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17 - Advancement Operator Equations

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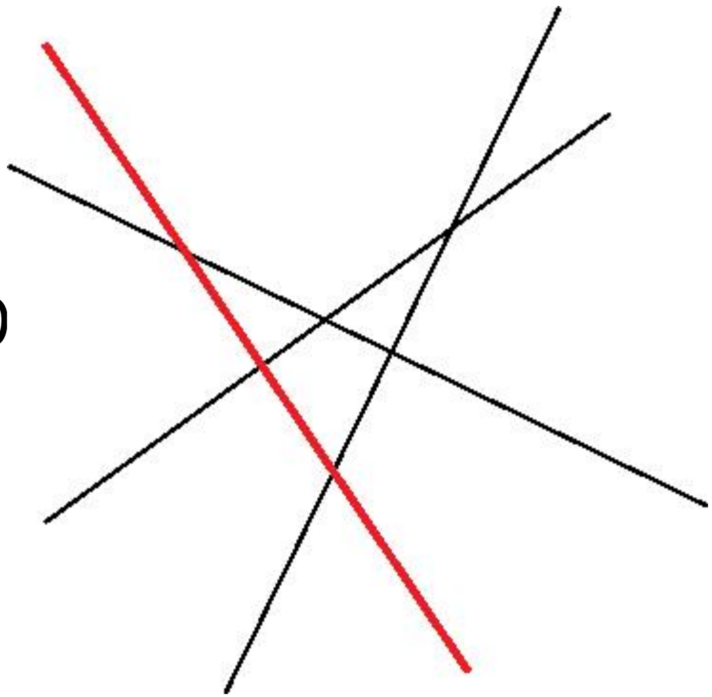
Review of Recurrence Equations (1)

Problem Let $r(n)$ denote the number of regions determined by n lines that intersect in general position.



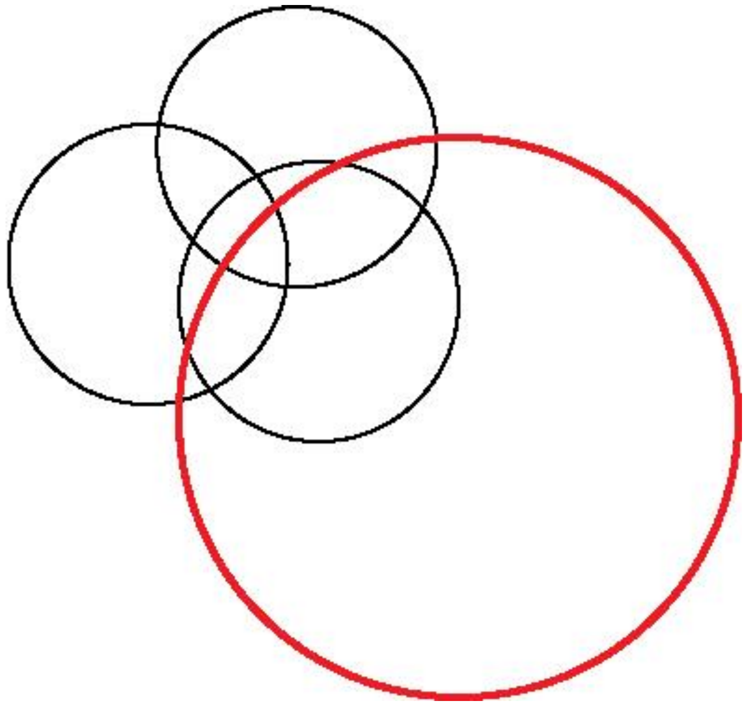
Solution

$r(1) = 2$

$r(n + 1) = r(n) + n+1$ when $n \geq 0$

Review of Recurrence Equations (2)

Problem Let $s(n)$ denote the number of regions determined by n circles that intersect in general position.



Solution

$s(1) = 2$

$s(n+1) = s(n) + 2n$ when $n \geq 0$.

Review of Recurrence Equations (3)

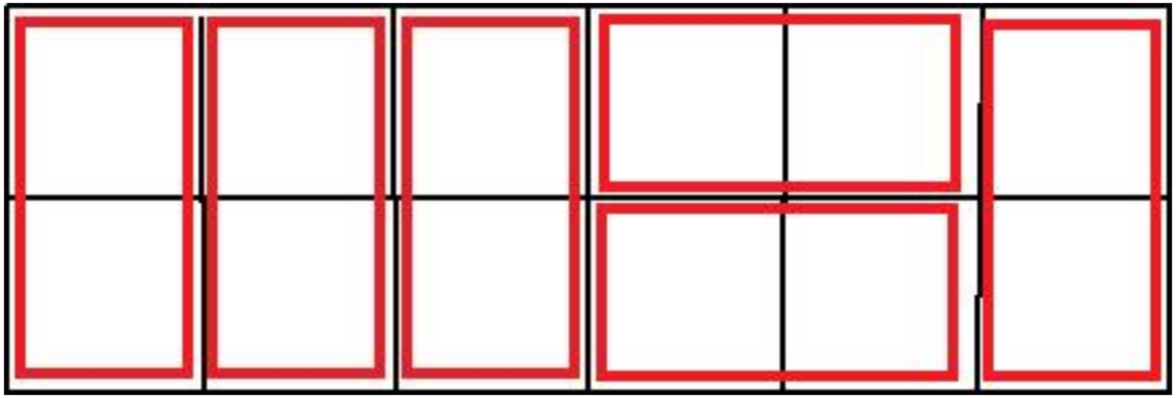
Problem Let $t(n)$ denote the number of ways to tile a $2 \times n$ grid with dominoes of size 1×2 and 2×1.

Solution

$t(1) = 1$

$t(2) = 2$

$t(n+2) = t(n+1) + t(n)$ when $n \geq 0$.



Review of Recurrence Equations (4)

Problem Let $u(n)$ denote the number of ternary sequences that do not contain 01 in consecutive positions.

Solution

$u(1) = 3$

$u(2) = 8$

$u(n+2) = 3u(n+1) - u(n)$

Review of Recurrence Equations (5)

Summary The recurrence equations in the last four examples are:

$$r(n+1) - r(n) = n+1$$

$$s(n+1) - s(n) = 2n$$

$$t(n+2) - t(n+1) - t(n) = 0$$

$$u(n+2) - 3u(n+1) + u(n) = 0$$

Developing a General Framework (1)

Observation We consider the family **V** of all functions which map the set **Z** of all integers (positive, negative and zero) to the set **C** of complex numbers. This is a more general framework than we first studied, but as will become clear, we need this additional structure to make the form of general solutions relatively easy to obtain.

Note Each of the four examples presented above have involved functions with range and domain being the set **N** of positive integers, so **V** is a more general setup.

Developing a General Framework (2)

Fact The family **V** is an infinite dimensional vector space over the field ***C*** of complex numbers, with $(f + g)(n) = f(n) + g(n)$ and $(a\, f)(n) = a(f(n))$.

Note Students should spot the "operator overloading" in these two equations, even when one of the two operators (multiplication) is indicated simply by adjacent symbols, one a scalar and the other a vector.

Note The "zero" of **V** is the constant function which maps all integers to the "zero" in ***C***.

Developing a General Framework (3)

Observation We will first focus on homogeneous linear recurrence equations. These have the following form:

$$a_0 f(n+d) + a_1 f(n+d-1) + a_2 f(n+d-2) + \dots + a_{d-1} f(n+1) + a_d f(n) = 0$$

Note The coefficients $a_0, a_1, a_2, \dots, a_d$ are complex numbers. Without loss of generality $a_0 \neq 0$. For the time being, we will also assume that $a_d \neq 0$.

Developing a General Framework (4)

Example A homogeneous equation:

$$(2+3i)g(n+3) - (8-7i)g(n+2) + 42g(n+1) - (5i)g(n) = 0$$

Example A non-homogeneous equation:

$$(2+3i)g(n+3) - (8-7i)g(n+2) + 42g(n+1) - (5i)g(n) = (2-i)(3+i)^n + 12n^3$$

Remark In order to fully understand the homogeneous case, we will need to discuss the non-homogeneous case concurrently.

Developing a General Framework (5)

Alternate Notation We define the advancement operator A on the vector space V by the rule $A\,f(n) = f(n+1)$. Note that $A^2 f(n) = f(n+2)$, $A^3 f(n) = f(n+3)$, etc. So our linear homogeneous equation

$$a_0 f(n+d) + a_1 f(n+d-1) + a_2 f(n+d-2) + \dots + a_{d-1} f(n+1) + a_d f(n) = 0$$

can then be rewritten as:

$$(a_0 A^d + a_1 A^{d-1} + a_2 A^{d-2} + \dots + a_{d-1} A + a_d)\, f(n) = 0.$$

Remark The "polynomial form" is significant!

Developing a General Framework (6)

Theorem The set S of all solutions to a homogeneous linear recurrence equation of the form:

$$(a_0A^d + a_1A^{d-1} + a_2A^{d-2} + \ldots + a_{d-1}A + a_d)\, f_n = 0$$

is a d-dimensional subspace of $\mathbf{V}$ provided both a_0 and a_d are non-zero

Conclusion The solution space can be specified entirely just by providing a basis for the subspace S.

The General Theorem

Theorem The solution space S of the operator equation:

$$(a_0 A^d + a_1 A^{d-1} + a_2 A^{d-2} + \ldots + a_{d-1} A + a_d)\, f(n) = 0$$

is a d-dimensional subspace of $\mathbf{V}$, provided both a_0 and a_d are non-zero. Furthermore, a basis for S can be formed by taking functions of the form $n^i r^n$ where $r \neq 0$ is a root of the associated polynomial and $0 \leq i < m$, with m the multiplicity of r.

An Example

Example The solution space S to

$$((A - 3)^4(A - 7 + 2i)^3(A + 5 - 8i)^2)\, f(n) = 0$$

is a 9 – dimensional subspace of V and the following 9 functions form a basis for S:

3^n	$n3^n$	$n^2 3^n$	$n^3 3^n$
$(7 - 2i)^n$	$n(7 - 2i)^n$	$n^2(7 - 2i)^n$	
$(-5 + 8i)^n$	$n(-5 + 8i)^n$		

An Example - Alternate Statement

Example The general solution to

$$((A - 3)^4(A - 7 + 2i)^3(A + 5 - 8i)^2)\, f(n) = 0$$

is:

$$\begin{aligned} f(n) = {} & c_1 3^n + c_2 n 3^n + c_3 n^2 3^n + c_4 n^3 3^n \\ & + c_5 (7 - 2i)^n + c_6 n (7 - 2i)^n + c_7 n^2 (7 - 2i)^n \\ & + c_8 (-5 + 8i)^n + c_9 n (-5 + 8i)^n \end{aligned}$$

Analogies with Partial Fractions

Example Given a proper rational function $p(x)/q(x)$ whose denominator polynomial $q(x)$ can be factored as

$$q(x) = (x - 3)^3 x^2(x^2 + 2x + 9)$$

there are constants $c_1, c_2, c_3, c_4, c_5, c_6$ and c_7 so that

$$\begin{aligned} p(x)/q(x) = {} & c_1/(x - 3) + c_2/(x - 3)^2 + c_3/(x - 3)^3 \\ & + c_4/x + c_5/x^2 \\ & + c_6/(x^2 + 2x + 9) + c_7x/(x^2 + 2x + 9) \end{aligned}$$

Analogies with Differential Equations

Example Let D be the differential operator, i.e., D f is the derivative of f. Then the solution to the equation:

$$(D - 3)^3 D^2(D^2 + 16) f = 0$$

has the form:

$$f(x) = c_1 e^{3x} + c_2 x e^{3x} + c_3 x^2 e^{3x} + c_4 + c_5 x + c_6 e^{4i} + c_7 e^{-4i}$$

where c_6 and c_7 are complex conjugates.

The Case $d = 1$

Theorem Let a_0 and a_1 be non-zero complex numbers, and set $r = (-a_1/a_0)$. Then the solution space S of the advancement operator equation $(a_0A + a_1)f(n) = 0$ is a 1-dimensional subspace of V and the function r^n is a basis, i.e., every solution is of the form $f(n) = c_1 r^n$ where c_1 is a constant.

Proof Let f be any solution to $(a_0A + a_1)f(n) = 0$, and let $c_1 = f(0)$. We show that $f(n) = c_1 r^n$ for all integers n. We first show that $f(n) = c_1 r^n$ when $n \geq 0$. We do this by induction on n.

The Case d = 1 (Part 2)

Base Case The base case is $n = 0$, where the left hand side is $f(0) = c_1$, and the right hand side is $c_1\, r^0$. But since $r \neq 0$, the right hand side is $c_1 \cdot 1 = c_1$. So the statement holds when $n = 0$.

The Inductive Step Now assume that $f(k) = c_1\, r^k$ for some $k \geq 0$. Since $(a_0A + a_1)f(n) = 0$ for all integers n, we know that:

$$(a_0A + a_1)f(k) = 0$$
$$(a_0f(k+1) + a_1f(k) = 0$$
$$f(k+1) = (-a_1/a_0)c_1r^k$$
$$f(k+1) = r\; c_1\; r^k$$
$$f(k+1) = c_1r^{k+1}$$

The Case d = 1 (Part 3)

Negative Integers It remains only to show that $f(n) = c_1 r^n$ for all integers $n \leq 0$. This is equivalent to showing that $f(-n) = c_1 r^{-n}$ for all $n \geq 0$. This is done by induction and the argument is a trivial modification of what we have just done.

Conclusion We have verified the assertion that the solution space to the homogeneous equation

$$(a_0 A^d + a_1 A^{d-1} + a_2 A^{d-2} + \ldots + a_{d-1} A + a_d)\, f(n) = 0$$

with a_0 and a_d non-zero is a d-dimensional subspace of **V** when $d = 1$.

Towards the General Case (1)

Exercise Note that

$A^2 + 2A - 35 = (A + 7)(A - 5)$

Example The functions $(-7)^n$ and 5^n are solutions to the equation:
$(A^2 + 2A - 35) f(n) = 0$.

Observation If $r \neq 0$ and r is a root of the advancement operator polynomial, then r^n is a solution.

Towards the General Case (2)

Exercise Show that

$$A^2 + (-12 + i)A + 41 - i = (A - 5 - 2i)(A - 7 + 3i)$$

Example The functions $(5 + 2i)^n$ and $(7 - 3i)^n$ are solutions to the equation:
$(A^2 + (-12 + i)A + 41 - i)\, f(n) = 0.$

Observation If $r \neq 0$ and r is a root of the advancement operator polynomial, then r^n is a solution.

Towards the General Case (3)

Example Note that $A^2 - 10A + 25 = (A - 5)^2$.

Also note that the functions 5^n and $n5^n$ are solutions to the equation:

$$(A - 5)^2 f(n) = 0.$$

Observation If $r \neq 0$ and r is a root of multiplicity 2, then r^n and $n r^n$ are solutions.

Towards the General Case (4)

Example The functions $(5 - 2i)^n$ and $n(5 - 2i)^n$ are solutions to the equation:

$$(A - 5 + 2i)^2 f(n) = 0.$$

Observation If $r \neq 0$ and r is a root of multiplicity 2, then r^n and $n\, r^n$ are solutions.

Towards the General Case (5)

Lemma If $p(A)$ is a polynomial in the advancement operator A, $r \neq 0$ and r is a root of multiplicity m, then each of the following functions is a solution of the equation: $p(A)\, f(n) = 0$

$$r^n \quad n\, r^n \quad n^2\, r^n \quad n^3\, r^n \quad n^4\, r^n \quad \ldots \quad n^{m-1}\, r^n$$

Proof This is an easy exercise using an inductive argument.

Towards the General Case (6)

Example The general solution to

$$((A - 3)^4(A - 7 + 2i)^3(A + 5 - 8i)^2)\, f(n) = 0$$

is:

$$\begin{aligned} f(n) = {} & c_1 3^n + c_2 n 3^n + c_3 n^2 3^n + c_4 n^3 3^n \\ & + c_5(7 - 2i)^n + c_6 n(7 - 2i)^n + c_7 n^2(7 - 2i)^n \\ & + c_8(-5 + 8i)^n + c_9 n(-5 + 8i)^n \end{aligned}$$

Towards the General Case (7)

Example The solution space to:

$$((A - 3)^4(A - 7 + 2i)^3(A + 5 - 8i)^2)\, f(n) = 0$$

is a 9-dimensional subspace of V and the following functions are a basis:

3^n $n3^n$ $n^2 3^n$ $n^3 3^n$

$(7 - 2i)^n$ $n(7 - 2i)^n$ $n^2(7 - 2i)^n$

$(-5 + 8i)^n$ $n(-5 + 8i)^n$

Analogies with Partial Fractions

Example Given a proper rational function $p(x)/q(x)$ whose denominator polynomial $q(x)$ can be factored as

$$q(x) = (x - 3)^3 x^2(x^2 + 2x + 9)$$

there are constants $c_1, c_2, c_3, c_4, c_5, c_6$ and c_7 so that

$$\begin{aligned} p(x)/q(x) = {} & c_1/(x - 3) + c_2/(x - 3)^2 + c_3/(x - 3)^3 \\ & + c_4/x + c_5/x^2 \\ & + c_6/(x^2 + 2x + 9) + c_7x/(x^2 + 2x + 9) \end{aligned}$$

Analogies with Differential Equations

Example Let D be the differential operator, i.e., $D f$ is the derivative of f. Then the solution to the equation:

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has the form:

$$f(x) = c_1 e^{3x} + c_2 x e^{3x} + c_3 x^2 e^{3x} + c_4 + c_5 x + c_6 e^{4i} + c_7 e^{-4i}$$

where c_6 and c_7 are complex conjugates.

The Non-Homogeneous Case

Theorem Let $p(A)\,f = g$ be a non-homogeneous equation. If h_0 is any solution to this equation, then the general solution is $h_0 + f$ where f is a solution to the associated homogeneous equation $p(A)\,f = 0$.

Note The proof of this theorem is relatively straightforward.

Terminology The function h_0 is referred to as a particular solution to $p(A)\,f = g$.

The Non-Homogeneous Case (2)

Example For the non-homogeneous equation $(A - 3)\, f(n) = 8\,(5)^n$, the function $h_0 = 4 \cdot 5^n$ is a particular solution. Accordingly, the general solution has the form:

$$f(n) = c_1 3^n + 4 \cdot 5^n$$

The Non-Homogeneous Case (3)

Strategy To solve the non-homogeneous equation $p(A)\, f(n) = g(n)$, first find by hook or crook a particular solution $h_0(n)$. Then solve the homogenous equation $p(A) = 0$ to find a d-dimensional solution space S where d is the degree of the polynomial p. It follows that all solutions have the form:

$$h_0(n) + f(n)$$

where $f(n)$ is in S.

A Detail on Zero as a Root

Observation Suppose $p(A) = A^m q(A)$ where $m \geq 1$ and the constant term in $q(A)$ is non-zero. Then the dimension of the solution space is the degree of the polynomial $q(A)$ and solutions to the equation $p(A) f(n) = 0$ are just shifts of solutions to $q(A) f(n) = 0$.

Applying the Theorem

Example The general solution to

$$((A - 3)^4(A - 7 + 2i)^3(A + 5 - 8i)^2(A - 1)^5)\ f(n) = 0$$

is:

$$\begin{aligned} f(n) = {} & c_1 3^n + c_2 n 3^n + c_3 n^2 3^n + c_4 n^3 3^n \\ & + c_5(7 - 2i)^n + c_6 n(7 - 2i)^n + c_7 n^2(7 - 2i)^n \\ & + c_8(-5 + 8i)^n + c_9 n(-5 + 8i)^n \\ & + c_{10} + c_{11} n + c_{12} n^2 + c_{13} n^3 + c_{14} n^4 \end{aligned}$$

Using Initial Conditions

Example Find the solution to $(A^2 - 7A + 10)\,f(n) = 0$ with $f(0) = 9$ and $f(1) = 27$.

Solution The general solution is $f(n) = c_1 2^n + c_2 5^n$. So our constraints become:

$$c_1 + c_2 = 9$$
$$2c_1 + 5c_2 = 27$$

This forces $c_1 = 6$ and $c_2 = 3$, so the answer is

$$f(n) = 6 \cdot 2^n + 3 \cdot 5^n$$

Using Initial Conditions (2)

Example For the non-homogeneous equation $(A - 3)\, f(n) = 8\,(5)^n$, the function $h_0 = 4 \cdot 5^n$ is a particular solution. Accordingly, the general solution has the form:

$$f(n) = c_1 3^n + 4 \cdot 5^n$$

Exercise Find the solution to $(A - 3)\, f(n) = 8\,(5)^n$ subject to the requirement that $f(3) = 118$.
This requires $118 = 9c_1 + 100$, so $c_1 = 2$ and the answer is $f(n) = 2 \cdot 3^n + 4 \cdot 5^n$

When 0 is a root

Observation Consider the equation $A^m f(n) = 0$. A solution must satisfy $f(n+m) = 0$ for all integers n. This forces $f(n) = 0$ for all n, i.e., the only solution is the zero function.

Consequence If $p(A) = A^m q(A)$ where $q(A)$ is a polynomial of degree $d \geq 1$, then the solution space of the equation $p(A) f(n) = 0$ will be a d-dimensional subspace of $\mathbf{V}$.

Remark This explains why we have focused on the form:

$$(a_0 A^d + a_1 A^{d-1} + a_2 A^{d-2} + \ldots + a_{d-1} A + a_d) f(n) = 0$$

with both a_0 and a_d non-zero.

The Non-Homogeneous Case

Theorem Let $p(A)\, f = g$ be a non-homogeneous equation. If h_0 is any solution to this equation, then the general solution is $h_0 + f$ where f is a solution to the associated homogeneous equation $p(A)\, f = 0$.

Note The proof of this theorem is relatively straightforward.

Terminology The function h_0 is referred to as a particular solution to $p(A)\, f = g$.

Proof of the General Theorem

Theorem The solution space S of the operator equation:

$$(a_0A^d + a_1A^{d-1} + a_2A^{d-2} + \ldots + a_{d-1}A + a_d)\, f(n) = 0$$

is a d-dimensional subspace of **V**, provided both a_0 and a_d are non-zero. Furthermore, a basis for S can be formed by taking functions of the form $n^i r^n$ where $r \neq 0$ is a root of the associated polynomial and $0 \leq i < m$, with m the multiplicity of r.

A Key Lemma

Theorem Let $d \geq 1$, let $r, s \neq 0$. Then let

$$p(n) = a_0 n^d + a_1 n^{d-1} + a_2 n^{d-2} + \ldots + a_{d-1} n + a_d$$

be a complex polynomial of degree d, i.e., the leading coefficient $a_0 \neq 0$. Then $(A - r)\, p(n) r^n = q(n) r^n$ for some polynomial $q(n)$ of degree $d - 1$.

Furthermore, if $s \neq r$, then $(A - s)\, p(n) r^n = q'(n) r^n$ for some polynomial $q'(n)$ of degree d.

A Useful Corollary

Corollary Let $d \geq 0$ and let

$$p(n) = a_0 n^d + a_1 n^{d-1} + a_2 n^{d-2} + \ldots + a_{d-1} n + a_d$$

be a complex polynomial of degree d, i.e., the leading coefficient $a_0 \neq 0$. Then there is a uniquely determined polynomial $q(n)$ of degree $d + 1$ so that

$$(A - r)\, q(n) r^n = p(n) r^n$$

Outline of Arguments

Theorem Let $m \geq 1$ and let $r \neq 0$. Then the solution space S of the equation $(A - r)^m f(n)$ is an m-dimensional subspace of $\mathbf{V}$ and the following functions form a basis for S:

$$r^n \quad n\, r^n \quad n^2\, r^n \quad n^3\, r^n \quad n^4\, r^n \quad \ldots \quad n^{m-1}\, r^n$$

Remark There are three parts to the proof. First is showing that each of these functions is a solution. Second is showing that every solution is a linear combination of these functions. Third is showing that they are linearly independent. We will sketch these arguments in class.

Analysis of Solutions

Questions Consider the Fibonacci sequence:

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, …

Is the 1000th term more or less than 10^{300} ?

Does the ratio $f(n+1)/f(n)$ tend to a limit.

Answers The equation is $(A^2 - A - 1) f(n) = 0$.
There are two roots: $(1 + \sqrt{5})/2$ and $(1 - \sqrt{5})/2$
and the initial conditions are $f(0) = f(1) = 1$.

Analysis of Solutions (2)

Answer For the advancement operator equation $(A^2 - A - 1) f(n) = 0$, the ratio of $f(n+1)/f(n)$ tends to $(1 + \sqrt{5})/2$ independent of the initial conditions and the values of c_1 and c_2. As a result, the 1000^{th} term is very close to $c_1 ((1+\sqrt{5})/2)^{1000}$. With this information in hand and knowing the value of c_1, we can easily compare this term with 10^{300}.