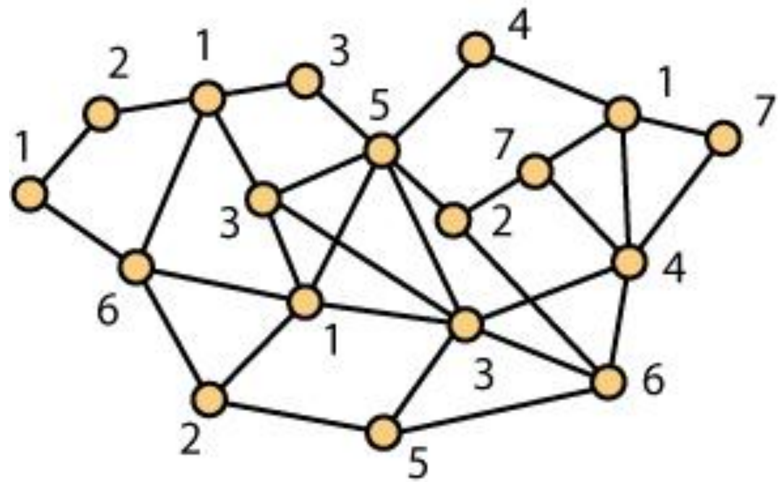
November 14, 2017

7 - Graph Coloring

William T. Trotter
trotter@math.gatech.edu

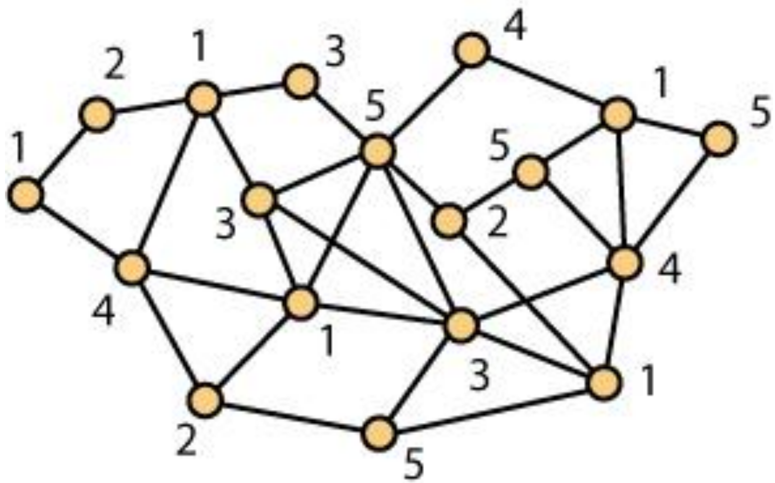
Chromatic Number

Definition A t-coloring of a graph G is an assignment of integers (colors) from $\{1, 2, \ldots, t\}$ to the vertices of G so that adjacent vertices are assigned distinct colors. We show a 7-coloring of the graph below.



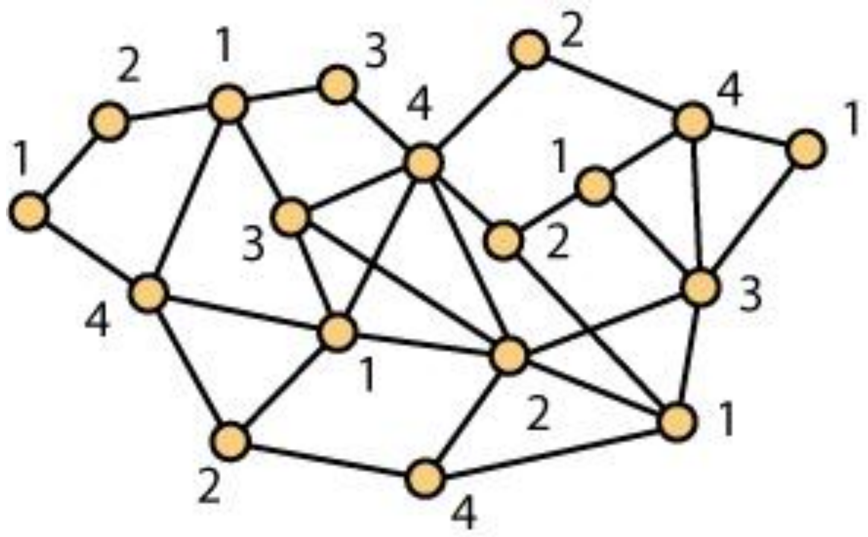
Chromatic Number (2)

Optimization Problems Given a graph G, what is the least t so that G has a t-coloring? This integer is called the chromatic number of G and is denoted $\chi(G)$. The coloring below is the same graph but now we illustrate a 5-coloring, so $\chi(G) \le 5$.



Chromatic Number (3)

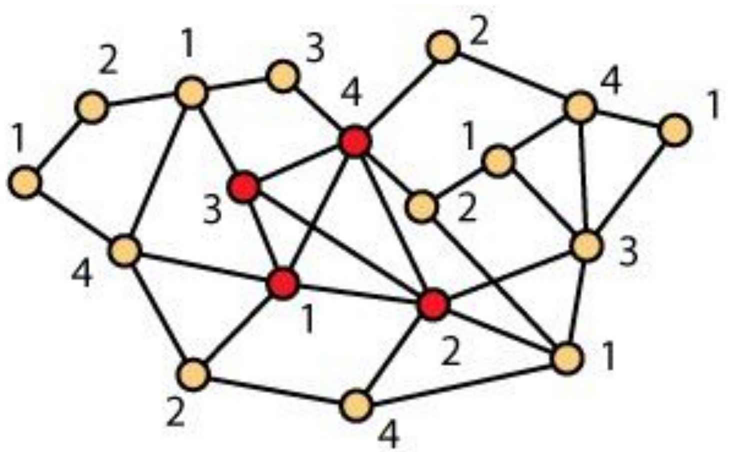
Optimization Problems The coloring below is the same graph but now we illustrate a 4-coloring, so $\chi(G) \leq 4$.



Maximum Clique Size

Definition Given a graph G, the maximum clique size of G, denoted $\omega(G)$, is the largest integer k for which G contains a clique (complete subgraph) of size k.

Trivial Lower Bound $\chi(G) \geq \omega(G)$ so in this case, we know $\chi(G) = \omega(G) = 4$.



Maximum Clique Size (2)

Observation When $n \geq 2$, the odd cycle C_{2n+1} satisfies $\chi(C_{2n+1}) = 3$ and $\omega(C_{2n+1}) = 2$ so the inequality

$$\chi(G) \geq \omega(G)$$

need not be tight. Later, we will say much more about this inequality.

Computing Chromatic Number

Computational Complexity Detail Given a graph G and an integer t, the yes-no question: "Is $\chi(G) \leq t$?" belongs to the class **NP**.

Explanation It is obvious that a "yes" answer has a certificate that can be checked very efficiently. The certificate is just the assignment of colors to vertices.

Chromatic Number – A Special Case

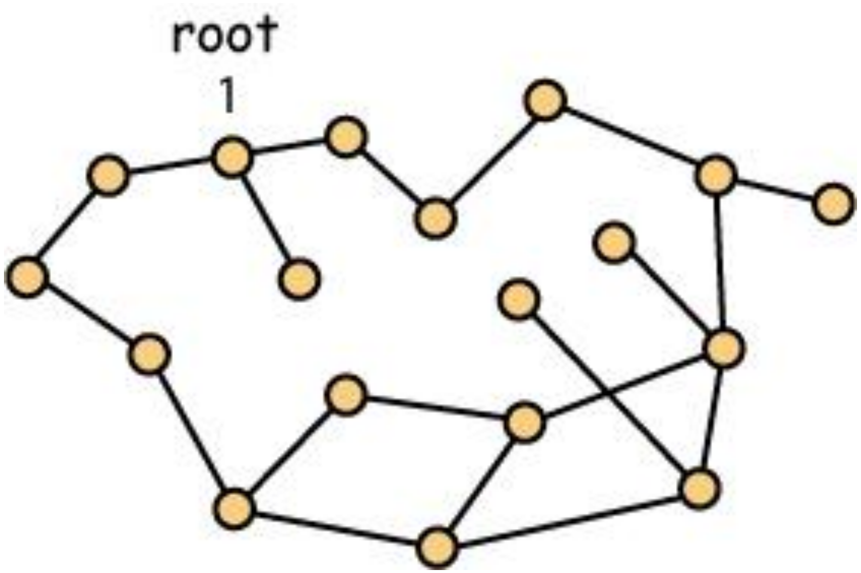
Computational Complexity Detail Given a graph G and an integer t, the yes-no question: "Is $\chi(G) \leq 2$?" belongs to the class **P**.

Basic Idea It is easy to see that $\chi(G) \geq 3$ when G contains an odd cycle. The algorithm we present will show that $\chi(G) \leq 2$ if and only if G does not contain an odd cycle. CS students will recognize that the algorithm uses "breadth-first" search. We will revisit this concept in greater detail later in the course.

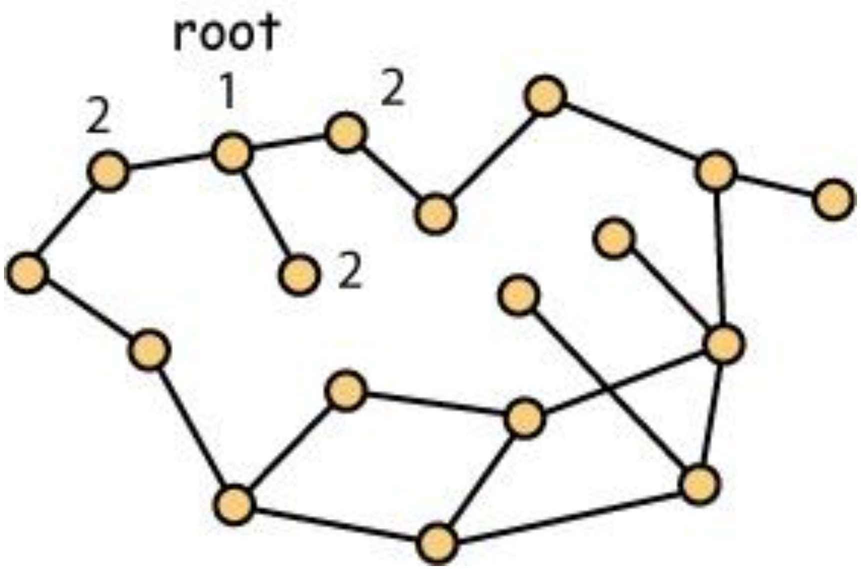
Chromatic Number – A Special Case (2)

Algorithm Choose an arbitrary vertex x and color it 1. Then find all uncolored vertices that are neighbors of colored vertices and color them with 2. Pause to check if you have an edge among the vertices colored 2. If yes, there is a triangle, so $\chi(G) \geq 3$ and the answer is "no". If no, find all uncolored neighbors of colored neighbors and color them 1. Pause to see if there are any edges among the vertices just colored. If yes, there is 5-cycle in G and the answer is "no". If yes, continue, alternating colors 1 and 2. Either the graph will be eventually 2-colored or we will find an odd cycle.

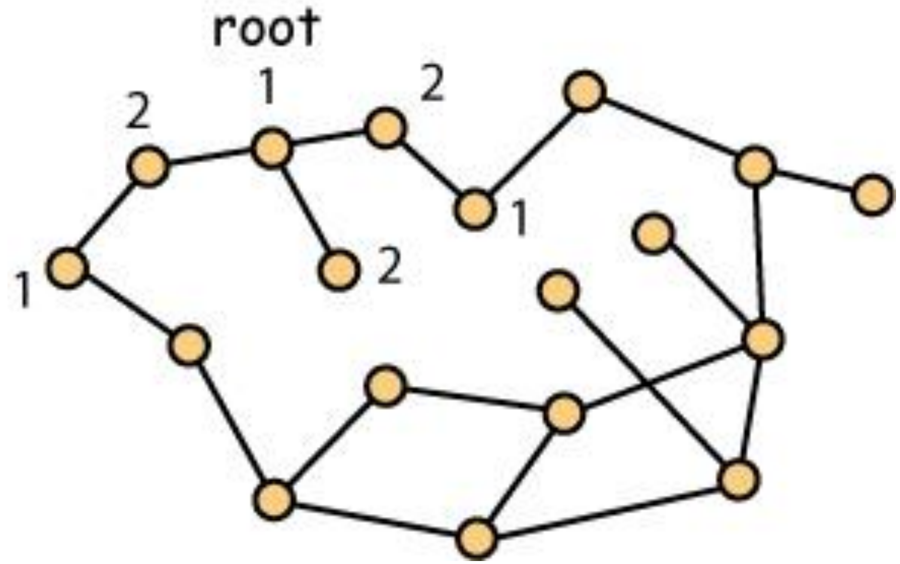
Applying the Algorithm



Applying the Algorithm (2)

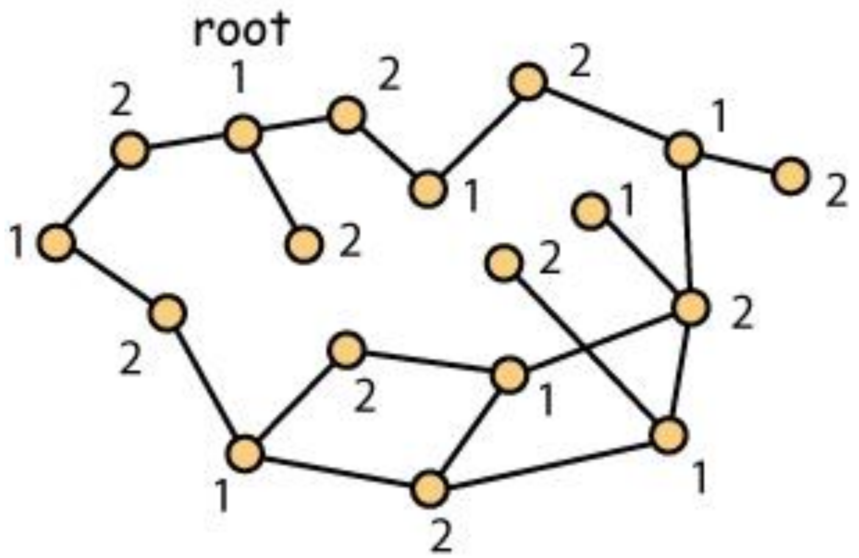


Applying the Algorithm (3)



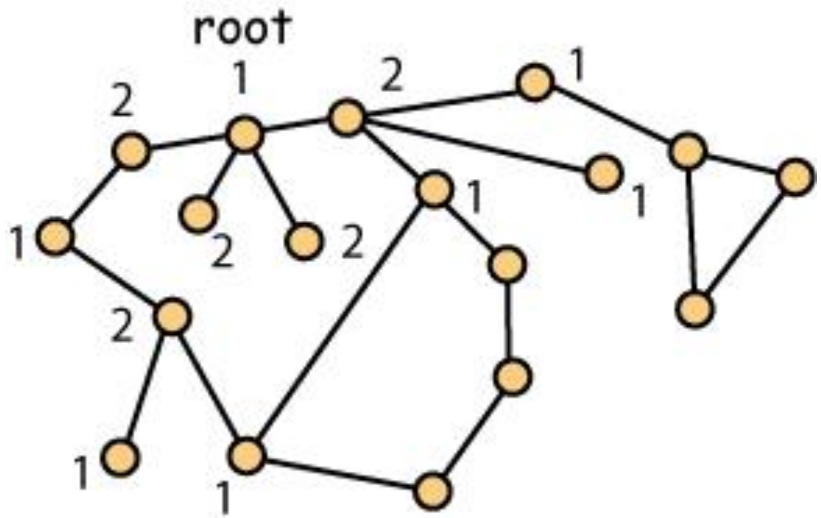
Applying the Algorithm (4)

Observation After several more steps, the algorithm halts with a 2-coloring of *G*.



Applying the Algorithm (5)

Observation Here's an example (using a different graph) of how the algorithm will detect an odd cycle.



Another Way to Earn a Million Bucks!!

Computational Complexity Question Given a graph G and an integer t, the yes-no question: "Is $\chi(G) \leq 3$?" belongs to the class **NP**. Does it also belong to **P**?

Remark As was stated explicitly in our lectures, I am not encouraging Math 3012 students to ponder on this question, as the greatest minds in the world have spent enormous amounts of time on it without success. However, it does represent just how challenging the delightful world of combinatorics can be.

The Inequality Can Fail Arbitrarily

Observation We have previously noted that $\chi(G) \geq \omega(G)$ for every graph G.

Now will given three different explanations for the following result.

Theorem For every $t \geq 3$, there is a graph G with $\chi(G) = t$ and $\omega(G) = 2$.

Note A clique of size 3 is also called a triangle. Graphs with $\omega(G) \leq 2$ are said to be triangle-free. So the fact can be rephrased as asserting that there are triangle-free graphs with arbitrarily large chromatic number.

A Construction Using the Pigeon-Hole Principle

Basic Idea Proceed by induction. When $t = 3$, take G as the odd cycle C_5. Now suppose that for some $t \geq 3$, we have a triangle-free graph G with $\chi(G) = t$. Here's how we build a new triangle-free graph whose chromatic number is $t + 1$. Suppose G has m vertices labelled $x_1, x_2, \ldots, x_m$.

Start with a "large" independent set Y. For each m-element subset $\{y_1, y_2, \ldots, y_m\}$ of Y, attach a copy of G with x_i adjacent to y_i for each $i = 1, 2, \ldots, m$. This works if Y has size at least $t(m - 1) + 1$ by the Pigeon-Hole principle.

The Mycielski Construction

Basic Idea Proceed by induction. When $t = 3$, take G as the odd cycle C_5. Now suppose that for some $t \geq 3$, we have a triangle-free graph G with $\chi(G) = t$. Here's how we build a new triangle-free graph whose chromatic number is $t + 1$.

Start with a copy of G. Then add an independent set Y containing a "mate" y_x for every vertex x of G. The mate y_x has exactly the same neighbors in G as does x.

Then add one new vertex x_0 which is adjacent to every vertex in Y but to none of the vertices in G.

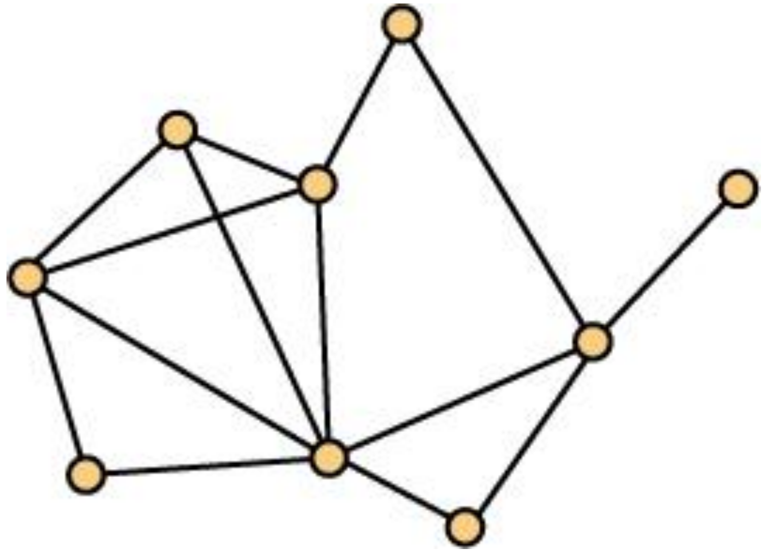
Shift Graphs

Definition When $n \geq 2$, the shift graph S_n has $C(n, 2)$ vertices and these are the 2-element subsets of $\{1, 2, \ldots, n\}$. For each 3-element subset $\{i, j, k\}$ of $\{1, 2, \ldots, n\}$, with $i < j < k$, the vertex $\{i, j\}$ is adjacent to the vertex $\{j, k\}$ in S_n.

Theorem For every $n \geq 2$, the chromatic number of the shift graph S_n is the least positive integer t so that $2^t \geq n$.

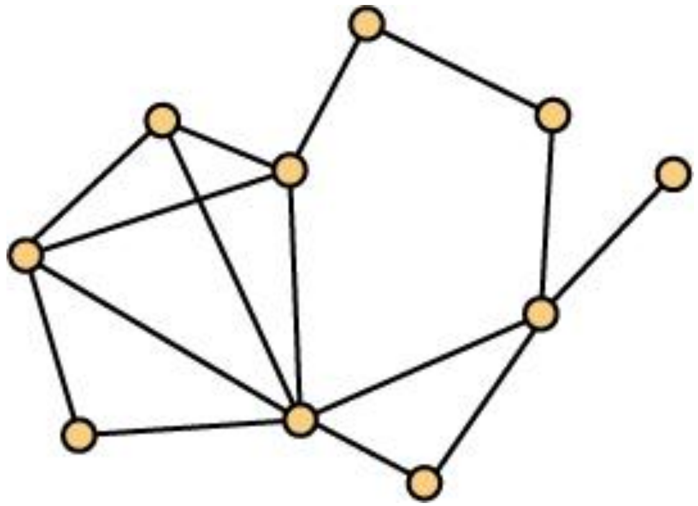
Perfect Graphs

Definition A graph G is perfect if $\chi(H) = \omega(H)$ for every induced subgraph H of G. The graph shown below is perfect.



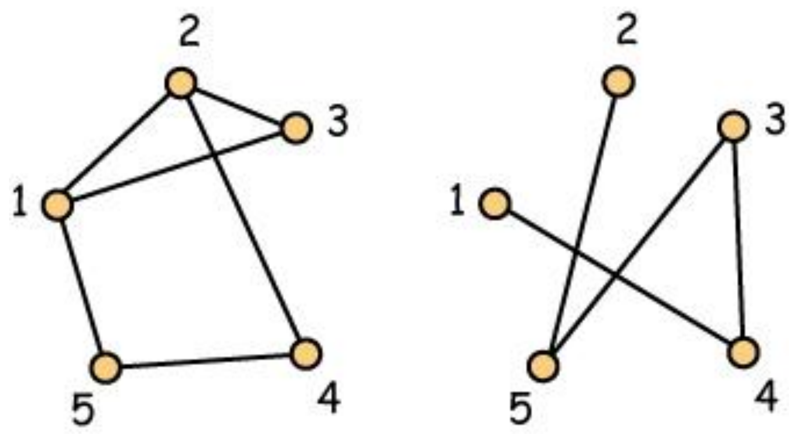
Perfect Graphs and Odd Cycles

Observation A graph G is not perfect if contains an odd cycle as an induced subgraph. The graph shown below is not perfect. Note that it contains C_5 as an induced subgraph.



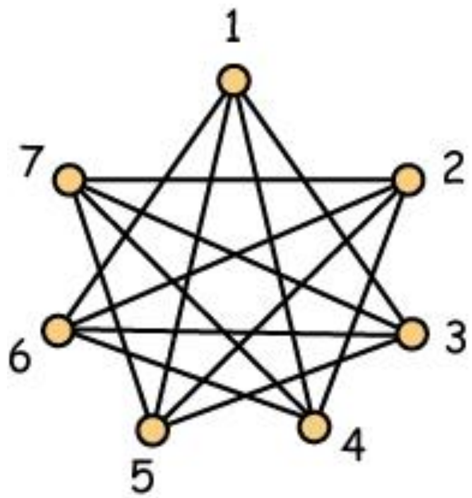
The Complement of a Graph

Definition The complement of a graph G, denoted G^c is the graph having the same vertex as G but a pair xy of distinct vertices forms an edge in G^c if and only if it does not form an edge in G. In the figure below, we show two graphs with vertex set $\{1, 2, 3, 4, 5\}$. Each is the complement of the other.



The Complement of a Graph

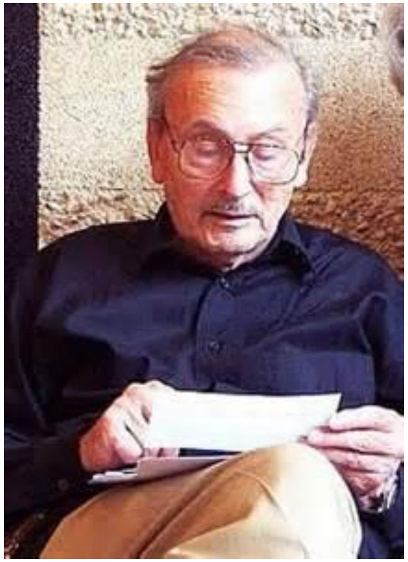
Observation A graph G is not perfect if its complement contains an odd cycle as an induced subgraph.



Observation When G is the complement of C_{2n+1}, with $n \geq 2$, $\chi(G) = n + 1$ and $\omega(G) = n$.

Berge's Perfect Graph Conjecture

Conjecture (Claude Berge, 1961) *A* graph *G* is perfect if and only if neither the graph nor its complement contains an odd cycle as an induced subgraph.



The Perfect Graph Theorem

Historical Note The following result was proven by Laszlo Lovász in 1972. Lovász has won numerous international prizes, including the 2010 Kyoto Prize (50 million yen ≈ USD 550K), the Wolf Prize, the Fulkerson Prize (twice), the Polya Prize and the Gödel Prize. As a youngster, he won three consecutive gold medals in the Math Olympiad.

Theorem A graph G is perfect if and only if its complement is perfect.

The Strong Perfect Graph Theorem

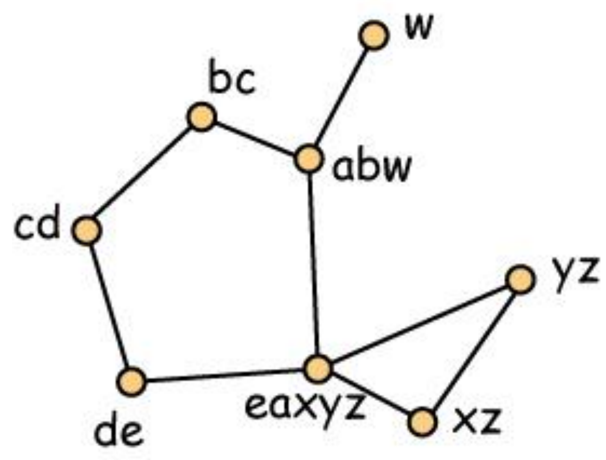
Historical Note The following result is proven in a 178 page paper appeared in the *Annals of Mathematics* in 2006 and won the 2009 Fulkerson Prize and a cash award of $10,000.

Theorem (Chudnovsky, Robertson, Seymour, Thomas) *A* graph *G* is perfect if and only if neither the graph nor its complement contains an odd cycle as an induced subgraph.



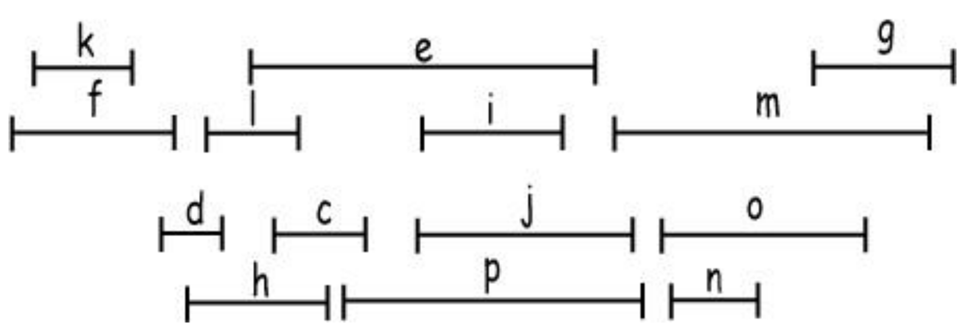

Intersection Graphs

Definition Let $F = \{A_x : x \in X\}$ be a family of sets. We associate with F an intersection graph G where the vertices of G are the elements of X and xy is an edge in G when the sets A_x and A_y intersect.



Interval Graphs

Definition A graph G is called an interval graph when it is the intersection graph of a family of closed intervals of $\mathbb{R}$. For the family shown below, c and p intersect while c and n do not.



Interval Graphs are Perfect

Algorithm Given a representation of an interval graph, apply First Fit (Greedy) and color in the order of left end points.