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16 - Generating Functions

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Generating Functions - Fun!!??

Observation

$1/(1-x) = 1 + x + x^2 + x^3 + x^4 + \ldots$

So

$1/(1 - 1/x) = 1 + 1/x + 1/x^2 + 1/x^3 + 1/x^4 + \ldots$

But $1/(1 - x) + 1/(1 - 1/x) = 1$ so

$\ldots + 1/x^4 + 1/x^3 + 1/x^2 + 1/x + 1 + x + x^2 + x^3 + x^4 + \ldots = 0$

Challenge Explain this equation to your high school math teacher!!

Generating Functions - Introduction

Example Given a sequence $\{a_n: n \geq 0\}$ of real numbers, the "function" $f(x) = \sum_{n=0}^{\infty} a_n x^n$ is called the generating function of the sequence. This function may or may not have meaning in the sense of an infinite series like you studied in calculus classes. For our purposes, the emphasis is on the role of the function in coding information about the sequence of coefficients.

In particular, generating functions can be added, subtracted, multiplied and divided. When they are real functions, they can be differentiated and integrated.

Generating Functions - Examples

Example The generating function of the constant sequence $a_n = 1$ for all $n \geq 0$ is $1/(1 - x)$.

Example The generating function of the sequence $a_n = n + 1$ is for all $n \geq 0$ is $1/(1 - x)^2$.

Example The generating function of the constant sequence $a_n = 1/n!$ for all $n \geq 0$ is e^x.

Note All three of these examples come from calculus. In the first two, the radius of convergence is 1, and in the third example, the radius is infinite.

Generating Functions – More Calculus

Example The generating function of the sequence $a_n = (-1)^n$ is $1/(1 + x)$, i.e.,

$1/(1 + x) = 1 - x + x^2 - x^3 + x^4 - x^5 + \ldots$

Example $\ln(1 + x) = x - x^2/2 + x^3/3 - x^4/4 + \ldots$

Example $[\ln(1 + x)]/x = 1 - x/2 + x^2/3 - x^3/4 + \ldots$
so $[\ln(1 + x)]/x$ is the generating function for the sequence $a_n = (-1)^n/(n + 1)$.

Example Then generating function for the sequence $a_n = n!$ has radius of convergence 0.

Generating Functions – An Old Friend

Example The generating function of the sequence $a_n = C(n+r-1,r-1)$ is $1/(1-x)^r$.

Explanation The coefficient of x^n is the number of ways we can write $n = p_1 + p_2 + \ldots + p_r$ where each $p_i \geq 0$. This is a problem from the first week of our class and the answer is $C(n+r-1,r-1)$.

Partitions of an Integer

Notation

A distribution of n non-distinct objects into n non-distinct cells is called a partition of the integer n.

Example

$38 = 8 + 8 + 8 + 3 + 3 + 3 + 3 + 2 + 0 + \dots + 0$

is a partition of the integer 38

Observation It isn't really necessary to write the trailing zeroes, so we just abbreviate this to

$38 = 8 + 8 + 8 + 3 + 3 + 3 + 3 + 2$

Partitions of the Integer 6

Example

$6 = 6$

$= 5 + 1$

$= 4 + 2$

$= 3 + 3$

$= 4 + 1 + 1$

$= 3 + 2 + 1$

$= 2 + 2 + 2$

$= 3 + 1 + 1 + 1$

$= 2 + 2 + 1 + 1$

$= 2 + 1 + 1 + 1 + 1$

$= 1 + 1 + 1 + 1 + 1 + 1$

Partitions of the Integer 6

Example

$6 = 6$

$= 5 + 1$

$= 4 + 2$

$= 3 + 3$

$= 4 + 1 + 1$

$= 3 + 2 + 1$

$= 2 + 2 + 2$

$= 3 + 1 + 1 + 1$

$= 2 + 2 + 1 + 1$

$= 2 + 1 + 1 + 1 + 1$

$= 1 + 1 + 1 + 1 + 1 + 1$

11 partitions altogether

3 partitions into 2 parts

4 partitions into distinct parts

4 partitions into odd parts

Exercises

Exercise

Write all the partitions of the integer 7. What is the total number? Of these, how many are partitions into 3 parts. How many are partitions into distinct parts? How many are partitions into odd parts?

Exercise Repeat for partitions of the integer 8.

Question How would you like to have to do this for the integer 2339745007313?

A Fascinating Equation

Theorem For every positive integer n, the number of partitions of n into distinct parts is equal to the number of partitions of n into odd parts.

Observation Normally a statement like this is explained by finding a bijection between two sets. We will take a completely different approach!

A Generating Function

Notation Let $f(x)$ be the generating function whose n^{th} coefficient a_n is the number of partitions of the integer n into distinct parts.

Fact

$$f(x) = (1+x)(1+x^2)(1+x^3)(1+x^4)(1+x^5)(1+x^6)\ldots$$

Another Generating Function

Notation Let $g(x)$ be the generating function whose n^{th} coefficient b_n is the number of partitions of the integer n into distinct parts.

Fact

$$g(x) = 1/(1-x)\ 1/(1-x^3) 1/(1-x^5) 1/(1-x^7)\ 1/(1-x^9)\ \ldots$$

An Incredible Identity!!

Theorem $f(x) = g(x)$, i.e., the two generating functions we have just defined have the same coefficients.

Proof

$$f(x) = (1+x)(1+x^2)(1+x^3)(1+x^4)(1+x^5)(1+x^6)\ldots$$

$$= [(1-x^2)/(1-x)][(1-x^4)/(1-x^2)][(1-x^6)/(1-x^3)]\ldots$$

$$= 1/(1-x)\ 1/(1-x^3)1/(1-x^5)1/(1-x^7)\ 1/(1-x^9)\ \ldots$$

$$= g(x)$$

An Identity Using Calculus!!

Reminder When $f(x)$ is infinitely differentiable, and $f(x) = \Sigma_{n \geq 0}\, a_n\, x^n$ is the Taylor series (about $x = 0$), then:

$$a_n = f^{(n)}(x)|_{x = 0} / n!$$

Exercise When $f(x) = (1 - 4x)^{-1/2}$

$$f^{(n)}(x) = (1 - 4x)^{(-2n-1)/2}\, n!\, C(2n, n) \text{ and } a_n = C(2n, n)$$

An Identity Using Calculus!!

Reminder When $f(x) = \Sigma_{n \geq 0} a_n x^n$, $g(x) = \Sigma_{n \geq 0} b_n x^n$
$h(x) = f(x)\, g(x)$, then $h(x)= \Sigma_{n \geq 0} c_n x^n$, where

$$c_n = \Sigma_{0 \leq k \leq n} a_k b_{n-k}$$

Exercise When $f(x) = (1 - 4x)^{-1/2}$ and $h(x) = f(x)\, f(x)$, it follows that:

$$4^n = \Sigma_{0 \leq k \leq n} C(2k, k)\, C(2n - 2k, n - k)$$

Challenge Try to give a combinatorial proof.