k)\cos\psi = a[(x - h)\cos\psi + (y - k)\sin\psi]^2 \tag{1.40}$$

where $a > 0$ and ψ is, again, an angle of rotation. We know also the parabola is the set of points equidistant from a fixed focal point and a fixed line (directrix). The parabola defined by (1.40) has focal point

$$(h, k) + \frac{1}{4a}(-\sin\psi, \cos\psi)$$

and directrix

$$L = \left\{ (h,k) - \frac{1}{4a}(-\sin\psi, \cos\psi) + t(\cos\psi, \sin\psi) : t \in \mathbb{R} \right\}.$$

Again, one can verify via a lengthy calculation that (1.40) expresses the condition

$$\begin{aligned} &\left| (x-h, y-k) - \frac{1}{4a}(-\sin\psi, \cos\psi) \right| \\ &\qquad = -(x-h)\sin\psi + (y-k)\cos\psi + \frac{1}{4a}. \end{aligned} \tag{1.41}$$

Letting w be the complex focal point and

$$L \sim \left\{ w - \frac{1}{4a}(-\sin\psi + i\cos\psi) + t(\cos\psi + i\sin\psi) : t \in \mathbb{R} \right\},$$

we can write (1.41) as

$$|z - w| = \mathrm{Re}[(z - z_0)\overline{iu}] = \mathrm{Im}[(z - z_0)\bar{u}] \tag{1.42}$$

where z_0 is any point in the directrix, for example,

$$z_0 = w - \frac{1}{2a}(-\sin\psi + i\cos\psi) = w - \frac{i}{2a}\, u$$

with $u = \cos\psi + i\sin\psi$ as usual.

The right side of (1.42) may be written as

$$\begin{aligned} \mathrm{Re}[(z - z_0)\overline{iu}] &= \frac{1}{2}\left[(z - z_0)(-i\bar{u}) + (\bar{z} - \bar{z}_0)iu\right] \\ &= \frac{i}{2}\left[(\bar{z} - \bar{z}_0)u - (z - z_0)\bar{u}\right]. \end{aligned}$$

Starting with (1.41) the vertex is

$$w - \frac{i}{4a}u = h + ik$$

so

$$z - z_0 = z - w + \frac{i}{2a}\, u = z - (h + ik) + \frac{i}{4a}u,$$

and (1.41) becomes

$$\begin{aligned}&\left|z-(h+ik)-\frac{i}{4a}u\right|\\&\qquad=\frac{i}{2}\left[\left(\bar{z}-(h-ik)-\frac{i}{4a}\,\bar{u}\right)u-\left(z-(h+ik)+\frac{i}{4a}\,u\right)\bar{u}\right].\end{aligned}$$

To put this relation in familiar form, we again write

$$z-(h+ik)=(\alpha+i\beta)u.$$

Then we have

$$\begin{aligned}\left|\alpha+i\beta-\frac{i}{4a}\right|&=\frac{i}{2}\left[\left(\alpha-i\beta-\frac{i}{4a}\right)-\left(\alpha+i\beta+\frac{i}{4a}\right)\right]\\&=\beta+\frac{1}{4a}.\end{aligned}$$

Squaring both sides we find

$$\alpha^2+\left(\beta-\frac{i}{4a}\right)^2=\left(\beta+\frac{i}{4a}\right)^2.$$

This simplifies to $\beta=a\alpha^2$.

The hyperbola

Finally, we consider the hyperbola. Like the ellipse there are two focal points w_1 and w_2. Here there is a constant difference

$$\big|\,|z-w_1|-|z-w_2|\,\big|=2a. \tag{1.43}$$

Following the computation for the ellipse (with the geometry of the hyperbola) we have center

$$h+ik=\frac{w_1+w_2}{2},$$

focal length

$$c=\frac{|w_2-w_1|}{2},$$

and rotation angle

$$\psi=\arg(w_2-w_1).$$

The nondegeneracy condition is $0 < a < c$. With these assignments we set

$$z - (h + ik) = (\alpha + i\beta)u$$

with

$$u = \cos\psi + i\sin\psi = \frac{w_2 - w_1}{|w_2 - w_1|}.$$

Noting that

$$z - w_1 = z - (h + ik) + cu = (\alpha + c + i\beta)u \quad \text{and} \quad z - w_2 = z - (h + ik) - cu = (\alpha - c + i\beta)u,$$

the equation (1.43) can be written as

$$\big|\,|\alpha + c + i\beta| - |\alpha - c + i\beta|\,\big| = 2a.$$

Squaring gives

$$\alpha^2 + \beta^2 + c^2 - |\alpha^2 - \beta^2 - c^2 + 2i\alpha\beta| = 2a^2.$$

Rearranging and squaring again,

$$(\alpha^2 - \beta^2 - c^2)^2 + 4\alpha^2\beta^2 = (\alpha^2 + \beta^2 + c^2 - 2a^2)^2$$

or

$$(\alpha^2 + \beta^2)^2 - 2c^2(\alpha^2 - \beta^2) = (\alpha^2 + \beta^2)^2 + 2(c^2 - 2a^2)(\alpha^2 + \beta^2) - 4a^2(c^2 - a^2).$$

This becomes

$$(c^2 - a^2)\alpha^2 - a^2\beta^2 = a^2(c^2 - a^2)$$

which is the standard form of a hyperbola:

$$\frac{\alpha^2}{a^2} - \frac{\beta^2}{b^2} = 1$$

where $b^2 = c^2 - a^2$.

1.2.7 §2.3 exercises

Exercise 1

We are asked to determine when the set

$$S = \{z : az + b\bar{z} + c = 0\}$$

is a line in the complex plane. This is essentially an exercise in expressing the linear algebra of 2×2 real matrices in complex form. We have already seen the equation $az + b\bar{z} + c = 0$ in Exercise I 1.4.4 where we found it equivalent to the real 2×2 system

$$\begin{cases} (a_1 + b_1)x - (a_2 - b_2)y = -c_1 \\ (a_2 + b_2)x + (a_1 - b_1)y = -c_2. \end{cases} \tag{1.44}$$

If we let a and b range over all complex numbers, we see this system is, in turn, equivalent to the general system

$$\begin{cases} a_{11}x + a_{12}y = -c_1 \\ a_{21}x + a_{22}y = -c_2 \end{cases}$$

owing to the fact that the matrices

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

are invertible, and the systems

$$\begin{cases} a_1 + b_1 = a_{11} \\ a_1 - b_1 = a_{22} \end{cases} \qquad \text{and} \qquad \begin{cases} a_2 - b_2 = -a_{12} \\ a_2 + b_2 = a_{21} \end{cases}$$

are always uniquely solvable. As in the previous problem, however, we again prefer to avoid using the real and imaginary parts a_1 ,a_2, b_1, and b_2 explicitly. Proceeding as in the "better solution" of the previous problem we take the conjugate of the complex equation and arrive at a 2×2 complex coefficient system for z and $\bar{z}$:

$$\begin{cases} az + b\bar{z} = -c \\ \bar{b}z + \bar{a}\bar{z} = -\bar{c}. \end{cases}$$

Now we take the opposite point of view noting that in order to have *more than one unique solution*, we need $|a| = |b|$. If $|a| = |b| = 0$, then we either get *no solution* (if $c \neq 0$) or $S = \mathbb{C}$ (if $c = 0$). We conclude

$$|a| = |b| \neq 0 \tag{1.45}$$

is necessary for S to be a line. Assuming (1.45), we can multiply the first equation by $\bar{a}$, the second by b, and subtract to get

$$\bar{a}c = b\bar{c}. \tag{1.46}$$

This condition must be satisfied in order to get any solution whatsoever. Thus, for S to be a line, both conditions (1.45) and (1.46) are necessary. They are also sufficient. To see this clearly and point out some other aspects of the problem, it is convenient to consider two separate cases.

If $c = 0$, then (1.46) is trivially satisfied, and it remains to show that $S = \{az + b\bar{z} = 0\}$ is a line when (1.45) holds. There are various ways to see this. One way is to use our observation from Exercise I 1.4.4 that the defining equation $az + b\bar{z} = 0$ is equivalent to a 2×2 linear system of real equations for x and y. In this case it is a homogeneous system, and we know the solution set is a vector space, meaning we can either have $S = \{0\}$, $S = \mathbb{C}$, or S is a line (through the origin). The assumption (1.45) rules out having a unique solution, so $S \neq \{0\}$. Thus, showing S is a line reduces to showing S is not the entire complex plane. We can do this by contradiction:

$$\begin{array}{rcrcl} b \in S & \Rightarrow & ab + b\bar{b} = 0 & \Rightarrow & a + \bar{b} = 0 \\ ib \in S & \Rightarrow & abi - b\bar{b}i = 0 & \Rightarrow & a - \bar{b} = 0. \end{array}$$

Adding the last relations gives $a = 0$ which is a contradiction. A similar contradiction may be reached by assuming $a, ia \in S$.

The foregoing argument is somewhat unsatisfying because we have used the real and imaginary parts a_1 ,a_2, b_1, and b_2 (in the previous problem) to see the equivalence with the real system (1.44). Notice, however, the argument showing $S \neq \mathbb{C}$ is essentially independent of (1.44).

Here is another explanation which rests on two observations about real linear combinations of complex numbers:

Lemma 2 *$S = \{az + b\bar{z} = 0\}$ is closed under real linear combinations, i.e., if $z \in S$ and $w \in S$, then $tz + sw \in S$ for $t, s \in \mathbb{R}$.*

This is, more or less, obvious since $az+b\bar{z}$ is linear in z and $\bar{z}$ (and conjugation does not change real numbers).

Lemma 3 *If $\alpha \neq 0$ and there is no $t \in \mathbb{R}$ such that $\beta \neq t\alpha$, then*

$$\{t\alpha + s\beta : t, s \in \mathbb{R}\} = \mathbb{C}.$$

Proof: Writing $\alpha = \alpha_1 + i\alpha_2$ and $\beta = \beta_1 + i\beta_2$, we claim

$$\alpha_1\beta_2 - \alpha_2\beta_1 \neq 0 \tag{1.47}$$

so that the system

$$\begin{cases} \alpha_1 t + \beta_1 s = x \\ \alpha_2 t + \beta_2 s = y \end{cases}$$

always has a unique solution. To see (1.47), assume $\alpha_1\beta_2 - \alpha_2\beta_1 = 0$. If $\alpha_1 \neq 0$, then $\beta_2 = \alpha_2\beta_1/\alpha_1$ and

$$\beta = \beta_1 + \frac{\alpha_2}{\alpha_1}\beta_1 i = \frac{\beta_1}{\alpha_1}\,\alpha$$

which is a contradiction. The alternative is $\alpha_2 \neq 0$. In this case, $\beta_1 = \alpha_1\beta_2/\alpha_2$ and we obtain a similar contradiction:

$$\beta = \frac{\alpha_1}{\alpha_2}\beta_2 + \beta_2 i = \frac{\beta_2}{\alpha_2}\,\alpha. \qquad \square$$

Using these two observations, which are essentially translations of facts about real linear combinations in $\mathbb{R}^2$ into complex notation, we may argue as follows: If $a = -b$, then the equation reduces to $z - \bar{z} = 0$, i.e., $\mathrm{Im}(z) = 0$. Thus, S is the real axis, which is a line. If $a + b \neq 0$, then

$$\alpha = \frac{b}{a+b}i$$

is a nonzero solution. In fact,

$$a\frac{b}{a+b}i + b\frac{\bar{b}}{\bar{a}+\bar{b}}(-i) = ib\frac{a(\bar{a}+\bar{b}) - \bar{b}(a+b)}{|a+b|^2} = 0.$$

Since we also know $S \neq \mathbb{C}$ by the argument using b and ib above, we can use Lemmas 2 and 3 to conclude every solution of $az + \bar{z} = 0$ is a real multiple of the nonzero solution, i.e.,

$$S = \left\{ t\frac{b}{a+b}i : t \in \mathbb{R} \right\}$$

which is a line in $\mathbb{C}$. This completes the case $c = 0$.

If $c \neq 0$, we have the additional condition (1.46) from which we know

$$b = \frac{\bar{a}c}{\bar{c}}.$$

It follows that

$$\begin{aligned} S &= \left\{ z : az + \frac{\bar{a}c}{\bar{c}}\bar{z} + c = 0 \right\} \\ &= \left\{ z : a\bar{c}z + \overline{a\bar{c}z} = -|c|^2 \right\} \\ &= \left\{ z : \mathrm{Re}(a\bar{c}z) = -\frac{|c|^2}{2} \right\} \\ &= \left\{ z : a\bar{c}z = -\frac{|c|^2}{2} + ti,\ t \in \mathbb{R} \right\} \\ &= \left\{ -\frac{c}{2a} + \frac{t}{a\bar{c}}i :\ t \in \mathbb{R} \right\}. \end{aligned}$$

This is a line. □

Exercise 2

We have covered this exercise rather thoroughly in the lecture above. Let us record the complex forms of each conic section:
ellipse:

$$|z - w_1| + |z - w_2| = 2a$$

where the focal points are w_1 and w_2 and the major semi-axis has length $2a$ with $2a > |w_2 - w_1| = 2c$. This can also be written in "symmetric" form as

$$\sqrt{(z - w_1)(\bar{z} - \bar{w}_1)} + \sqrt{(z - w_2)(\bar{z} - \bar{w}_2)} = 2a.$$

hyperbola:

$$\big|\,|z - w_1| - |z - w_2|\,\big| = 2a$$

where w_1 and w_2 are the focal points and $2a$ is the distance between the vertices with $2a < |w_2 - w_1| = 2c$. The Symmetric form is

$$\left|\sqrt{(z - w_1)(\bar{z} - \bar{w}_1)} - \sqrt{(z - w_2)(\bar{z} - \bar{w}_2)}\right| = 2a.$$

parabola:

$$|z - w| = \mathrm{Im}[(z - z_0)\bar{u}]$$

or

$$|z-w| = \frac{i}{2}\left[(\bar{z}-\bar{z}_0)u - (z-z_0)\bar{u}\right]$$

where w is the focal point and $t \mapsto z_0 + tu$ parameterizes the directrix. For a symmetric form we have

$$\sqrt{(z-w)(\bar{z}-\bar{w})} = \frac{i}{2}\left[(\bar{z}-\bar{z}_0)u - (z-z_0)\bar{u}\right]$$

Exercise 3

The diagonals of a parallelogram bisect each other.

To see this, we may introduce complex coordinates so that two of the sides of the parallelogram are v and w. The diagonals may then be represented by $z+w$ and $z-w$. The condition that these vectors intersect in their midpoints is

$$\frac{z+w}{2} = z + \frac{w-z}{2}.$$

This is obviously the case.

The diagonals of a rhombus are orthogonal.

Again, we can take two sides of the rhombus to be $z, w \in \mathbb{C}$; in this case, we have $|z| = |w|$. Orthogonality of the diagonals is then

$$\operatorname{Re}\left[(z+w)\overline{(z-w)}\right] = 0.$$

In fact, $(z+w)(\bar{z}-\bar{w}) = |z|^2 - z\bar{w} + w\bar{z} - |w|^2 = 2i\,\operatorname{Im}(w\bar{z}) \in i\mathbb{R}$.

1.2.8 §2.4 Stereographic projection and the Riemann sphere

In the previous sections, we have taken advantage of the identification

$$\mathbb{C} \sim \mathbb{R}^2 \qquad \text{by} \qquad a + ib \sim (a, b)$$

of $\mathbb{C}$ with $\mathbb{R}^2$. We have seen how it is useful to represent $\mathbb{C}$ by the plane owing to the geometry of angles and vectors in the plane. We now introduce another useful representation of $\mathbb{C}$. Consider

$$\mathbb{S}^2 = \{\mathbf{x} = (x, y, z) : x^2 + y^2 + z^2 = 1\}.$$

$\mathbb{S}^2$ is just the unit sphere in three-dimensional Euclidean space. We can identify, trivially, the plane with the x, y-plane of $\mathbb{R}^3$ as well:

$$\mathbb{R}^2 = \{(x, y, z) : z = 0\}.$$

For each $(x, y, 0) \in \mathbb{R}^2$, there is a unique point

$$(x, y, 0) + t[(0, 0, 1) - (x, y, 0)] \in \mathbb{S}^2 \backslash \{(0, 0, 1)\}.$$

In fact, the condition $(x, y, 0) + t[(0, 0, 1) - (x, y, 0)] \in \mathbb{S}^2$ implies

$$(x^2 + y^2 + 1)t^2 - 2(x^2 + y^2)t + x^2 + y^2 - 1 = 0$$

which has solutions

$$t = 1, \frac{x^2 + y^2 - 1}{x^2 + y^2 + 1}.$$

The value $t = 1$ gives the north pole $\mathbf{e}_3 = (0, 0, 1)$, and the other value gives the point in the sphere we seek:

$$(x, y, z) = \frac{(2x, 2y, x^2 + y^2 - 1)}{x^2 + y^2 + 1}.$$

Geometrically, it is clear that we have a one-to-one correspondence between

Figure 1.18: stereographic projection

$\mathbb{R}^2$ and $\mathbb{S}^2\backslash\{(0,0,1)\}$. In fact, given $(x,y,z) \in \mathbb{S}^2\backslash\{(0,0,1)\}$, we may also seek $t \in \mathbb{R}$ for which $(1-t)(0,0,1) + t(x,y,z) \in \mathbb{R}^2$, i.e., $1 - t + tz = 0$. Taking the unique value $t = 1/(1-z)$, we obtain a mapping

$$\sigma : \mathbb{S}^2\backslash\{(0,0,1)\} \to \mathbb{R}^2 \qquad \text{by} \qquad \sigma(x,y,z) = \frac{(x,y)}{1-z}.$$

This mapping is called **stereographic projection**. The reverse identification is called **inverse stereographic projection:**

$$\sigma^{-1} : \mathbb{R}^2 \to \mathbb{S}^2 \qquad \text{by} \qquad \sigma^{-1}(x,y) = \frac{(2x, 2y, x^2+y^2-1)}{x^2+y^2+1}.$$

The formulas associated with σ and σ^{-1} are somewhat complicated, but the geometry is intuitively clear, and the mapping has many interesting properties.

First, there is a natural concrete representative for "the point at infinity" in $\mathbb{S}^2$, namely the north pole $(0,0,1)$. The stereographic projection of a circle C on $\mathbb{S}^2$ passing through $(0,0,1)$, i.e., ∞, is the intersection of a non-horizontal plane through $(0,0,1)$ with $\mathbb{S}^2$. Thus, the projection of such a circle gives a straight line in $\mathbb{R}^2$, which also passes through "the point at infinity." Analytically,

$$C = \mathbb{S}^2 \cap \{[(x,y,z) - (0,0,1)] \cdot N = 0\}$$

for some unit normal $N = (N_1, N_2, N_3) \in \mathbb{R}\backslash\{(0,0,\pm 1)\}$. Therefore, if $(x,y,z) \in C\backslash\{(0,0,1)\}$, then

$$\sigma(x,y,z) = \frac{(x,y)}{1-z} \quad \text{satisfies} \quad N_1\left(\frac{x}{1-z}\right) + N_2\left(\frac{y}{1-z}\right) = N_3$$

since $N_1 x + N_2 y = N_3(1-z)$. Thus, the stereographic projection of C, denoted $\sigma(C) = \{\sigma(\mathbf{x}) : \mathbf{x} \in C\backslash\{(0,0,1)\}\}$, is a straight line in $\mathbb{R}^2$.

Also, *every* straight line in $\mathbb{R}^2$ is given by such a projection. Geometrically, just take the plane passing through the line and $(0,0,1)$; intersect it with $\mathbb{S}^2$. Analytically, we have

$$L = \{(x,y) : ax + by = c\},$$

and

$$\begin{aligned} \sigma^{-1}(L) &= \left\{ \frac{(2x, 2y, x^2+y^2-1)}{x^2+y^2+1} : ax + by = c \right\} \\ &= \{(x,y,z) : ax + by + cz = c\} \cap \mathbb{S}^2. \end{aligned}$$

The stereographic projection of the **equator** $\{(x, y, 0) : x^2 + y^2 = 1\}$ is itself and, therefore, a circle. It is somewhat less obvious that the stereographic projection of *any* circles in $\mathbb{S}^3 \backslash \{(0, 0, 1)\}$ is a circle. To see this, let us use complex notation in the plane so that

$$\sigma^{-1}(z) = \sigma^{-1}(x + iy) = \frac{(z + \bar{z}, (z - \bar{z})/i, |z|^2 - 1)}{|z|^2 + 1}.$$

When $\mathbb{C}$ is used to provide coordinates for $\mathbb{S}^2$, i.e., $\mathbb{S}^2 \backslash \{(0, 0, 1)\}$, the sphere is called the **Riemann sphere**, and this is a first example (and a very simple example) of a **Riemann surface**.

To distinquish coordinates we now use x_1, x_2, x_3-coordinates in $\mathbb{R}^3$. A circle C in $\mathbb{S}^2$ determined by

$$N_1 x_1 + N_2 x_2 + N_3 x_3 = d$$

does not contain $(0, 0, 1)$ when $N_3 \neq d$. Also, by the Cauchy-Schwarz inequality in $\mathbb{R}^3$, we have

$$|d| \leq |N \cdot \mathbf{x}| \leq |N||\mathbf{x}|$$

with equality only if N and $\mathbf{x}$ are parallel. In the case of equality, therefore, the circle C degenerates to a single point. We may therefore assume $d < 1$. The image, $\sigma(C)$, corresponds to points $z \in \mathbb{C}$ with

$$N_1(z + \bar{z}) - iN_2(z - \bar{z}) + N_3(|z|^2 - 1) = d(|z|^2 + 1)$$

or

$$(N_3 - d)(x^2 + y^2) + 2N_1 x + 2N_2 y - (N_3 + d) = 0.$$

This is the equation of a circle since

$$\frac{N_3 + d}{N_3 - d} + \frac{N_1^2}{(N_3 - d)^2} + \frac{N_2^2}{(N_3 - d)^2} = \frac{1 - d^2}{(N_3 - d)^2} > 0.$$

Again, *every* circle in $\mathbb{R}^2$ is the stereographic projection of a circle in $\mathbb{S}^3$: If

$$(x - h)^2 + (y - k)^2 = r^2,$$

then $x^2 + y^2 + 2hx + 2ky + k^2 + h^2 - r^2 = 0$, and we can solve the system

$$\begin{cases} N_3 - d = 1 \\ N_3 + d = r^2 - h^2 - k^2. \end{cases}$$

Therefore,

$$\begin{aligned}
&\sigma^{-1}\left(\{(x,y):(x-h)^2+(y-k)^2=r^2\}\right)\\
&\qquad\qquad = \{\sigma^{-1}(z) : h(z+\bar{z}) - ik(z-\bar{z}) + c(|z|^2-1) = d(|z|^2+1)\}\\
&\qquad\qquad = \{(x_1,x_2,x_3) : hx_1 + kx_2 + cx_2 = d\} \cap \mathbb{S}^2.
\end{aligned}$$

We make one more calculation. As we have seen above, we can think of points on $\mathbb{S}^2$ as given by coordinates in the complex plane. In a similar way, except in the opposite direction, we can also induce a nonstandard metric on the plane by considering the distance between points z and w to be the distance between the inverse stereographic projections of the points. This allows us to take limits, as points z and w tend to infinity in the plane, and measure how the points get closer to one another because they are both getting close to the point at infinity. Notice the Euclidean distance of such points tending to infinity may become very large (or may get small).

Ahlfors uses the euclidean distance between $\sigma^{-1}(z)$ and $\sigma^{-1}(w)$, but we will use the intrinsic distance in the sphere. The two metrics are essentially equivalent, but we find the latter more natural. Since $\sigma^{-1}(z)$ and $\sigma^{-1}(w)$ lie in the unit sphere in $\mathbb{R}^3$, the distance between them (on the sphere) is the angle between the vectors, which is given by the dot product:

$$\begin{aligned}
\cos\theta &= \sigma^{-1}(z)\cdot\sigma^{-1}(w)\\
&= \frac{(z+\bar{z})(w+\bar{w}) - (z-\bar{z})(w-\bar{w}) + (|z|^2-1)(|w|^2-1)}{(|z|^2+1)(|w|^2+1)}\\
&= \frac{2z\bar{w} + 2\bar{z}w + |z|^2|w|^2 - |z|^2 - |w|^2 + 1}{(|z|^2+1)(|w|^2+1)}\\
&= 1 - \frac{2|z-w|^2}{(|z|^2+1)(|w|^2+1)}.
\end{aligned}$$

Thus, our new metric on $\mathbb{C}$ is

$$d(z,w) = \cos^{-1}\left(1 - \frac{2|z-w|^2}{(|z|^2+1)(|w|^2+1)}\right). \tag{1.48}$$

Notice this metric allows us to compute the distance from a point $z \in \mathbb{C}$ to the point at infinity:

$$\begin{aligned}
d(z,\infty) &= \lim_{|w|\nearrow\infty} d(z,w) \\
&= \lim_{|w|\nearrow\infty} \cos^{-1}\left(1 - \frac{2[(|z|^2 - z\bar{w} - \bar{z}w)/|w|^2 + 1}{(|z|^2+1)(1+1/|w|^2)}\right) \\
&= \cos^{-1}\left(1 - \frac{2}{|z|^2+1}\right) \\
&= \cos^{-1}\left(\frac{|z|^2-1}{|z|^2+1}\right).
\end{aligned}$$

If one prefers Ahlfors' Euclidean distance, it may be found via the law of cosines:

$$\tilde{d}(z,w) = |\sigma^{-1}(z) - \sigma^{-1}(w)| = \sqrt{2 - 2\cos\theta}$$

or

$$\tilde{d}(z,w) = \frac{2|z-w|}{\sqrt{(|z|^2+1)(|w|^2+1)}}. \tag{1.49}$$

Taking the same kind of limit

$$\tilde{d}(z,\infty) = \frac{2}{\sqrt{|z|^2+1}}.$$

It may be a little premature for me to state (and students to understand) the following principle, but this is an appropriate place because it is fundamentally related to stereographic projection.

Metaprinciple of complex numbers 1 *The point at infinity is just like any other point in the complex plane. This is the case, because the north pole is just like any other point in the sphere.*

1.3 §2.4 Exercises

Exercise 1

Two complex numbers z and w correspond to diametrically opposite points in the Riemann sphere, i.e.,

$$\sigma^{-1}(z) = -\sigma^{-1}(w), \tag{1.50}$$

if and only if $z\bar{w} = -1$.

The condition (1.50) is equivalent to three component equations:

$$\frac{z+\bar{z}}{|z|^2+1} = -\frac{w+\bar{w}}{|w|^2+1}, \qquad \frac{z-\bar{z}}{|z|^2+1} = -\frac{w-\bar{w}}{|w|^2+1}, \tag{1.51}$$

and

$$\frac{|z|^2-1}{|z|^2+1} = -\frac{|w|^2-1}{|w|^2+1}. \tag{1.52}$$

If $w = 0$, the condition (1.50) becomes $\sigma^{-1}(z) = (0,0,1)$ which is never satisfied for $z \in \mathbb{C}$. Likewise, the condition $z\bar{w} = -1$ cannot be satisfied by any $z \in \mathbb{C}$ when $w = 0$. We may therefore exclude $w = 0$ from consideration.

The equation in (1.52) implies $2|z|^2|w|^2 = 2$ which we can write as $|z| = 1/|w|$. Adding the equations in (1.51) and making the substitution $|z| = 1/|w|$, we find

$$\frac{z}{1/|w|^2+1} = -\frac{w}{|w|^2+1}.$$

This implies

$$z\bar{w} = -(1/|w|^2+1)\frac{|w|^2}{|w|^2+1} = -1.$$

We have thus established one direction.

Conversely, if $z\bar{w} = -1$, then it is easy to check each of the equations in (1.51-1.52) by substituting $z = -1/\bar{w}$ on the left. In fact,

$$\frac{z+\bar{z}}{|z|^2+1} = \frac{-1/\bar{w}-1/w}{1/|w|^2+1} = \frac{-w-\bar{w}}{|w|^2+1},$$

$$\frac{z-\bar{z}}{|z|^2+1} = \frac{-1/\bar{w}+1/w}{1/|w|^2+1} = \frac{-w+\bar{w}}{|w|^2+1},$$

and

$$\frac{|z|^2-1}{|z|^2+1} = \frac{1/|w|^2-1}{1/|w|^2+1} = \frac{1-|w|^2}{|w|^2+1}.$$

Exercise 2

The cube inscribed in $\mathbb{S}^2$ with sides parallel to the coordinate planes is shown in Figure 1.19. The two vertices that project into the first quadrant are

$$\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) \qquad \text{and} \qquad \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right).$$

Figure 1.19: cube in the sphere

The stereographic projections of these points are

$$\frac{(1/\sqrt{3}, 1/\sqrt{3})}{1 - 1/\sqrt{3}} = \frac{(1,1)}{\sqrt{3}-1} \qquad \text{and} \qquad \frac{(1/\sqrt{3}, 1/\sqrt{3})}{1 - 1/\sqrt{3}} = \frac{(1,1)}{\sqrt{3}+1}.$$

The remaining vertices are symmetric with these, namely,

$$\frac{(-1,-1)}{\sqrt{3}-1}, \qquad \frac{(-1,-1)}{\sqrt{3}+1}, \qquad \frac{(\pm 1, \mp 1)}{\sqrt{3}-1}, \qquad \text{and} \qquad \frac{(\pm 1, \mp 1)}{\sqrt{3}+1}$$

as indicated in Figure 1.20.

It is interesting (and perhaps fun) to centrally project the edges of the cube onto the sphere as well, and then stereographically project them. In order to organize the coding for visualization it is convenient to group the edges into three groups of four: (1) edges parallel to the x_1-axis, (2) those parallel to the x_2-axis, and (3) the vertical edges. The linear edges in $\mathbb{R}^3$ may be parameterized as:

$$p_{jk}(t) = \frac{1}{\sqrt{3}}\left[t\mathbf{e}_j - (1-t)\mathbf{e}_j + \sqrt{2}(\cos(k\pi/4)\mathbf{e}_\ell + \sin(k\pi/4)\mathbf{e}_m)\right]$$

where $k = 1, 3, 5, 7$ and $j = 1, 2, 3$ (as the edge is parallel to the x_j-axis) with $\ell = \ell(j) = (j+1) \mod 3$ and $m = m(j) = (j+2) \mod 3$. The centrally

Figure 1.20: stereographically projected vertices (and edges)

projected edges are then given by

$$q_{jk} = \frac{p_{jk}}{|p_{jk}|}$$

where $|p_{jk}|$ is the three-dimensional Euclidean norm. Notice that, while the linear edges intersect in pairs at right angles, the spherical edges intersect at 120 degrees. Thus, the central projection of the inscribed cube is a network of great circles meeting in threes at 120 degrees. The fact that there are exactly ten such networks is attributed to Fred Almgren and Jean Taylor. They state this fact in their 1976 paper The geometry of soap films and soap bubbles which appeared in *Scientific American* Vol. 235, no. 1. Almgren and Taylor were interested in the possible point singularities in soap films/minimal surfaces. Any such singularity produces (by scaling out with the singularity at the origin in $\mathbb{R}^3$) a network of great circles on the sphere meeting at 120 degree angles. Of the ten possible networks, only three correspond to singularities in soap films, and the cube is not one of them, i.e., if you connect all the points on the edges of the cube back to the origin in $\mathbb{R}^3$, then the network of intersecting planes you get does not correspond to a soap film singularity. The tetrahedron in the next exercise, however, is one of the three. That is, connecting all the points on the edges of a centrally situated regular tetrahedron to the origin produces a *singular minimal cone*.

The stereographic projections of the centrally projected edges are shown on the right in Figure 1.20. Each of the horizontal edges projects to an arc of a circle. Can you guess how the radius of the projections of the top edges compares to those of the bottom edges? Stereographic projection is a **conformal map**, that is, it preserves angles. The angles between the curves in the projection should all be 120 degrees. This may not look to be the case at first, but it is the case.

Exercise 3

Now we are to look at the regular tetrahedron (inscribed in $\mathbb{S}^2$) in "general position."

It was relatively easy to write down the coordinates for the vertices of the cube in the previous problem. (It seemed pretty obvious that the one in the first octant should be a multiple of $(1, 1, 1)$, and the others were just reflections of the first one.) The situation seems rather less obvious here. Let's reverse our point of view to find the centrally located regular tetrahedron in $\mathbb{S}^2$ in at least one position. Let's say we put a vertex at the north pole, then that one will not have a stereographic projection, but we can assume the projections of the other three vertices lie somewhere along the third roots of unity. That is, we may assume one is $\alpha > 0$ while the other two are

$$\alpha\left(\cos\frac{2\pi}{3} \pm i\sin\frac{2\pi}{3}\right) = \alpha\left(-\frac{1}{2} \pm i\frac{\sqrt{3}}{2}\right).$$

If we take the inverse stereographic projection of α and one of these other points we obtain two points in the sphere:

$$\frac{(2\alpha, 0, \alpha^2 - 1)}{\alpha^2 + 1} \qquad \text{and} \qquad \frac{(-\alpha, \alpha\sqrt{3}, \alpha^2 - 1)}{\alpha^2 + 1}.$$

In order for these to be vertices of a regular tetrahedron (along with the north pole), their dot product should be the same as the dot product of one of them with $(0, 0, 1)$:

$$\frac{-2\alpha^2 + (\alpha^2 - 1)^2}{(\alpha^2 + 1)^2} = \frac{\alpha^2 - 1}{\alpha^2 + 1}.$$

From this equation we conclude $\alpha = 1/\sqrt{2}$. The three vertices on the bottom of the tetrahedron are

$$\mathbf{v}_0 = \left(\frac{2\sqrt{2}}{3}, 0, -\frac{1}{3}\right) \qquad \text{and} \qquad \mathbf{v}_\pm = \left(-\frac{\sqrt{2}}{3}, \pm\frac{\sqrt{6}}{3}, -\frac{1}{3}\right), \tag{1.53}$$

and their stereographic projections are

$$\frac{\sqrt{2}}{2}, \qquad \text{and} \qquad -\frac{\sqrt{2}}{4} \pm i\frac{\sqrt{6}}{4}.$$

Centrally projecting the edges onto the sphere and stereographically projecting the resulting Steiner network, we obtain the subset of $\mathbb{C}$ (also a planar Steiner network) indicated on the right in Figure 1.21. Notice that with the

Figure 1.21: stereographically projected vertices (and edges)

tetrahedron in this position, one of the vertices is at the north pole and three of the (centrally projected) edges stereographically project to rays approaching the point at infinity.

At this point, it is not too difficult to write down expressions for the stereographic projection of the vertices (and/or the centrally projected eges) of a tetrahedron in any particular position. Presumably this is what Ahlfors means by "general position." A general rotation $\mathbb{R}^3$ may be expressed as a composition of three coordinate rotations determined by matrix multiplication with

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi_3 & -\sin\phi_3 \\ 0 & \sin\phi_3 & \cos\phi_3 \end{pmatrix} \begin{pmatrix} \cos\phi_2 & 0 & -\sin\phi_2 \\ 0 & 1 & 0 \\ \sin\phi_2 & 0 & \cos\phi_2 \end{pmatrix} \begin{pmatrix} \cos\phi_1 & -\sin\phi_1 & 0 \\ \sin\phi_3 & \cos\phi_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Letting this transformation be denoted by ρ, the stereographic projection of the vertices in "general position" would be

$$\sigma \circ \rho(0,0,1), \quad \sigma \circ \rho(\mathbf{v}_0) \quad \text{and} \quad \sigma \circ \rho(\mathbf{v}_\pm)$$

where $\mathbf{v}_0$ and $\mathbf{v}_\pm$ are the original vertices defined in (1.53). This formula does not strike me as particularly enlightening. What I think would be more interesting is to consider some particular projections with the tetrahedron in some particular positions. Let us begin by rotating the tetrahedron we found above about the x_2-axis until the the vertex projecting to the real line

$$\left(\frac{2\sqrt{2}}{3}, 0, -\frac{1}{3}\right)$$

has moved to the equator. Noting the coordinates of this vector, we see the appropriate rotation is one by an angle θ satisfying

$$\cos\theta = \frac{2\sqrt{2}}{3} \qquad \text{and} \qquad \sin\theta = \frac{1}{3}.$$

Thus, letting ρ denote the rotation about the x_2 axis associated with the matrix

$$\frac{1}{3}\begin{pmatrix} 2\sqrt{2} & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 2\sqrt{2} \end{pmatrix}$$

we find rotated vertices:

$$\rho(0,0,1) = \frac{(-1,0,\sqrt{2})}{3}, \quad rho(\mathbf{v}_0) = (0,0,1), \quad \text{and} \quad \rho(\mathbf{v}_\pm) = \frac{(-1,\pm\sqrt{6},\sqrt{2})}{3}.$$

All four vertices now project into $\mathbb{C}$ as indicated on the right in Figure 1.22, though one of the centrally projected edges still passes through the north pole, so one side passes through the point at infinity. We could not execute a rotation about the x_1-axis and see various stereographic projections with all the normally projected edges stereographically projecting into $\mathbb{C}$.

Instead we will consider rotating in the opposite direction about the x_2-axis until the original point $\mathbf{v}_0$ moves to the south pole $(0,0,-1)$. This is equivalent to using the negatives of our original vertices:

$$(0,0,-1), \qquad -\mathbf{v}_0 = \left(-\frac{2\sqrt{2}}{3}, 0, \frac{1}{3}\right) \qquad \text{and} \qquad -\mathbf{v}_\pm = \left(\frac{\sqrt{2}}{3}, \mp\frac{\sqrt{6}}{3}, \frac{1}{3}\right).$$

This is an "upside down" tetrahedron. It has an entirely nonsingular projection as shown in Figure 1.23.

Figure 1.22: stereographically projected vertices (and edges)

Figure 1.23: upside down tetrahedron

Exercise 5

Let $z_0 = h + ik$ be the center of a circle C in $\mathbb{C}$ of radius r. We are asked to find the radius of the inverse stereographic projection $\sigma^{-1}(C)$ of this circle. The question can have two meanings depending on whether one wants the intrinsic radius, with $\sigma^{-1}(C)$ considered as a circle in $\mathbb{S}^2$, or the extrinsic radius in $\mathbb{R}^3$. Both answers are easy to find from the induced distance formulas above.

Rotations of the complex plane (centered at the origin) correspond to rotations of the sphere about the z_3 axis and, thus, do not change the radius of either circle. This reduces the problem to consideration of a circle with center $|z_0|$. Two diametrically opposite points in $\sigma^{-1}(C)$ are given by

$$\sigma^{-1}(|z_0| - r) = \frac{(2(|z_0| - r), 0, (|z_0| - r)^2 - 1)}{(|z_0| - r)^2 + 1}$$

and

$$\sigma^{-1}(|z_0|+r) = \frac{(2(|z_0|+r), 0, (|z_0|+r)^2 - 1)}{(|z_0|+r)^2+1}.$$

Both the intrinsic distance and the extrinsic distance between two such points is calculated above. The intrinsic distance is calculated using (1.48):

$$d(|z_0|-r, |z_0|+r) = \cos^{-1}\left(1 - \frac{8r^2}{[(|z_0|-r)^2+1][(|z_1|+r)^2+1]}\right).$$

Therefore, the intrinsic radius of the circle on the sphere is

$$\frac{1}{2}\cos^{-1}\left(1 - \frac{8r^2}{[(|z_0|-r)^2+1][(|z_1|+r)^2+1]}\right).$$

Using the half angle formula for cosine, namely,

$$\cos\frac{\theta}{2} = \sqrt{\frac{1+\cos\theta}{2}},$$

this can also be written as

$$\cos^{-1}\sqrt{1 - \frac{4r^2}{[(|z_0|-r)^2+1][(|z_1|+r)^2+1]}}.$$

Similarly, the extrinsic distance is given in (1.49):

$$\tilde{d}(|z_0|-r, |z_0|+r) = \frac{4r}{\sqrt{[(|z_0|-r)^2+1][(|z_1|+r)^2+1]}},$$

so the extrinsic radius is

$$\frac{2r}{\sqrt{[(|z_0|-r)^2+1][(|z_1|+r)^2+1]}}.$$

I don't see any substantial simplification of this formula.

Chapter 2

Chapter II

2.1 Lecture 3: § 1.1-2; differentiability of complex functions

2.1.1 functions, continuity, and differentiability

Let $\mathcal{U}$ be an open subset of $\mathbb{C}$, and consider $f : \mathcal{U} \to \mathbb{C}$.

Figure 2.1: a complex mapping on an open domain in $\mathbb{C}$

(To be open in $\mathbb{C}$ means that for each $z \in \mathcal{U}$, there is some $r > 0$ such that $B_r(z) = \{\zeta : |\zeta - z| < r\} \subset \mathcal{U}$.)

Continuity at z means for any $\epsilon > 0$, there is some $\delta > 0$ such that

$$|\zeta - z| < \delta \quad \Rightarrow \quad |f(\zeta) - f(z)| < \epsilon.$$

Equivalently,

$$\lim_{\zeta \to z} f(\zeta) = f(z).$$

(Generally, $\lim_{\zeta \to z} f(z) = L \in \mathbb{C}$ means for any $\epsilon > 0$, there is some $\delta > 0$ such that

$$0 < |\zeta - z| < \delta \quad \Rightarrow \quad |f(\zeta) - L| < \epsilon.)$$

Remember $\mathbb{C} \sim \mathbb{R}^2$, and for everything we have said so far we can assume f is like a mapping (vector field) on $\mathbb{R}^2$. In principle $f : \mathcal{U} \to \mathbb{C}$ is equivalent to a mapping $\phi : \mathcal{U} \to \mathbb{R}^2$ (with $\mathcal{U} \subset \mathbb{R}^2$) and

$$\begin{cases} f(z) = f(z) + v(z)\, i \\ \phi(x, y) = u(x, y) + v(x, y)\, i. \end{cases}$$

For ϕ, however, **differentiability** means the existence of the four partial derivatives

$$D\phi = \begin{pmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ & \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{pmatrix}.$$

There is also a somewhat stricter notion involving *linear approximation*, but if we assume the four partial derivatives are continuous on an open set, i.e., $\phi \in C^1(\mathcal{U})$, then the two notions are equivalent. In any case, **differentiability** (complex differentiability) for f is something quite different:

$$f'(z) = \lim_{\zeta \to z} \frac{f(\zeta) - f(z)}{\zeta - z}. \tag{2.1}$$

That is to say, f is **differentiable** if the limit of the **difference quotient** in (2.1) exists and has a complex limit $f'(z)$.

Nexercise 10 *Show that if the four partials of $D\phi$ are continuous on $\mathcal{U}$, then the mapping $\phi : \mathcal{U} \to \mathbb{R}^2$ (with $\mathcal{U} \subset \mathbb{R}^2$) is continuous.*

Lemma 4 *If $f : \mathcal{U} \to \mathbb{C}$ is differentiable at z, then f is continuous at z.*

Proof:

$$\begin{aligned} \lim_{\zeta \to z} |f(\zeta) - f(z)| &= \lim_{\zeta \to z} \frac{|f(\zeta) - f(z)|}{|\zeta - z|} \cdot |\zeta - z| \\ &= |f'(z)| \cdot 0 \qquad \Box. \end{aligned}$$

Notice this lemma parallels the real case. Now we will start to see some differences.

2.1.2 Cauchy-Riemann equations

Let us calculate the complex derivative by letting ζ tend to z in the difference quotient in two different ways.

$$\begin{aligned} f'(z) &= \lim_{h\searrow 0} \frac{f(z+h)-f(z)}{h} \\ &= \lim_{h\searrow 0} \frac{u(x+h,y)+iv(x+h,y)-u(x,y)-iv(x,y)}{h} \\ &= \lim_{h\searrow 0} \left[\frac{u(x+h,y)-u(x,y)}{h} + i\frac{v(x+h,y)-v(x,y)}{h}\right] \\ &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}. \end{aligned}$$

Notice the existence of the complex derivative f' implies the existence of the partial derivatives here. Similarly, we find

$$\begin{aligned} f'(z) &= \lim_{h\searrow 0} \frac{f(z+hi)-f(z)}{hi} \\ &= \lim_{h\searrow 0} \left[-i\frac{u(x,y+h)-u(x,y)}{h} + \frac{v(x,y+h)-v(x,y)}{h}\right] \\ &= \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}. \end{aligned}$$

We have established the following fundamental result:

Theorem 1 *Differentiability of $f : \mathcal{U} \to \mathbb{C}$ implies differentiability of $\phi : \mathcal{U} \to \mathbb{R}^2$ (with $\mathcal{U} \subset \mathbb{R}^2$) and $\phi(x,y) = (u,v)$, i.e., the partial derivatives*

$$\frac{\partial u}{\partial x},\ \frac{\partial u}{\partial y},\ \frac{\partial v}{\partial x},\ \text{and}\ \frac{\partial v}{\partial y} \quad \textit{all exist.}$$

Moreover, if f' exists, then

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \tag{2.2}$$

These are called the **Cauchy-Riemann Equations**.

Nexercise 11 *Give an example of a complex valued function $f : \mathcal{U} \to \mathbb{C}$ which is* **not** *differentiable, but the associated mapping $\phi : \mathcal{U} \to \mathbb{R}^2$ (with $\mathcal{U} \subset \mathbb{R}^2$) is differentiable.*

2.1.3 regularity and harmonic functions

Here is a result that shows even more clearly how different complex differentiable functions are from any real counterpart.

Theorem 2 (the regularity theorem) *If $f' : \mathcal{U} \to \mathbb{C}$ exists at each point in $\mathcal{U}$, then the real and imaginary parts u and v are in $C^\infty(\mathcal{U})$, i.e., all partial derivatives of all orders exist and are continuous. Consequently, all higher order complex derivatives of f also exist.*

Nexercise 12 *Give an example of a real mapping $\phi : \mathcal{U} \to \mathbb{R}^2$ (with $\mathcal{U} \subset \mathbb{R}^2$) which is continuously differentiable but not twice differentiable.*

Definition/terminology 1 *A function $f : \mathcal{U} \to \mathbb{C}$ which is differentiable on an open set $\mathcal{U}$ (as opposed to the associated mapping $\phi : \mathcal{U} \to \mathbb{R}^2$) is called* **analytic** *or* **holomorphic**.

Assuming the regularity theorem above, we can use the Cauchy-Riemann equations to prove the following important result.

Theorem 3 (harmonic conjugates) *If $f = u + iv$ is analytic, then the real an imaginary parts are harmonic, i.e.,*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{and} \quad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0. \tag{2.3}$$

Proof:

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}\right) = \frac{\partial}{\partial y}\left(-\frac{\partial u}{\partial y}\right) = -\frac{\partial^2 u}{\partial y^2}.$$

We have used the Cauchy-Riemann equations and the theorem that if a real function $v \in C^2(\mathcal{U})$, then the mixed partials of v are equal. A similar calculation using the Cauchy-Riemann equations and that the mixed partials of u are equal shows that v is harmonic. □

Example 1 $f(z) = z = x + iy$.

$$u(x,y) = x \quad \text{and} \quad v(x,y) = y \qquad \textit{are harmonic.}$$

$f(z) = z^2 = (x + iy)^2 = x^2 - y^2 + 2xy\, i$.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2 - 2 = 0 \quad \text{and} \quad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 + 0 = 0.$$

Note: Two harmonic functions are not necessarily related as the real and imaginary parts of a complex analytic function. If we take

$$u(x,y) = x \quad \text{and} \quad v(x,y) = 2xy,$$

Then the function $f(z) = \mathrm{Re}(z) + 2Re(z)\,\mathrm{Im}(z)\,i$ *is not analytic. (Why?)*

Definition 2 *If u and v are harmonic (real functions of two real variables on an open set $\mathcal{U} \subset \mathbb{R}^2$), then u and v are called* **harmonic conjugates** *if they satisfy the Cauchy-Riemann equations:*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Example 2 *$u(x,y) = x$ and $v(x,y) = 2xy$ are not harmonic conjugates.*

$$\frac{\partial u}{\partial x} = 1 \neq \frac{\partial v}{\partial y} = 2x.$$

Theorem 4 (harmonic conjugates) *If u and v are continuously differentiable on $\mathcal{U}$ and satisfy the Cauchy-Riemann equations, then*

$$f = u + iv \quad \textit{is analytic.}$$

Proof:

$$u(x+h, y+k) = u(x,y) + \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) \cdot (h,k) + \circ\left(\sqrt{h^2+k^2}\right)$$

where

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial x}(x,y) \quad \text{and} \quad \frac{\partial u}{\partial y} = \frac{\partial u}{\partial y}(x,y).$$

The notation $g = \circ\left(\sqrt{h^2+k^2}\right)$ denotes a function $g = g(h,k)$ for which

$$\lim_{(h,k)\to(0,0)} \frac{g(h,k)}{\sqrt{h^2+k^2}} = 0.$$

Thus, we are saying

$$\lim_{(h,k)\to(0,0)} \frac{u(x+h,y+k) - u(x,y) - \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right)\cdot(h,k)}{\sqrt{h^2+k^2}} = 0.$$

Similarly,

$$v(x+h,y+k) = v(x,y) + \left(\frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}\right) \cdot (h,k) + \circ\left(\sqrt{h^2+k^2}\right).$$

Therefore,

$$\frac{f(z+h+ik)-f(z)}{h+ik} = \frac{\left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) \cdot (h,k) + i\left(\frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}\right) \cdot (h,k) + \circ\left(\sqrt{h^2+k^2}\right)}{h+ik}.$$

Expanding the inner products appearing in the numerator and using the Cauchy-Riemann equations, we find

$$\frac{\partial u}{\partial x} h - \frac{\partial v}{\partial x} k + i\left(\frac{\partial v}{\partial x} h + \frac{\partial u}{\partial x} k\right) = \left(\frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x}\right)(h+ik).$$

Therefore,

$$\frac{f(z+h+ik)-f(z)}{h+ik} = \left(\frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x}\right) + \frac{\circ\left(\sqrt{h^2+k^2}\right)}{h+ik}.$$

Finally,

$$\lim_{h+ik\to 0}\left|\frac{\circ\left(\sqrt{h^2+k^2}\right)}{h+ik}\right| = \lim_{(h,k)\to(0,0)} \frac{\left|\circ\left(\sqrt{h^2+k^2}\right)\right|}{\sqrt{h^2+k^2}} = 0.$$

We conclude that $f'(z)$ exists and

$$f'(z) = \frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x}$$

as we already knew. □

In summary, we can write the following:

Theorem 5 *f is analytic in $\mathcal{U}$ if and only if $f = u+iv$ with u and v ($C^1(\mathcal{U})$) harmonic conjugates.*

2.2 §1.2 exercises

Exercise 3

$u = ax^3 + bx^2y + cxy^2 + dy^3$.

$$\frac{\partial u}{\partial x} = 3ax^2 + 2bxy + cy^2 \quad \text{and} \quad \frac{\partial u}{\partial y} = bx^2 + 2cxy + 3dy^2.$$

$$\frac{\partial^2 u}{\partial x^2} = 6ax + 2by \quad \text{and} \quad \frac{\partial^2 u}{\partial y^2} = 2cx + 6dy.$$

Thus, for u to be harmonic, we need $c = -3a$ and $b = -3d$. Assuming this is the case, any harmonic conjugate v must satisfy

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 3ax^2 - 6dxy - 3ay^2 \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = 3dx^2 + 6axy - 3dy^2.$$

Integrating the first equation, we find $v = 3ax^2y - 3dxy^2 - ay^3 + c(x)$ where $c(x)$ now represents some function of x. Differentiating with respect to x, we must also have

$$c'(x) = 3dx^2 \quad \text{or} \quad c(x) = dx^3 + c$$

where c is a constant. Therefore, the most general form of v is

$$v(x) = dx^3 + 3ax^2y - 3dxy^2 - ay^3 + c.$$

Exercise 4

Show that if $|f| = r$ (constant) and f is analytic, then f must be constant.

This is a very typical complex analysis assertion very analogous to the facts in the text that if an analytic function is real (or purely imaginary) then it must be constant. Let us review the argument for these other facts: If f is purely real, then $f = u + iv = u$ and taking the limit $\lim_{\zeta\to z}[f(\zeta) - f(z)]/(\zeta - z)$ with $\zeta = z + h$ and $h \in \mathbb{R}$, we find

$$f'(z) = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} i = \frac{\partial u}{\partial x} \in \mathbb{R}.$$

On the other hand, taking the same limit with $\zeta = z + ih$ and $h \in \mathbb{R}$, we get

$$f'(z) = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = -i\frac{\partial u}{\partial y} \in i\mathbb{R}.$$

Thus, $f'(z) \in \mathbb{R} \cap i\mathbb{R} = \{0\}$ and $f \equiv c$ (constant). This is Ahlfors' elementary proof. From the perspective of the Cauchy-Riemann equations, one can conclude immediately from the fact that $v \equiv 0$ that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \equiv 0 \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \equiv 0,$$

and hence $f = u \equiv c$ (constant). This problem, in any case, seems to require a little more work. Differentiating the condition $|f|^2 = u^2 + v^2 = r^2$ with respect to x and y we get

$$u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x} = 0 \quad \text{and} \quad u\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y} = 0.$$

On the other hand, we can use the Cauchy-Riemann equations to get

$$\begin{aligned}
\bar{f}f' &= (u - iv)\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) \\
&= u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x} + i\left(u\frac{\partial v}{\partial x} - v\frac{\partial u}{\partial x}\right) \\
&= 0 + i\left(-u\frac{\partial u}{\partial y} - v\frac{\partial v}{\partial y}\right) \\
&= 0.
\end{aligned}$$

With this in mind, let $A = \{z : f'(z) \neq 0\}$. This is an open set. If $z_0 \in A$, then there is some ball $B_r(z_0) \subset A$. From the fact that $\bar{f}f' = 0$, it follows that $f \equiv 0$ on A. But this would mean $f' \equiv 0$ on A which is is a contradiction.

We conclude that $f' \equiv 0$. That is,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 0 \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = 0.$$

Thus, $f \equiv c$ (constant).

Exercise 5

Show $f = u + iv$ is analytic if and only if

$$g(z) = \overline{f(\bar{z})} = u(x, -y) - iv(x, -y)$$

is analytic. I'm skeptical of the "Cauchy-Riemann equations" solution to this problem offered by Dustin Smith. It shouldn't be the case that $f(\bar{z})$ is analytic. One should be able to give a solution using the Cauchy-Riemann equations, but this requires $u, v \in C^1$. A more economical proof seems to be the one using the definition of differentiability directly.

The difference quotient for g at $x + iy$ is given by

$$\begin{aligned}\frac{g(z+h+ik)-g(z)}{h+ik} &= \frac{u(x+h,-y-k)-iv(x+h,-y-k)-u(x,-y)+iv(x,-y)}{h+ik}\\ &= \frac{u(x+h,-y-k)-u(x,-y)-i[v(x+h,-y-k)-v(x,-y)]}{h+ik}.\end{aligned}$$

Notice the functions f and g are not necessarily defined on the same domain. It may be assumed, however, that if z is a point where g is defined, then f should be defined (and differentiable) at $\bar{z} = x - iy$. If we assume differentiability for f at $\bar{z}$, then we have

$$\lim_{h-ik\to 0}\frac{f(\bar{z}+h-ik)-f(\bar{z})}{h-ik} = \frac{\partial u}{\partial x}(x,-y) + i\,\frac{\partial v}{\partial x}(x,-y).$$

Multiplying the right side of this equation by $(h - ik)/(h - ik)$ we get

$$\frac{1}{h-ik}\left[h\frac{\partial u}{\partial x} + k\frac{\partial v}{\partial x} + \left(h\frac{\partial v}{\partial x} - k\frac{\partial u}{\partial x}\right) i\right].$$

where

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial x}(x,-y) \quad \text{and} \quad \frac{\partial v}{\partial x} = \frac{\partial v}{\partial x}(x,-y).$$

Subtracting the right side limit from the left, we obtain terms

$$u(x+h,-y-k)-u(x,-y)-\left(h\frac{\partial u}{\partial x}+k\frac{\partial v}{\partial x}\right)$$

and

$$i\left[v(x+h,-y-k)-v(x,-y)-\left(h\frac{\partial v}{\partial x}-k\frac{\partial u}{\partial x}\right)\right]$$

the sum of which is $\circ(h - ik)$ as $h - ik \to 0$. That is,

$$\begin{aligned}\lim_{h-ik\to 0}\frac{1}{h-ik}\Bigg\{&u(x+h,-y-k)-u(x,-y)-\left(h\frac{\partial u}{\partial x}+k\frac{\partial v}{\partial x}\right)\\ &+ i\left[v(x+h,-y-k)-v(x,-y)-\left(h\frac{\partial v}{\partial x}-k\frac{\partial u}{\partial x}\right)\right]\Bigg\} = 0.\end{aligned}$$

Equivalently, the limit of the modulus of this complex number is zero. And since the modulus of a quotient of complex numbers is the quotient of the moduli of those numbers, we can take the conjugate of the numerator and denominator and get the same limit. That is,

$$\lim_{h-ik\to 0}\frac{1}{h+ik}\left\{u(x+h,-y-k)-u(x,-y)-\left(h\frac{\partial u}{\partial x}+k\frac{\partial v}{\partial x}\right)\right.$$
$$\left.-\,i\left[v(x+h,-y-k)-v(x,-y)-\left(h\frac{\partial v}{\partial x}-k\frac{\partial u}{\partial x}\right)\right]\right\}=0.$$

Noting that

$$(h+ik)\left(\frac{\partial u}{\partial x}-i\frac{\partial v}{\partial x}\right)=h\frac{\partial u}{\partial x}+k\frac{\partial v}{\partial x}+i\left(k\frac{\partial u}{\partial x}-h\frac{\partial v}{\partial x}\right),$$

we see the limit above is exactly the limit

$$\lim_{h-ik\to 0}\frac{1}{h+ik}\left\{g(z+h+ik)-g(z)-(h+ik)\left(\frac{\partial u}{\partial x}-i\frac{\partial v}{\partial x}\right)\right\}$$

involving the difference quotient for g. Taking the limit as $h-ik\to 0$ is the same as taking the limit as $(h,k)\to(0,0)$ or as $h+ik\to 0$. We conclude g' exists and

$$g'(z)=\overline{f'(\bar{z})}.$$

We have shown that when f is analytic, then g is also analytic on the appropriate domain. Dustin Smith rightly points out that this computation essentially completes the problem since

$$f(z)=\overline{g(\bar{z})}.$$

That is, when g is analytic, then f is analytic by the same argument.

2.3 Lecture 4: § 1.3-4 polynomials and rational functions

2.3.1 polynomials

A complex polynomial of **degree** n is a function of the form

$$P(z)=a_nz^n+a_{n-1}z^{n-1}+\cdots+a_1z+a_0,\qquad a_n\neq 0.$$

The complex number a_0 is called the **leading coefficient**; the polynomial is called **monic** if $a_n = 1$.

Nexercise 13 *Show the polynomial above is analytic with*

$$P'(z) = \sum_{j=0}^{n} j a_j z^{j-1}.$$

If P and Q are polynomials, we say P is **divisible** by Q if there is some polynomial $q = q(z)$ such that $P = qQ$. In this case, we write

$$Q \big| P \qquad \text{"}Q \textbf{ divides } P.\text{"}$$

Notice that if Q divides P, then $\deg(Q) \leq \deg(P)$. More generally, we have the **division algorithm** for polynomials:

Proposition 2 *If* $\deg(Q) \leq \deg(P)$ *and Q is monic, then there are unique polynomials $q = q(z)$ and $r = r(z)$ with* $\deg(r) < \deg(Q)$ *and*

$$P = qQ + r.$$

The polynomial q is called the **quotient** *and the polynomial r is called the* **remainder**.

Example 3 $P = z^4 + 4$; $Q = z^2 + 1$. $q = z^2 - 1$; $r = 5$.

$$z^4 + 4 = (z^2 - 1)(z^2 + 1) + 5.$$

Nexercise 14 *You should make sure that given any two polynomials P and Q with* $\deg(P) > \deg(Q)$*, you can find polynomials q and r with* $\deg(r) < \deg(Q)$ *and*

$$P = qQ + r.$$

Nexercise 15 *The division algorithm still holds if* $\deg(Q) > \deg(P)$*. (Explain.)*

Nexercise 16 *Prove the division algorithm by induction on the degree of P.*

roots

If $P(z_0) = 0$, then z_0 is called a **root** of P.

Theorem 6 *If $P(z_0) = 0$, then $(z - z_0)\big|P$.*

Proof: By the division algorithm

$$P = q(z - z_0) + r$$

where r is a constant. Since $P(z_0) = 0$, we know the constant $r = 0$. $\square$

Theorem 7 (fundamental theorem of algebra) *We state the result in two equivalent ways:*

(version 1) *Given a polynomial P with degree n, there are n complex roots, $\alpha_1, \ldots, \alpha_n$ of P such that*

$$P(z) = a_n \prod_{j=1}^{n} (z - \alpha_j). \tag{2.4}$$

(version 2) *If P is a polynomial with $\deg(P) \geq 1$, then there is a complex number α such that $P(\alpha) = 0$, i.e., P has a (complex) root.*

multiplicity

The roots of a polynomial may not be all different from one another, i.e., *distinct*, i.e., there may be **repeated roots**. But the roots of a polynomial $\{\alpha_1, \ldots, \alpha_n\}$ are unique as a set, even counted with multiplicites. That is, up to ordering, the product in (2.4) is unique. In view of this situation, let us change notation slightly. Let the distinct roots be $\alpha_1, \ldots, \alpha_k$ with $k \leq n$, and let the root α_j have **multiplicity** m_j. Then the product expression for P becomes

$$P(z) = a_n \prod_{j=1}^{k} (z - \alpha_j)^{m_j}.$$

The power/multiplicity m_j is also called the **order of the zero** α_j, and the zero is said to be **simple** if $m_j = 1$.

Note that we have

$$\sum_{j=1}^{k} m_j = n.$$

The order of a zero m_j is related to differentiation of P. For example, if α_j is a simple zero, then

$$P(z) = (z - \alpha_j)Q(z) \qquad \text{with } Q(\alpha_j) \neq 0.$$

This means

$$P'(z) = Q(z) + (z - \alpha_j)Q'(z) \qquad \text{and} \qquad P'(\alpha)Q(\alpha_j) \neq 0.$$

More generally, we have the following result:

Lemma 5 *The zero α of a polynomial P has order m if and only if*

$$P^{(j)}(\alpha) = 0,\ j = 0, \dots, m-1 \qquad \text{but} \qquad P^{(m)}(\alpha) \neq 0. \tag{2.5}$$

Proof: We first assume α has order m. Then $P = (z - \alpha)^m Q$ where Q is a polynomial with $Q(\alpha) \neq 0$. This means, for example,

$$P' = m(z-\alpha)^{m-1}Q + (z-\alpha)^m Q' \qquad \text{and} \qquad P'(\alpha) = m(\alpha-\alpha)^{m-1}Q(\alpha) = 0$$

unless $m = 1$. If $m = 1$, then the conclusion (2.5) of the theorem holds. If $m > 1$, we can continue and write

$$P' = (z - \alpha)^{m-1}[mQ + (z - \alpha)Q'] = (z - \alpha)^{m-1}Q_1$$

where Q_1 is a polynomial with $Q_1(\alpha) \neq 0$. Evidently, we can repeat this argument to find

$$P^{(j)} = (z - \alpha)^{m-j}Q_j \quad \text{with} \quad Q_j(\alpha) \neq 0 \quad \text{for} \quad j = 0, \dots, m-1.$$

Differentiating once more, we obtain

$$P^{(m)} = Q_{m-1} + (z - \alpha)Q'_{m-1}.$$

Plugging in $z = \alpha$ into these relations, we get the conclusion (2.5) of the theorem.

Conversely, once we have established (2.5), it is clear that this conclusion can hold for at most one integer m. Thus, the integer m for which this relation holds must be the order of α. $\square$

Lucas' theorem

This result may be viewed as a kind of exercise in understanding the structure of polynomials and the properties of complex numbers.

Theorem 8 (Lucas' theorem) *Let z_0 and w be fixed (determining and open left half plane in $\mathbb{C}$). If P is a polynomial and* $\mathrm{Re}[(\alpha - z_0)iw] > 0$, *i.e.,*

$$\mathrm{Im}[(\alpha - z_0)w] < 0 \qquad \textit{for all roots } \alpha \textit{ of } P,$$

then

$$\mathrm{Im}[(\beta - z_0)w] < 0 \qquad \textit{for all roots } \beta \textit{ of } P'.$$

Proof: We start with the product representation

$$P(z) = a_n \prod_{j=1}^{k} (z - \alpha_j)^{m_j}.$$

Differentiating, we see

$$P' = a_n \sum_{\ell=1}^{k} m_\ell (z - \alpha_\ell)^{m_\ell - 1} \prod_{j \neq \ell} (z - \alpha_j)^{m_j}.$$

From this expression we see

$$\frac{P'}{P} = \sum_{\ell=1}^{k} \frac{m_\ell}{z - \alpha_\ell} = \sum_{\ell=1}^{k} m_\ell \frac{\overline{z - \alpha_\ell}}{|z - \alpha_\ell|^2}.$$

If β is any complex number in the complementary closed half space, then β is not a root of P, and the function P'/P is well-defined and finite valued at β and has value

$$\frac{P'(\beta)}{P(\beta)} = \sum_{\ell=1}^{k} m_\ell \frac{\overline{\beta - \alpha_\ell}}{|\beta - \alpha_\ell|^2}.$$

On the other hand, we also have

$$\mathrm{Im}[(\beta - z_0)w] \geq 0,$$

but for each ℓ

$$\beta - \alpha_\ell = (\beta - z_0)w - (\alpha_\ell - z_0)w,$$

so

$$\operatorname{Im}[(\beta-\alpha_\ell)w] = \operatorname{Im}[(\beta-z_0)w] - \operatorname{Im}[(\alpha_\ell - z_0)w] > 0$$

or

$$\operatorname{Im}\left[\overline{(\beta-\alpha_\ell)w}\right] < 0 \qquad \text{for} \quad \ell = 1,\ldots,k.$$

Therefore, $\bar{w}P'(\beta)/P(\beta)$ satisfies

$$\begin{aligned}\operatorname{Im}\left[\bar{w}\frac{P'(\beta)}{P(\beta)}\right] &= \operatorname{Im}\left[\sum_{\ell=1}^{k} m_\ell \frac{\overline{w(\beta-\alpha_\ell)}}{|\beta-\alpha_\ell|^2}\right]\\ &= \sum_{\ell=1}^{k}\frac{\operatorname{Im}\left[\overline{(\beta-\alpha_\ell)w}\right]}{|\beta-\alpha_\ell|^2}\\ &< 0.\end{aligned}$$

Therefore, $P'(\beta) \neq 0$, and any root of P' must actually lie in the same open half plane with the roots of P. □

This means that if one takes the collection of roots $\{\alpha_1,\ldots,\alpha_k\}$ of a polynomial P and constructs from them the minimal convex polygon K containing them, i.e., the convex hull of the roots, then the roots of P' also lie in K.

2.3.2 rational functions

A rational function is a function of the form

$$R(z) = \frac{P(z)}{Q(z)} \qquad \text{where } P \text{ and } Q \text{ are polynomials.}$$

We can, and do, assume P and Q have no common factors, i.e., no common zeros. If $\beta \in \mathbb{C}$ is a root of Q, then we say R has a **pole** at $z = \beta$, and we define

$$R(\beta) = \infty.$$

This makes perfectly good sense on the Riemann sphere. That is to say, more generally,

> *rational functions are naturally considered with domain and range in the Riemann sphere.*

But there is a question:

Can we define $R(\infty)$?

The answer is "yes," but it will take a little work and terminology to do so. Before we address the definition of $R(\infty)$ directly, let us focus on a finite pole.

First of all, the poles of R have an order, like the order of the zeros of P or of R. If β is a pole of R, then the **order of the pole** β is the order of β as a zero of Q. Let us consider the behavior of R in a neighborhood of a pole β of order m. The picture we should ultimately have in mind is that of the mapping $\tilde{R} = \sigma^{-1} \circ R \circ \sigma$. On the other hand, for the mapping R, we can write

$$R = \frac{P}{Q} = \frac{P}{(z-\beta)^m Q_1} = \frac{1}{(z-\beta)^m} \frac{P}{Q_1}$$

where Q_1 is a polynomial with a (finite) nonzero value at $z = \beta$. Thus,

$$R_1 = \frac{P}{Q_1}$$

is a rational function with a finite nonzero value $L \in \mathbb{C}$ at $z = \beta$. As $z \to \beta$, the factor $1/(z-\beta)^m$ satisfies

$$\left| \frac{1}{(z-\beta)^m} \right| \geq \frac{1}{|z-\beta|} \to +\infty.$$

This means for any $M > 0$, there is some $r > 0$ such that

$$|z-\beta| < r \qquad \text{implies} \qquad \left|\frac{P}{Q}\right| = \left| \frac{1}{(z-\beta)^m} \right| \left| \frac{P}{Q_1} \right| \geq \frac{2M}{|L|} \frac{|L|}{2} = M.$$

This implies that for any $\epsilon > 0$ there is some r_* so that all points $\zeta \in \mathbb{S}^2$ with $0 < d(\zeta, \sigma^{-1}(\beta)) < r_*$ we have

$$d(\tilde{R}(\zeta), (0,0,1)) < \epsilon.$$

This is a fundamental observation about poles. Before we leave the topic, let us briefly consider the derivative of a rational function (at a pole). The quotient rule applies, and we have

$$R' = \frac{QP' - PQ'}{Q^2} = \frac{(z-\beta)^m Q_1 P' - PQ'}{(z-\beta)^{2m} Q_1^2}.$$

On the other hand,

$$Q' = m(z-\beta)^{m-1}Q_1 + (z-\beta)^m Q_1' = (z-\beta)^{m-1}Q_{11}$$

where $Q_{11} = mQ_1 + (z-\beta)Q_1'$ is a polynomial with $Q_{11}(\beta) \neq 0$. Thus,

$$R' = \frac{(z-\beta)Q_1P' - PQ_{11}}{(z-\beta)^{m+1}Q_1^2}$$

has numerator $P_1 = (z-\beta)Q_1P' - PQ_{11}$ which is a polynomial with $P_1(\beta) = -P(\beta)Q_{11}(\beta) \neq 0$. We conclude that R' is a rational function with a pole of order $m+1$ at β.

Now, let us consider $R(\infty)$. In order to see what is happening, we make a change of variables and look at $R(1/\zeta)$ for ζ near $\zeta = 0$. We find

$$R\left(\frac{1}{\zeta}\right) = \frac{P(1/\zeta)}{Q(1/\zeta)}.$$

This is, of course, a rational function of ζ, and we claim the basic behavior of R at $z = \infty$ is determined, first of all, by the relative orders of P and Q. Let's say

$$P = \sum_{j=0}^{n} a_j z^j \qquad \text{and} \qquad Q = \sum_{j=0}^{m} b_j z^j \qquad \text{with} \qquad n > m.$$

Then

$$\begin{aligned} R\left(\frac{1}{\zeta}\right) &= \frac{a_n/\zeta^n + a_{n-1}/\zeta^{n-1} + \cdots + a_1/\zeta + a_0}{b_m/\zeta^m + b_{m-1}/\zeta^{m-1} + \cdots + b_1/\zeta + b_0} \\ &= \frac{a_0\zeta^n + a_1\zeta^{n-1} + \cdots + a_{n-1}\zeta + a_n}{\zeta^{n-m}(b_0\zeta^m + b_1\zeta^{m-1} + \cdots + b_{m-1}\zeta + b_m)}. \end{aligned}$$

Since a_n and b_m are nonzero, $R(1/\zeta)$ has a pole of order $n-m$ at $\zeta = 0$. With this in mind let's think about what happens to points near the north pole under the mapping $\tilde{R} = \sigma^{-1} \circ R \circ \sigma$, but taking a detour via the reciprocal map $z \mapsto \zeta = 1/z$. Points near the north pole correspond, under the reciprocal map to points near $\zeta = 0$. We know the images of these points tend to $w = \infty$ satisfying an estimate

$$|R(z)| \geq \frac{c}{|\zeta|^{n-m}} \qquad \text{as } \zeta \to 0$$

where $c > 0$ is a constant. Looking back at the spherical metric (1.48), we see

$$d(z,\infty) = d(\zeta,0) = \cos^{-1}\left(1 - \frac{2|\zeta|^2}{|\zeta^2|^1}\right) = \cos^{-1}\left(\frac{1-|\zeta|^2}{1+|\zeta|^2}\right)$$

where ∞ means the north pole $(0,0,1) \in \mathbb{S}^2$.

Nexercise 17 *There is a positive constant m such that for ζ near $0 \in \mathbb{C}$,*

$$\frac{1}{m} d(\zeta,0) \leq |\zeta| \leq m d(\zeta,0).$$

Hint: $1-x^2/2 \leq \cos x \leq 1-x^2/4$ *and* $1-2|\zeta|^2 \leq (1-|\zeta|^2)/(1+|\zeta|^2) \leq 1-|\zeta|^2$.

Therefore, we have an estimate

$$|R(z)| \geq \frac{mc}{d(z,\infty)^{n-m}}.$$

By a similar argument, this also implies

$$d(R(z),\infty) \leq \frac{mc}{d(z,\infty)^{n-m}}.$$

In such a situation, we say R has a pole of order $n-m$ at $z = \infty$. We can summarize our discussion as follows:

Definition-Proposition 3 *A rational function $R = P/Q$ has* **a pole at infinity of order** $\ell > 0$ *if any one of the following three equivalent conditions holds.*

1. *$R(1/\zeta)$ has a pole of order ℓ at $\zeta = 0$.*

2. $\deg(P) - \deg(Q) = \ell$.

3. *There is a constant $C > 0$ such that*

$$d(R(z),\infty) \leq \frac{C}{d(z,\infty)^{\ell}} \qquad \text{for } z \text{ in some neighborhood of } \infty.$$

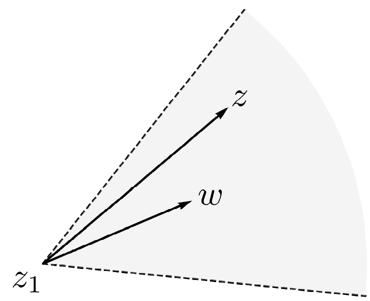

Figure 2.2: a convex angle domain

2.3.3 Abel limit theorem

An **angle domain** (or Stolz angle or sector) with vertex at $z_1 \in \mathbb{C}$ and reference direction $w \in \mathbb{S}^1$ is defined by

$$A_m = \{z \in \mathbb{C} : \mathrm{Re}[(z - z_1)\bar{w}/|z - z_1| > m\}.$$

The number m is the cosine of the half angle of the domain. If $m > 0$, then the half angle is less than $\pi/2$ and the domain is **convex**. If $-1 < m < 0$, then the domain is said to be **reentrant**. We will only be dealing with convex sectors here.

The following result allows us to find the value of a convergent series at a boundary point by taking a limit from within the disk of convergence. Consequently, it gives a kind of one-sided continuity at the boundary when the series converges.

Theorem 9 (Abel limit theorem) *If $f(z)$ is defined in $B_R(z_0)$ by*

$$f(z) = \sum_{j=0}^{\infty} a_j (z - z_0)^j$$

and $z_1 \in \partial B_R(z_0)$ satisfies

$$f(z_1) = \sum_{j=0}^{\infty} a_j (z_1 - z_0)^j \in \mathbb{C},$$

i.e., the series is convergent at $z = z_1$, then for any $\epsilon > 0$ and $m > 0$, there is some $\delta > 0$ such that

$$|f(z) - f(z_1)| < \epsilon \qquad \text{for all } z \in B_\delta(z_1) \cap A_m$$

where

$$A_m = \left\{ z \in \mathbb{C} : \mathrm{Re}\left[\frac{(z - z_1)\overline{(z_0 - z_1)}}{|z - z_1||z_0 - z_1|} \right] > m \right\}.$$

Proof: We will give the proof in the case $z_0 = 0 \in \mathbb{C}$, $R = 1$, and $z_1 = 1$. In this case, the angle domain becomes

$$A_m = \left\{ z \in \mathbb{C} : \mathrm{Re}\left[\frac{(1 - z)}{|1 - z|} \right] = \frac{1 - \mathrm{Re}(z)}{|1 - z|} > m \right\}.$$

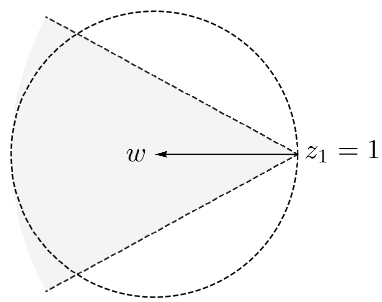

Figure 2.3: a convex angle domain at the boundary point $z_1 = 1$

Lemma 6 *For any μ with $0 < \mu < 1$, there is some $\delta_1 > 0$ such that*

$$\frac{1 - |z|}{|1 - z|} > \mu m \qquad \text{for } z \in B_{\delta_1}(1) \cap A_m.$$

Proof: Let $z = x + iy \in B_{\delta_1}(1) \cap A_m$ and note that for δ_1 small we must have $0 < x < 1$. Also, the condition

$$\frac{1 - \mathrm{Re}(z)}{|1 - z|} > m$$

can be written as

$$1 - x > m\sqrt{(1 - x)^2 + y^2} \qquad \text{which implies} \qquad y^2 < \frac{(1 - m^2)}{m^2}(1 - x)^2.$$

From this, we see

$$1 - |z| = 1 - \sqrt{x^2 + y^2} > 1 - \sqrt{x^2 + \frac{(1 - m^2)}{m^2}(1 - x)^2} = g(x).$$

We can compute

$$g'(x) = -\frac{x-(1-m^2)}{m^2}(1-x)\sqrt{x^2 + \frac{(1-m^2)}{m^2}(1-x)^2}.$$

Therefore $g(z)$ satisfies

$$g(1) = 0, \qquad \text{and} \qquad g'(1) = 1.$$

On the other hand, $h(x) = \mu(1-x)$ satisfies

$$h(1) = 0 \qquad \text{and} \qquad h'(1) = -\mu > -1.$$

Thus, for some $\delta_1 > 0$, we have

$$h(x) < g(x) \qquad \text{for } 1-\delta_1 < x < \delta_1.$$

In particular, for $z \in B_{\delta_1}(1) \cap A_m$ we have

$$1 - |z| > g(x) > h(x) = \mu(1-x) > \mu m|1-z|. \qquad \Box$$

Corollary 1 *There is some $\delta_1 > 0$ and some $M < \infty$ such that*

$$\frac{|1-z|}{1-|z|} < M \qquad \text{for } z \in B_{\delta_1}(1) \cap A_m. \tag{2.6}$$

Proof: $M = 1/(\mu m)$. $\Box$

Returning to the proof of the Abel limit theorem: Set

$$f(1) = \sum a_j = \alpha.$$

Let $\epsilon > 0$. We wish to show $|f(z) - f(1)| < \epsilon$ for $z \in B_\delta(1) \cap A_m$ (for some $\delta > 0$ yet to be identified).

Let us restrict, for the moment, to $zB_{\delta_1}(1) \cap A_m$. As usual, we have the partial sums

$$s_k = a_0 + a_1 z + z_2 z^2 + \cdots + a_k z^k. \tag{2.7}$$

Set

$$b_k = a_0 + a_1 + z_2 + \cdots + a_k.$$

Then $b_k - b_{k-1} = a_k$. Notice the coefficients a_k in (2.7) may be replaced by means of this relation. Let us modify this a little:

$$\tilde{s}_k = s_k - \alpha = a_0 - \alpha + a_1 z + z_2 z^2 + \cdots + a_k z^k;$$

$$\tilde{b}_k = b_k - \alpha = a_0 - \alpha + a_1 + z_2 + \cdots + a_k.$$

Then

$$\begin{aligned}\tilde{s}_k &= \tilde{b}_0 + (\tilde{b}_1 - \tilde{b}_0)z + (\tilde{b}_2 - \tilde{b}_1)z^2 + + \cdots + (\tilde{b}_k - \tilde{b}_{k-1})z^k \\ &= \tilde{b}_0(1-z) + \tilde{b}_1(1-z)z + \cdots + \tilde{b}_{k-1}(1-z)z^{k-1} + \tilde{b}_k z^k \\ &= (1-z)\sum_{j=0}^{k-1} \tilde{b}_j z^j + \tilde{b}_k z^k.\end{aligned}$$

On the other hand, $b_k \to \alpha$ as $k \to \infty$, so $\tilde{b}_k \to 0$ as $k \to \infty$. This means the last term tends to zero, and the series $\sum \tilde{b}_j z^j$ has a finite limit:

$$f(z) - \alpha = (1-z)\sum_{j=0}^{\infty} \tilde{b}_j z^j \qquad \text{for} \qquad z \in B_1(0).$$

Again, since $\tilde{b}_j \to 0$, there is some N so that

$$j > N \qquad \text{implies} \qquad |\tilde{b}_j| < \frac{\epsilon}{2M}$$

where M comes from Corollary 1. Taking N as an integer, we can now estimate as follows:

$$\begin{aligned}|f(z) - f(1)| &= |f(z) - \alpha| \\ &= \left|(1-z)\sum_{j=0}^{\infty} \tilde{b}_j z^j\right| \\ &= \left|(1-z)\sum_{j=0}^{N} \tilde{b}_j z^j + (1-z)\sum_{j=N+1}^{\infty} \tilde{b}_j z^j\right| \\ &\le |1-z|\sum_{j=0}^{N} |\tilde{b}_j| + |1-z|\frac{\epsilon}{2M}\sum_{j=N+1}^{\infty} |z|^j \\ &\le |1-z|\sum_{j=0}^{N} |\tilde{b}_j| + |1-z|\frac{\epsilon}{2M}\sum_{j=0}^{\infty} |z|^j \\ &= |1-z|\sum_{j=0}^{N} |\tilde{b}_j| + |1-z|\frac{\epsilon}{2M}\frac{1}{1-|z|}.\end{aligned}$$

Since $\sum_{j=0}^N |\tilde{b}_j|$ is fixed, there is some $\delta < \delta_1$ for which

$$z \in B_\delta(1) \qquad \text{implies} \qquad |1-z| \sum_{j=0}^N |\tilde{b}_j| < \frac{\epsilon}{2}.$$

Also, from Corollary 1 we know

$$\frac{|1-z|}{1-|z|} < M.$$

Therefore, we have that for $z \in B_\delta(1) \cap A_m$ there holds

$$|f(z) - f(1)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \qquad \square$$

Nexercise 18 *State and prove a version of Lemma 6 which applies in an arbitrary ball $B_R(z_0)$ with an arbitrary boundary point $z_1 \in \partial B_R(z_0)$ and a convex angle domain determined by $w = (z_0 - z_1)/|z_0 - z_1|$.*

Nexercise 19 *Fill in the details to show Abel's theorem holds in the general case for $z_1 \in \partial B_R(z_0)$ as stated.*

Exercise 4 from Chapter II § 1.4

Our starting point is the example

$$R(z) = \frac{z-\alpha}{1-\bar{\alpha}z}.$$

This function may be found though experimentation (as described above). Ahlfors also tries to lead us to this function in Chapter I; see Exercise 3 of § 1.4 and Exercise 1 of § 1.5. In any case, this problem is still relatively difficult even with this example. What we should notice is that there is one root and one pole, and they are related by reflection across the circle $\mathbb{S}^1$. The basic question is

Can there be anything else?

For example, instead of introducing a pole at the reflection $1/\bar{\alpha}$ of the root α, we might try to see if we could put a second root somewhere and create "balance." That is: Is there an example which is a quadratic polynomial

$$P(z) = a_2(z-\alpha_1)(z-\alpha_2)$$

with α_1 and α_2 both finite roots in $\mathbb{C}$, so that there is a double pole at $z = \infty$? Now, maybe we suspect this won't work, and there should really be a pole at $1/\bar{\alpha}_1$ (and a another at $1/\bar{\alpha}_2$. (Note that we could, by a rotation, assume $\alpha_1 \in \mathbb{R}$ or even $0 < \alpha_1 < 1$. Those might be good assumptions, but it turns out they won't be necessary.)

The first crucial observation/trick is that it is a good idea to consider $P(1/\bar{z})$. I do not see a compelling reason to look at this function, but except that we are clearly interested in $1/\bar{\alpha}_j$ for $j = 1, 2$, and if you substitute these values alone into P, then you can't say much of anything, so considering $P(1/\bar{z})$ is a more general approach one might think might give something. In fact it does.

$$P(1/\bar{z}) = a_2 \left(\frac{1}{\bar{z}} - \alpha_1 \right) \left(\frac{1}{\bar{z}} - \alpha_2 \right) = \frac{a_2}{\bar{z}^2}(1-\alpha_1\bar{z})(1-\alpha_2\bar{z}) = \frac{a_2}{|z|^2}(z-\alpha_1|z|^2)(z-\alpha_2|z|^2).$$

Another thing to notice is that this function has zeros at $z = 1/\bar{\alpha}_j$ for $j = 1, 2$. Not poles, but zeros. Actually, it wouldn't really make sense to think about this function having poles becuase it's not a rational function of z.

And maybe that's the next crucial thing to realize:

This is **not** *an analytic function (or rational function) of z.*

If we take the conjugate of the whole thing, however, we get a rational function of z:

$$\overline{P\left(\frac{1}{\bar{z}}\right)} = \frac{\bar{a}_2}{|z|^2}(\bar{z} - \bar{\alpha}_1|z|^2)(\bar{z} - \bar{\alpha}_2|z|^2).$$

Now we might be stuck. For lack of something better to do, we might look at $1/P(z)$. This is a rational function. It's analytic/meremorphic. It also has poles where P had zeros. This too turns out to be a good function to consider:

$$\frac{1}{P(z)} = \frac{1}{a_2}\,\frac{1}{z - \alpha_1}\,\frac{1}{z - \alpha_2} = \frac{\bar{a}_2(\bar{z} - \bar{a}_1)(\bar{z} - \bar{a}_1)}{|P(z)|^2}.$$

Now, here's the third thing (and this is a big one): If we look at the two expressions for these functions above, they are equal when $|z| = 1$. That is, they are equal on the entire unit circle. In particular, if we took one and divided by the other, then the result would be a rational function which was **identically** 1 on the whole unit circle. But nonconstant rational functions can take a particular value, like 1, at most finitely many times. This means one of these functions is a constant multiple of the other and, in fact, they are equal:

$$\overline{P\left(\frac{1}{\bar{z}}\right)} = \frac{1}{P(z)}.$$

Cosine

The basic formula for the complex cosine is given by

$$\begin{aligned}\cos z &= \cos(x+iy)\\ &= \frac{1}{2}\left(e^{i(x+iy)} + e^{-i(x+iy)}\right)\\ &= \cosh y \cos x - i \sinh y \sin x.\end{aligned}$$

If we begin at the origin and take $z = x$ incerasing along the real axis, the cosine decreases from $w = 1$ to $w = -1$. At each of these points $z = 0$ and $z = 1$, the derivative vanishes and moving $z = x$ further along the axis will result in a repitition of the values between -1 and 1.

For fixed x with $0 < x < \pi/2$ or $\pi/2 < x < \pi$, we see

$$\frac{(\cosh y \cos x)^2}{\cos^2 x} - \frac{(-\sinh y \sin x)^2}{\sin^2 x} = 1.$$

This means the image of the vertical lines $z = x + iy$ with $x \neq k\pi/2$ fixed lie along hyperbolic curves centered at the origin with $y > 0$ corresponding to $\text{Im}(\cos z) < 0$ and $y > 0$ corresponding to $\text{Im}(\cos z) > 0$ as indicated in Figure 2.4.

The definition of a **closed** set is that it is **the complement of an open set**.

Nexercise 20 *If A is connected and $A = C_1 \cup C_2$ where C_1 and C_2 are closed disjoint sets (in A), then either $C_1 = \phi$ or $C_2 = \phi$.*

Theorem 10 *Invervals in $\mathbb{R}$ (including $\mathbb{R} = (-\infty, \infty)$) are connected. If $A \subset \mathbb{R}$ is connected, then A is an interval.*

Theorem 11 *Let $\mathcal{U} \subset \mathbb{C}$ be open. Then:*

> *$\mathcal{U}$ is connected if and only if for each $z, w \in \mathcal{U}$, there is a polygonal path $\Gamma \subset \mathcal{U}$ with endpoints z and w.*

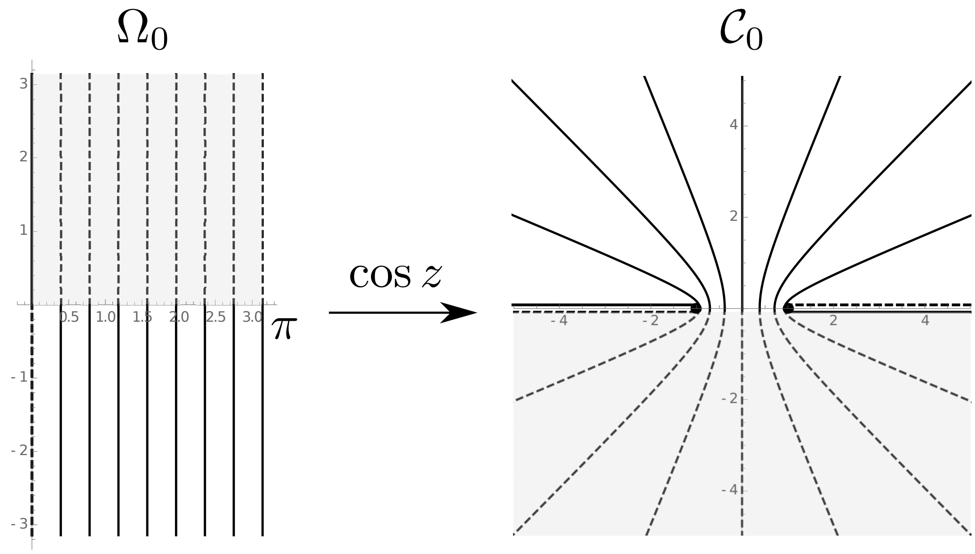

Figure 2.4: a fundamental domain for cosine

By a **polygonal path***, we mean a finite collection of concatenated segments*

$$\{(1-t)z_1 + tz_2 : 0 \le t \le 1\}.$$

Proof: If $\mathcal{U}$ is open and connected, then for any fixed point $z \in \mathcal{U}$ we consider

$$\mathcal{U}_1 = \{w \in \mathcal{U} : w \text{ is connected to } z \text{ by a polygonal path}\}$$

and $\mathcal{U}_2 = \mathcal{U} \backslash \mathcal{U}_1$.

If $w_0 \in \mathcal{U}_1$, then there is some $\epsilon > 0$ such that $B_\epsilon(w_0) \subset \mathcal{U}$. Let $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_k$ be a concatenation of segments in $\mathcal{U}$ starting at z and ending at w_0. Concatenating

$$\Gamma_{k+1} = \{(1-t)w_0 + tw\} \subset B_\epsilon(w_0)$$

for any $w \in B_\epsilon(0)$, we see $B_\epsilon(0) \subset \mathcal{U}_1$. Therefore, $\mathcal{U}_1$ is open.

If there is some $w_0 \in \mathcal{U} \backslash \mathcal{U}_1$, then again, there is some $\epsilon > 0$ so that $B_\epsilon(w_0) \subset \mathcal{U}$. If Γ is a polygonal path connecting z to any $w \in B_\epsilon(w_0)$, then concatenating

$$\{(1-t)w + tw_0\} \subset B_\epsilon(w_0)$$

to Γ we have $w_0 \in \mathcal{U}$ (a contradiction). Therefore, $B_\epsilon(w_0) \subset \mathcal{U}_2 = \mathcal{U}\backslash\mathcal{U}_1$. This means $\mathcal{U}_2$ is open.

Since $\mathcal{U} = \mathcal{U}_1 \cup \mathcal{U}_2$, both $\mathcal{U}_1$ and $\mathcal{U}_2$ are open, and $\mathcal{U}_1 \cap \mathcal{U}_2 = \phi$ (and $\mathcal{U}_1 \neq \phi$ unless $\mathcal{U} = \phi$), we must have $\mathcal{U}_2 = \phi$ and $\mathcal{U}_1 = \mathcal{U}$ is path connected.

The fact that (polygonal) path connectedness implies connectedness is a consequence of the fact that intervals are connected in $\mathbb{R}$. This is less important for us, so we omit the details. $\square$

Definition-Proposition 4 *Given a set A (a subset of a metric space) and $z_0 \in A$,*

1. *The union of all connected sets C containing z_0 is a connected set.*

   This is called the **component** *of z_0 in A, and may be denoted by* $\mathrm{comp}(z_0)$.

2. *If C is any connected set in A, then either*

$$C \cap \mathrm{comp}(z_0) = C \qquad \text{or} \qquad C \cap \mathrm{comp}(z_0) = \phi.$$

3. *In particular, the set A has a unique decomposition into components.*

2.3.4 Chapter III § 2.2 (exercises)

Exercise 1

Give a precise definition of a single valued branch of $\sqrt{1+z} + \sqrt{1-z}$. Prove it is analytic.

We can (and should) take a branch of the square root defined on the Riemann surface

$$\Sigma = \{\zeta \in \mathbb{C} : -\pi < \arg \zeta \leq \pi\}$$

with a branch cut along the negative real axis. We can then define $\sqrt{1+z}$ on $p^{-1}(\Sigma)$ where $p(z) = 1 + z$. Since $z = p^{-1}(\zeta) = \zeta - 1$, the set $p^{-1}(\Sigma)$ is

$$p^{-1}(\Sigma) = \{z \in \mathbb{C} : -\pi < \arg(z+1) \leq \pi\}.$$

Similarly, We can define $\sqrt{1-z}$ on $q^{-1}(\Sigma)$ where $q(z) = 1 - z$. Since $z = q^{-1}(\zeta) = 1 - \zeta$, the set $q^{-1}(\Sigma)$ is

$$q^{-1}(\Sigma) = \{z \in \mathbb{C} : 0 < \arg(z-1) \leq 2\pi\}.$$

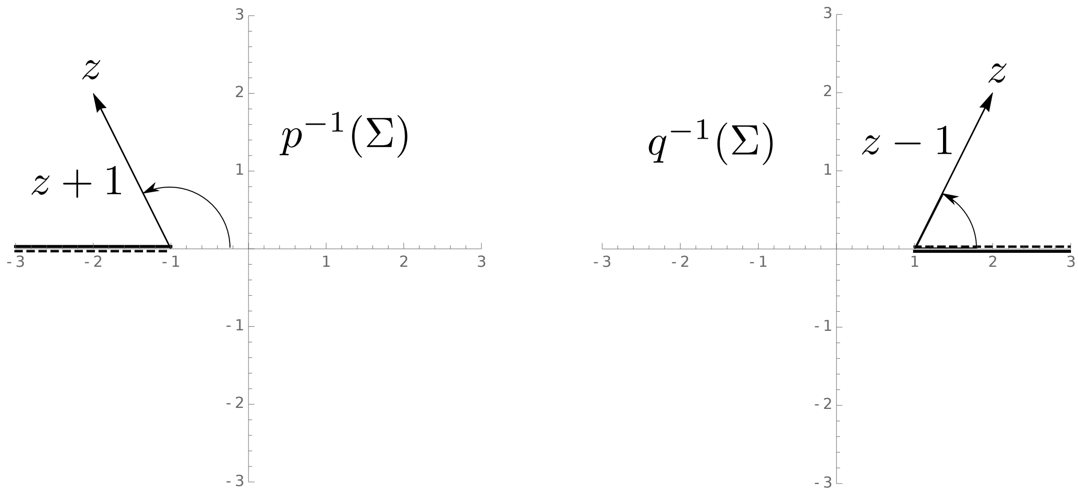

Figure 2.5: domains for $\sqrt{1+z}$ (left) and $\sqrt{1-z}$ (right)

These sets are indicated in Figure 2.5 with an indication of how the arguments are measured for a typical point. The intersection $\Omega = p^{-1}(\Sigma) \cap q^{-1}(\Sigma)$ of these two domains, or if one wishes to have an open set the interior of this set, gives a set on which the function

$$f(z) = \sqrt{1+z} + \sqrt{1-z}$$

has a well-defined branch. The derivative

$$f'(z) = \frac{1}{2\sqrt{1+z}} - \frac{1}{2\sqrt{1-z}}$$

is well-defined on the same region, so f is analytic there. Notice that aside from the excluded vertices at $z = \pm 1$ where the square root requires branch cuts and is singular, we have $f'(0) = 0$.

Nexercise 21 *Show that $z = 0$ is the only interior point where $f'(z) = 0$.*

This singularity/non-conformality makes the image somewhat difficult to determine/visualize. As Ahlfors says, you can only really understand such a mapping when it is one-to-one, and this mapping is not one-to-one on Ω. For example, the segments $[-1, 0]$ and $[0, 1]$ map to the same segment $[\sqrt{2}, 2]$ with $0 \mapsto 2$ and $\pm 1 \mapsto \sqrt{2}$.

Careful consideration reveals that the open right half of Ω, that is,

$$\Omega^+ = \{z = x + iy : x > 0\} \backslash \{x \in \mathbb{R} : x \geq 1\}$$

maps (one-to-one and onto) the cut hyperbolic region

$$f(\Omega^+) = \{z = x + iy : x > \sqrt{2 - y^2}\} \backslash \{x \in \mathbb{R} : x \geq 2\}.$$

The correspondence is indicated in Figure 2.6. Consider the boundary a,

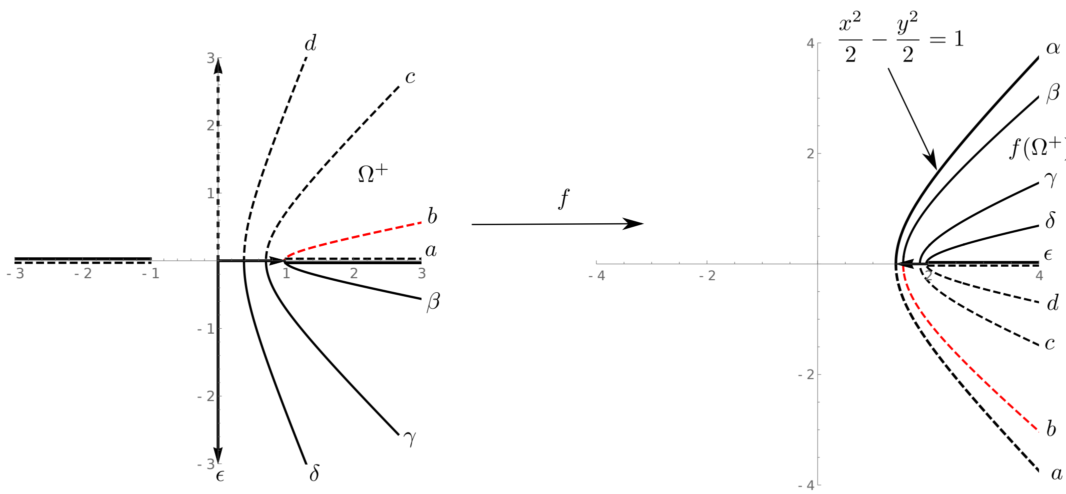

Figure 2.6: a fundamental domain for the map f

by which we mean approaching the real axis along $x > 1$ from the first quadrant. Let us denote such a point by $t = t^+ \in \mathbb{R}$. To determine $f(t)$, note that $\sqrt{1+t}$ is simply the real square root, but $\sqrt{1-t}$ is purely imaginary and determined as follows: We negate $t = t^+$ to attain a point $-t^+$ on the lower boundary along $x < -1$. We add 1 to move the end (-1) of the left branch cut to the origin and take the square root, which in this case, since we have an argument close to $\arg(1-t^+)$ is just greater than $-\pi$. Therefore, $\sqrt{1-t} = -i\sqrt{t-1}$. Thus, we have a parameterization of this boundary given by

$$f(t^+) = \sqrt{1+t} - i\sqrt{t-1} \qquad \text{for } t > 1.$$

Setting $x = \sqrt{1+t}$ and $y = -\sqrt{t-1}$, we see $x^2 - y^2 = 2$. Thus, the image of this boundary is along the hyperbola labeled "a" on the right in Figure 2.6.

We noted above, in our discussion of the cosine function that this region Ω^+ is nicely foliated by hyperbolas which are the image under the cosine function of the vertical lines $x + it$ where $0 < x < \pi/2$ and $t \in \mathbb{R}$; see Figure 2.4. The domain curve labeled "b" is given by

$$t \mapsto \cos x \cosh t - i \sin x \sinh t \qquad \text{for some } x \text{ with } 0 < x < \pi/2 \text{ and } t < 0.$$

We can calculate

$$\sqrt{1 + \cos x \cosh t - i \sin x \sinh t} = r(\cos\theta + i \sin\theta)$$

where

$$r = \sqrt[4]{(1 + \cos x \cosh t)^2 + \sin^2 x \sinh^2 t}$$

and

$$\theta = -\frac{1}{2} \tan^{-1} \frac{\sin x \sinh t}{1 + \cos x \cosh t}$$

with θ satisfying $0 < \theta < \pi/4$ since $t < 0$. Similarly,

$$\sqrt{1 - \cos x \cosh t + i \sin x \sinh t} = R(\cos\Theta + i \sin\Theta)$$

where

$$R = \sqrt[4]{(1 - \cos x \cosh t)^2 + \sin^2 x \sinh^2 t}$$

and

$$\Theta = \frac{\arg(1 - \cos x \cosh t + i \sin x \sinh t)}{2}$$

with Θ satisfying $-\pi/2 < \Theta < 0$ since $t < 0$. We wish to show the image points

$$\begin{aligned} f(\cos x \cosh t - i \sin x \sinh t) = & \\ & \sqrt{1 + \cos x \cosh t - i \sin x \sinh t} \\ & \quad + \sqrt{1 - \cos x \cosh t + i \sin x \sinh t} \end{aligned}$$

lie along a portion of a hyperbola like the curve marked "b" on the right in Figure 2.6. Note that

$$(r\cos\theta + R\cos\Theta)^2 = r^2\cos^2\theta + 2rR\cos\theta\cos\Theta + R^2\cos^2\Theta.$$

Also,

$$(r\sin\theta + R\sin\Theta)^2 = r^2\sin^2\theta + 2rR\sin\theta\sin\Theta + R^2\sin^2\Theta.$$

Therefore, this will take a long calculation, but I'll guess it's true.

We consider one more pair of boundary curves which together make up the imaginary axis. The point ti with $t > 0$ has

$$\sqrt{1+ti} = \sqrt[4]{1+t^2}(\cos\theta/2 + i\sin\theta/2)$$

where

$$\cos\theta = \frac{1}{\sqrt{1+t^2}} \qquad \text{and} \qquad \sin\theta = \frac{t}{\sqrt{1+t^2}}.$$

Also,

$$\sqrt{1-ti} = \sqrt[4]{1+t^2}(\cos\theta/2 - i\sin\theta/2).$$

Therefore, $f(ti)$ is real with $\operatorname{Re} f(ti) = 2\cos(\theta/2)\sqrt[4]{1+t^2} > 2$.

There is a branch point in the image at $f(0) = 2$ and the points $z \in \Omega$ with $\operatorname{Re} z < 0$ cover the same cut region bounded by the right portion of the hyperbola again with the branch cut $z < -1$ having two boundary sides that correspond to the hyperbolic boundary.

Exercise 2

2.3.5 Chapter III § 3.1 (exercises)

Exercise 1

Show the reflection $z \mapsto \bar{z}$ is not a linear fractional transformation.

The linear fractional transformations are degree one rational functions and, therefore, analytic/holomorphic at most points. If $f(x+iy) = x - iy$, then it's easy to see the Cauchy-Riemann equations are not satisfied. We have $u = x$ and $v = -y$. Therefore,

$$u_x = 1 \neq v_y = -1.$$

On the other hand,

$$u_x = -v_y \text{and} \qquad u_y = 0 = v_x.$$

These are the version of the Cauchy-Riemann equations for anti-holomorphic functions, and $z \mapsto \bar{z}$ is anti-holomorphic.

Exercise 2

$$T_1(z) = \frac{z+2}{z+3} \qquad \text{and} \qquad T_2(z) = \frac{z}{z+1}.$$

$$T_1(T_2(z)) = \frac{\frac{z}{z+1}+2}{\frac{z}{z+1}+3} = \frac{3z+2}{4z+3}.$$

$$T_2(T_1(z)) = \frac{z+2}{z+2+z+3} = \frac{z+2}{2z+5}.$$

Setting $\zeta = T_1(z)$, we find

$$T_1^{-1}(\zeta) = \frac{-3\zeta+2}{\zeta-1}.$$

Therefore,

$$T_1^{-1}(T_2(z)) = \frac{-3(z+2)+2(z+3)}{z+2-(z+3)} = \frac{-z}{-1} = z.$$

This means T_1^{-1} and T_2^{-1} are the same. That doesn't seem correct. I should double check that.

Exercise 3

2.3.6 Chapter IV § 1.3 (exercises)

Exercise 1